1.) $\phi^4$ FIELD THEORY FOR AN ISING FERROMAGNET

The $\phi^4$ field theory discussed in class with a Euclidean action of the form,
\[ S = \int d^d x \int_0^\beta d\tau L, \]  
with Lagrangian density,
\[ L = \frac{1}{2}[(\partial_\tau \phi)^2 + (\nabla \phi)^2 + r\phi^2] + \frac{u}{4}\phi^4, \]
can be used to extract properties of the transverse field quantum Ising ferromagnet both at $T = 0$ and $T \neq 0$. Within a simple mean-field treatment, the ferromagnetic to paramagnetic phase transition boundary in the $h$-$T$ plane (where $h$ is the transverse field) occurs when $r(h,T) = 0$. For temperatures near the finite temperature transition, $T \approx T_c$, one can expand as $r = a(T - T_c) + ...$, whereas at $T = 0$ near the quantum transition one has $r = a_0(h - h_c) + ...$.

a.) Within a saddle-point plus Gaussian fluctuation treatment (setting $u = 0$) of the partition function, $Z = \int [D\phi] \exp(-S)$, obtain an expression for the spin-structure function, $C(\vec{k},\omega)$, in the paramagnetic phase ($r > 0$) as a function of the temperature $T$. Here $C(\vec{k},\omega)$ is the space-time Fourier transform of the spin-spin correlation function,
\[ C(\vec{x} - \vec{x}',t - t') = \langle \hat{\phi}(\vec{x},t)\hat{\phi}(\vec{x}',t') \rangle. \]  
[Hint: First calculate the thermal Green’s function (as a function of $\vec{k}$ and $\omega_n$), analytically continue to extract the response function, and then use the FDT theorem to get the spin structure function]. The spin structure function in a quantum magnet can be measured rather directly via neutron scattering.

b.) Fourier transform your result from part (a) to obtain an expression for the equal time spin-spin correlation function,
\[ C(\vec{x},t = 0) = \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} C(\vec{k},\omega), \]  
and extract the large $|\vec{x}|$ asymptotic behavior at both the quantum ($T = 0$) and classical ($T \neq 0$) phase transitions in general dimension $d$.

c.) In the 2d quantum paramagnet at $T = 0$, use your result from part (b) to calculate the equal-time spin-spin correlation function, and show that it vanishes exponentially at large $\vec{x}$: $C(x) \sim e^{-|x|/\xi}$. How does the correlation length, $\xi$, diverge with $r \rightarrow 0$ as the transition is approached?

In Ising ferromagnets, the long-range dipolar interaction favors the formation of domains of positive or negative magnetization, which are separated by domain walls. For a flat domain wall with normal in the...
x-direction, the magnetization profile depends on \( x \), with \( \langle \phi(x) \rangle = M(x) \) approaching the equilibrium values \( \pm M_0 \) for \( x \to \pm \infty \). The interfacial profile can be extracted from \( \phi^4 \) theory.

d.) Show that in the ferromagnet with \( r < 0 \), a saddle point evaluation of the action leads to a simple differential equation for the interfacial profile which can be cast in the form,

\[
(-\xi^2 \partial_x^2 - 1 + f^2) f = 0,
\]

where \( f(x) = M(x)/M_0 \) with \( M_0 = \sqrt{|r|/u} \) and \( \xi = \sqrt{1/|r|} \).

e.) Noting that this equation is equivalent to a classical Newtonian equation of motion, with \( x \) playing the role of time and \( f \) a spatial coordinate, show that it can be cast in the form:

\[
(\partial_x f)^2 = \frac{(1 - f^2)^2}{2\xi^2}.
\]

f.) Solve for the interfacial profile \( f(x) \). What happens to the interfacial width as the ferromagnetic-paramagnetic phase boundary is approached, \( |r| \to 0 \)?

2.) \( O(N) \phi^4 \) FIELD THEORY AT LARGE N

In class, we studied the scalar \( \phi^4 \) theory in the mean-field approximation and corrections to it. We saw that fluctuation effects are large in the critical region for dimensions \( D < 4 \). One approach to get around this problem is to introduce another small parameter to control the behavior in this region. Consider the so-called \( O(N) \) model,

\[
Z = \int [D\vec{\phi}] e^{-S[\vec{\phi}]},
\]

with action,

\[
S[\vec{\phi}] = \int dx \left\{ \frac{1}{2} |\nabla \vec{\phi}|^2 + \frac{u}{2} |\vec{\phi}|^2 + \frac{u}{4N} (|\vec{\phi}|^2)^2 \right\},
\]

where the integration is over \( D \)-dimensional space (or space-time) with position vector \( \vec{x} = (x_1, x_2, ..., x_D) \). Here \( \vec{\phi} \) is an \( N \)-component real vector with components denoted \( \phi_\mu \) and \( \mu = 1, 2, 3, ..., N \). Here, we have used the notation, \( |\vec{\phi}|^2 = \sum_{\mu=1}^N \phi_\mu^2 \) and \( |\nabla \vec{\phi}|^2 = \sum_{\mu=1}^N (\nabla \phi_\mu)^2 \), where \( \nabla \) denotes a derivative in \( D \)-dimensions.

The action for the \( O(N) \) model is invariant under global (spatially-independent) rotations of the vector field, \( \phi_\mu(\vec{x}) \to \mathcal{O}_{\mu\nu} \phi_\nu(\vec{x}) \), for arbitrary \( N \times N \) orthogonal matrix, \( \mathcal{O} \), satisfying \( \mathcal{O}^T = \mathcal{O}^{-1} \). For all \( N \geq 2 \) this is a continuous symmetry (since the rotations can be by continuous angles), in contrast to the discrete \( \mathbb{Z}_2 \) symmetry of the Ising model. This will have important implications which we will find below. In this problem, we will be interested in exploring the properties of the \( O(N) \) model in the limit \( N \to \infty \).

As in class, we will consider the Green’s function of the field \( \vec{\phi} \), defined as the average,

\[
G(\vec{x} - \vec{x}') = N^{-1} \langle \vec{\phi}(\vec{x}) \cdot \vec{\phi}(\vec{x}') \rangle,
\]

and its Fourier transform, denoted \( G(\vec{k}) \), defined through,

\[
G(\vec{k}) = \int d\vec{x} \ G(\vec{x}) e^{-i\vec{k} \cdot \vec{x}} \quad G(\vec{x}) = \int_{\vec{k}} \ G(\vec{k}) e^{i\vec{k} \cdot \vec{x}},
\]
where \( \mathbf{k} \) is a \( D \)-component momentum-space vector and we have employed the shorthand notation, \( \int_{\mathbf{k}} \frac{3}{(2\pi)^D} \int d\mathbf{k}. \) In the absence of the interaction, \( u = 0 \), the “bare” Green’s function is given by, \( G_0(\mathbf{k}) = (r + \mathbf{k}^2)^{-1} \), with \( r \geq 0 \). The exact Green’s function in the paramagnetic phase, \( \langle \hat{\phi} \rangle = 0 \), can be formally expressed as, \( G^{-1}(\mathbf{k}) = G_0^{-1}(\mathbf{k}) - \Sigma(\mathbf{k}), \) where \( \Sigma(\mathbf{k}) \) is the self-energy.

Here, the first goal will be to obtain a closed form integral equation for the full Green’s function at \( N = \infty \) by exploring the diagrammatic expansion in powers of the interaction \( u \). We will then analyze the integral equation to extract the properties of the model, focussing on the fate/properties of the paramagnet to ferromagnet phase transition.

Since, for \( u = 0 \), the action \( S[\phi, u = 0] \) is a sum of decoupled terms for each component of \( \mathbf{\phi} \), the perturbative expansion in powers of \( u \) can be represented by a series of diagrams in which each line carries a component index \( \mu = 1 \ldots N \). The interaction in this representation is indicated by the “vertex” sketched in the figure. Notice that we have introduced a wavy line into the vertex in order to separate the two factors of \( |\mathbf{\phi}|^2 \) in the \( |\mathbf{\phi}|^2 \) interaction. Thus the two lines on either side of the wavy line carry the same component index (\( \mu \) or \( \nu \)).

### a.)
In terms of the \( u = 0 \) propagator \( G_0(\mathbf{k}) = (r + \mathbf{k}^2)^{-1} \) and the interaction vertex above, draw all of the diagrams up to second order in \( u \) for \( G(\mathbf{k}) \).

### b.)
The term represented by any given diagram is proportional to some power of \( N \), i.e. \( N^\ell \). Determine a “rule” giving \( \ell \) in terms of graphical properties of the diagram. [Hint: Since the interaction strength is \( u/N \) each vertex carries an inverse power of \( N \). Closed loops will involve summations over the component indices.]

### c.)
Collect those terms in the expansion which are largest (i.e. \( O(1) \)) as \( N \rightarrow \infty \). By observing them, graphically sketch the expansion of the diagrams contributing to \( G(\mathbf{k}) \) which survive in this limit. We will denote this Greens function as \( G_\infty(\mathbf{k}) \).

### d.)
These diagrams can be organized as a geometric series in terms of the self-energy, as discussed in class. Obtain the graphical expansion of diagrams that contribute to the self energy at \( N = \infty \), denoted \( \Sigma_\infty \).

### e.)
Finally, show that the series of diagrams for the self-energy can be re-expressed in terms of the full \( (N = \infty) \) Greens function, \( G_\infty(\mathbf{k}) \), as,

\[
\Sigma_\infty = -u \int_{\mathbf{k}}^{\Lambda} G_\infty(\mathbf{k}).
\]

The \( \Lambda \) on the integral sign indicates that the momentum integration is to be performed over values of \( \mathbf{k} \) which satisfy, \( |\mathbf{k}| \leq \Lambda \). Notice that the self energy is actually independent of momentum at \( N = \infty \). Since \( G_\infty = r + \mathbf{k}^2 - \Sigma_\infty \), this is a closed form integral equation for the self-energy, which then determines the Greens function.
We will now explore the properties of this integral equation, which can be used to infer the properties of the phase transition in the \(O(N = \infty)\) model. To this end, it is useful to first define a quantity \(m^2 = r - \Sigma_{\infty} \geq 0\), so that \(G_{\infty}(k) = (m^2 + k^2)^{-1}\), and the integral equation can be expressed as,

\[
    r = m^2 - u \int_{k}^{\Lambda} \frac{1}{m^2 + k^2}.
\]  

(12)

As we vary the bare parameter, \(r\), we will seek to determine whether this equation has a solution for \(m\), and if it does we will denote the solution as \(m(r)\). We now argue that the (putative) phase transition from the paramagnet to the ferromagnet should occur if/when there is a value of \(r = r_c\) that satisfies \(m(r_c) = 0\).

f.) To see that \(m = 0\) would correspond to the phase transition, one can Fourier transform the Green’s function back to real space, \(G_{\infty}(x) = \int_{k}^{\Lambda} G_{\infty}(k) e^{i k \cdot x}\). Specializing for the moment to \(D = 3\), perform the momentum integration (setting \(\Lambda = \infty\)) to obtain an explicit expression for \(G_{\infty}(x)\), and show that for large \(|x|\) the Green’s function decays exponentially, \(G_{\infty} \sim e^{-|x|/\xi}\) with correlation length \(\xi\). Verify that \(\xi = 1/m\). This dependence holds in general dimension \(D\). Since the correlation length diverges when \(m \to 0\), we can argue that the (putative) phase transition occurs at \(m = 0\).

If the self-consistency equation does describe a phase transition, we can locate the critical value of \(r\), denoted \(r_c\), by putting \(m = 0\) in Eq. 12,

\[
    r_c = -u \int_{k}^{\Lambda} \frac{1}{k^2}.
\]  

(13)

In the absence of interactions, \(u = 0\), the critical value is \(r_c = 0\), as expected, and this value is pushed negative for non-zero \(u\).

g.) Show that in low dimensions, \(D \leq 2\), the above equation implies that \(r_c = -\infty\). This indicates that it is not possible to tune \(r\) to reach a transition from the paramagnet into a ferromagnet. The inference is that the \(O(N = \infty)\) model does not have a ferromagnet phase for all \(D \leq 2\), i.e., the \(O(N = \infty)\) symmetry is not spontaneously broken. It turns out that this is a general result - models with continuous symmetries cannot have spontaneous symmetry breaking for all \(D \leq 2\) - and goes under the name of the Mermin-Wagner theorem. In contrast, the Ising model does have a stable ferromagnet in \(D = 2\), with the discrete \(Z_2\) symmetry spontaneously breaking at low temperature.

The ferromagnetic phase does exist in the \(O(N = \infty)\) model for \(D > 2\). We now seek to analyze the nature of the paramagnetic-ferromagnetic phase transition, focusing on the properties near the phase transition on the paramagnetic side, that is with \(t = r - r_c\) small and positive.

h.) Show that the self-consistency equation can be cast in the form,

\[
    t = m^2 + u m^2 \int_{k}^{\Lambda} \frac{1}{k^2(m^2 + k^2)}.
\]  

(14)

By analyzing the asymptotic small \(m\) behavior of the above integral, extract the scaling behavior of \(m(t)\), that is show that \(m(t) \sim t^{\nu}\) as \(t \to 0^+\), and deduce the value of \(\nu\) for all \(D > 2\).

Combining your \(m(t)\) with the result from part (f), shows that the correlation length diverges algebraically upon approaching the transition from the paramagnetic phase, \(\xi \sim 1/t^{\nu}\). And \(\nu\) is the correlation length critical exponent. The upper critical dimension, \(D_c\), is defined as the dimension above which \(\nu\) equals the mean field value, \(\nu_{MF} = 1/2\). What is \(D_c\) for the \(O(N = \infty)\) model?
3). $O(N = \infty)$ MODEL REDUX: SOLUTION BY AUXILIARY FIELD (SADDLE-POINT) METHOD

Here one uses a clever trick to re-write the partition function by decoupling the interaction term via an “auxiliary field”, using the so-called Hubbard-Stratonovich identity,

$$e^{-\frac{u}{N} \int dx |\vec{\phi}|^2} = C \int [D\sigma] e^{-\int dx \left[ i\sigma \vec{\phi}^2 + \frac{N}{4} u \sigma^2 \right]},$$

where $C$ is a ($x$-independent) constant. Here $\sigma(x)$ is the “auxiliary” real field, and $[D\sigma]$ denotes a functional integral.

a.) Use the above identity to re-write the partition function as a functional integral over both $\vec{\phi}$ and $\sigma$. Argue that (formally) performing the functional integral over $\vec{\phi}$ will lead to an expression for the partition function of the form,

$$Z = C \int [D\sigma] e^{-NS_{\text{eff}}[\sigma]},$$

with $S_{\text{eff}}$ independent of $N$. For the special case that $\sigma(x)$ is a spatially independent constant, obtain an explicit form for $S_{\text{eff}}[\sigma]$ by performing the functional integral over $\vec{\phi}$. [Hint: Work in momentum space, and integrate over each Fourier mode, $\vec{\phi}(k)$, independently.]

b.) From the form of Eq. 16, argue that a saddle-point approximation is exact in the limit $N \to \infty$. (Note that this is a saddle-point in the field $\sigma$, not $\vec{\phi}$). Assuming that the saddle-point occurs for a spatially constant (and purely imaginary) value $\sigma(x) = i\Sigma$, show that the saddle point equation for $\Sigma$ is exactly the same as the self-consistency equation for the self-energy $\Sigma_{\infty}$ in Eq. 11. [Such an imaginary saddle-point solution is not uncommon in such problems, and is mathematically similar to the steepest descent method of analyzing oscillatory Fourier integrals where one must deform the integration contour off the real axis.] This establishes the equivalence between the auxiliary-field and the diagrammatic methods of solving the $O(N = \infty)$ model.