Chapter 2

1. Hamiltonian $H = \omega S_z$. The Heisenberg equations of motion are:

$$
\dot{S}_x = (1/i\hbar)[S_x, H] = (\omega/i\hbar)[S_x, S_z] = -i\omega S_y \\
\dot{S}_y = (1/i\hbar)[S_y, H] = (\omega/i\hbar)[S_y, S_z] = i\omega S_x \\
\dot{S}_z = 0.
$$

Hence $\dot{S}_x + i\dot{S}_x = -i\omega S_y + i\omega S_x = i\omega(S_x + iS_y)$ and $\dot{S}_x - i\dot{S}_y = -i\omega S_y - i\omega(S_x - iS_y)$, so $(S_x + iS_y)$ and $\omega$ are identical, and we have finally $S_z(t) = S_z(0) = S_z(0) = S_z(0)$.

2. The Hamiltonian is obviously not Hermitian. Physically, the particle can go from state 2 to state 1 but not from state 1 to state 2. Because $H$ is not Hermitian, the time evolution operator is not unitary. Since unitarity is important for probability conservation, we suspect this probability conservation is violated.

To illustrate this point, set $H_{11} = H_{22} = 0$ for simplicity. For the time evolution operator, we get, as usual, $U(t, t_0) = \lim_{N \to \infty} (1 - itH/\hbar\Omega)^N$ where $U$ is actually not unitary. But $H^2 = H_{12}^2 |1\rangle < 2|1\rangle < 2| < 2| = 0$, hence $H^n = 0$ for $n > 1$.

This means that $U(t, t_0) = 1 - (itH/\hbar\Omega)|1\rangle < 2|$ even for a finite time interval. Now the most general initial state is $c_1|1\rangle + c_2|2\rangle$. After a later time we have $|1\rangle < 2| c_1|1\rangle + c_2|2\rangle = (itH_12/\hbar\Omega)c_2|1\rangle$.

Hence the probability for being found in $|1\rangle$ is $|c_1|^2$ and the probability for being found in $|2\rangle$ is $|c_2|^2$. But the total probability is $|c_1|^2 - 2\text{Im}(c_1c_2^*)H_{12}t/\hbar\Omega + |c_2|^2H_{12}^2t^2/\hbar\Omega^2 + |c_2|^2 - |c_1|^2 + |c_2|^2$ in general, and in fact $<\alpha, t_0 = 0 | \alpha, t_0 = 0 > <\alpha, t_0 = 0 | t \alpha, t_0 = 0 > t$, so probability conservation is violated!

3. At time $t = 0$, $\hat{n} = \sin \theta \hat{x} + \cos \theta \hat{z}$, and $\hat{z} = \hbar \hat{\sigma}_z$, and $\hat{\sigma}_z \hat{n} = \frac{\hbar}{2} (\sin \theta \hat{\sigma}_x + \cos \theta \hat{\sigma}_z)$.

The eigenvalue equation at $t = 0$ $\hat{z} \hat{n} |\psi\rangle = \frac{\hbar}{2} |\psi\rangle$ where $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ leads to
acosθ + bsinθ = a, and a normalized eigenstate of form

\[
(1 + \cosθ)^{1/2} \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
\sinθ/(1 + \cosθ)
\end{pmatrix}.
\]

The Hamiltonian \( H = -\mathbf{\hat{z}} \cdot \mathbf{\hat{S}} = (g_\mu B/2)\mathbf{\hat{S}} \cdot \mathbf{\hat{b}} \) is that under consideration.

(a) The time dependence of \( \psi(t) \) is governed by \( H |\psi\rangle = i\hbar \partial/\partial t |\psi\rangle \) or

\[
-\omega \begin{pmatrix}
A(t) \\
-B(t)
\end{pmatrix} = \partial/\partial t \begin{pmatrix}
A(t) \\
B(t)
\end{pmatrix}
\]

where \( \omega = g_\mu B/2\hbar \). This leads to two equations

\[-\omega A(t) = \partial A(t)/\partial t \text{ and } +\omega B(t) = \partial B(t)/\partial t, \]

thus \( A(t) = A(0)e^{-i\omega t} \) and \( B(t) = B(0)e^{i\omega t} \). Compare with (1) above, we have

\[
\psi(t) = \begin{pmatrix}
[(1 + \cosθ)^{1/2}/2]^k e^{-i\omega t} \\
[(\sinθ/2)(1 + \cosθ)^{1/2}]e^{+i\omega t}
\end{pmatrix}.
\]

Next we express \( |\psi(t)\rangle \) in the \( |s_x;+\rangle \) basis as \( a_1^*|s_x;+\rangle + a_2|s_x;-> \) where \( |s_x;+\rangle \) are given explicitly by (1.4.17a) and \( a_1 = \frac{1}{\sqrt{2}}(1, 1) \begin{pmatrix}
Ae^{-i\omega t} \\
Be^{+i\omega t}
\end{pmatrix} = (1/\sqrt{2})Ae^{-i\omega t} + 
\]

\( (1/\sqrt{2})Be^{i\omega t} \) and \( a_2 = \frac{1}{\sqrt{2}}(1, -1) \begin{pmatrix}
Ae^{-i\omega t} \\
Be^{+i\omega t}
\end{pmatrix} = (1/\sqrt{2})Ae^{-i\omega t} - (1/\sqrt{2})Be^{+i\omega t} \) (for short we have written \( A(0) = A \) and \( B(0) = B \)). Hence probability of finding the electron in \( s_x = \hbar/2 \) state is \( a_1^*a_1 + a_2^*a_2 = (A^2 + B^2 + AB(e^{2i\omega t} + e^{-2i\omega t})) = \hbar(1 + \sinθ\cos2\omega t) \).

(b) \( <s_x | = <\psi(t)|s_x|\psi(t)> = (A^*(t), B^*(t)) \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
A(t) \\
B(t)
\end{pmatrix} = \frac{\hbar}{2} (A^*(t)B(t) + B^*(t)A(t)) = (\hbar/2)\sinθ\cos2\omega t \).

(c) In case (i) \( \beta = \gamma \), \( a_1^*a_1 = 1/4 \) and \( <s_x | = 0 \); in case (ii) \( a_1^*a_1 = 1/2(1 + \cos2\omega t) = \cos^2\omega t \) while \( <s_x | = \hbar(\cos^2\omega t - 1/4) \). These answers are eminently sensible since for \( \beta = 0 \) \( \hat{n} \) is along the z-axis, hence there is equal probability of being found in \( |s_x;+\rangle \) (i.e. \( a_1^*a_1 \)) and in \( |s_x;-> \) (i.e. \( a_2^*a_2 \)) - both being \( 1/4 \). Yet \( <s_x >\)
4. First work out \( x(t) \) and \( p(t) \) in the Heisenberg picture. Evidently \( \hat{x} = (1/\hbar)\{x, p^2/2m\} = p/m \), and \( \hat{p} = (1/\hbar)\{p, p^2/2m\} = 0 \). So \( p(t) = p(0) \) and is independent of time, while \( x(t) = x(0) + (p(0)/m)t \). Hence \( \{x(t), x(0)\} = (t/m)[x(0), p(0)] = i\hbar t/m \).

5. \( [\hat{H}, x] = [p^2/2m + \hat{V}(\hat{x}), x] = -i\hbar p/m \), therefore \( [[\hat{H}, x], x] = -\hbar^2/m \). Take the expectation value of \( [[\hat{H}, x], x] \) w.r.t. an energy eigenket \( |a''\rangle \), we have

\[
\langle a''| \{\hat{H}, x\} | a''\rangle = 2\langle a''| x\hat{H}\rangle |a''\rangle + \langle a''| xx\hat{H}|a''\rangle = -\hbar^2/m. \tag{1}
\]

Use next \( \hat{H}|a''\rangle = E_{a''}|a''\rangle \) and \( \langle a''| \hat{H} = E_{a''}|a''\rangle \), (1) becomes

\[
E_{a''}\langle a''| xx|a''\rangle - 2\langle a''| x\hat{H}|a''\rangle + \langle a''| xx\hat{H}|a''\rangle = -\hbar^2/m \tag{2a}
\]

or

\[
E_{a''}\langle a''| xx|a''\rangle + \langle a''| x\hat{H}|a''\rangle = \hbar^2/2m \tag{2b}
\]

Now using closure property, we have \( \langle a''| x\hat{H}|a''\rangle = \sum_a E_a \langle a''| x|a\rangle \langle a'|x|a''\rangle = \sum_a E_a |\langle a''| x|a\rangle|^2 \) and \( \langle a''| xx\hat{H}|a''\rangle = \sum_a E_a |\langle a''| x|a\rangle|^2 \). Equation (2b) becomes

\[
\sum_a |\langle a''| x|a\rangle|^2 \left( E_a - E_{a''} \right) = \hbar^2/2m. \tag{3}
\]

6. Let \( \hat{H} = \hat{p}^2/2m + \hat{V}(\hat{x}) \), and we compute \( [\hat{x}, \hat{p}, \hat{H}] \) through the following steps.

\[
[x, \hat{p}, \hat{H}] = [\hat{x}, \hat{p}, \hat{p}^2/2m + \hat{V}(\hat{x})] = (1/2m)[\hat{x}, \hat{p}, \hat{p}^2] + [\hat{x}, \hat{p}, \hat{V}(\hat{x})] = (1/2m) \sum_{i,j} \varepsilon_{i,j} [x_i p_j, \hat{p}] + [x_i, \hat{p}, \varepsilon_{i,j} x_j]
\]

\[
= (1/2m) \sum_{i,j} \varepsilon_{i,j} \{x_i, \hat{p} p_j, \varepsilon_{i,j} x_j\} = (1/2m) \sum_{i,j} \varepsilon_{i,j} \{x_i, \hat{p} p_j, \varepsilon_{i,j} x_j\} + [x_i, \hat{p}, \varepsilon_{i,j} x_j]
\]

\[
= (1/2m) \sum_{i,j} \varepsilon_{i,j} \{x_i, \hat{p} p_j, \varepsilon_{i,j} x_j\} + [x_i, \hat{p}, \varepsilon_{i,j} x_j]
\]

Hence \( [\hat{x}, \hat{p}, \hat{H}] = \)

- \( \langle \hat{x}\hat{p} - \hat{p}\hat{x} \rangle = \hbar \)
reads \( \frac{0}{\delta} \Delta A(t) \) \( \frac{\delta}{0} \Delta B(t) \) = \( i \hbar d/dt (A(t) \delta B(t)) \) or \( \delta B(t) = \frac{i \hbar dA(t)}{dt} \) and \( \Delta A(t) = \frac{i \hbar dB(t)}{dt} \).

Thus \( A(t) = -\frac{\hbar}{\delta d} \frac{d^2 A(t)}{dt^2} \) and \( B(t) = -\frac{\hbar}{\delta d} \frac{d^2 B(t)}{dt^2} \), and \( A(t) = A_1 \cos \omega t + A_2 \sin \omega t \), \( B(t) = B_1 \cos \omega t + B_2 \sin \omega t \) are the simple harmonic solutions with \( \omega = \sqrt{\delta/\hbar} \).

It is evident that \( |a\rangle \rightarrow |\psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) hence \( A_1 = 1 \), \( \omega t = 0 \) and from normalization \( B_2 = 1 \), \( A_2 = 0 \). So \( |\psi(t)\rangle = \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix} \).

(c) We need to evaluate \( \langle a''|\psi(t)\rangle^2 \) where \( a'' = (0,1) \). Evidently probability is \( \sin^2 \omega t \).

(d) The Hamiltonian \( H = \frac{\delta}{1 0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is for a spin \( \frac{1}{2} \) system if \( \delta = \hbar \), hence \( |\psi(t)\rangle = \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix} \) describes the evolution of a spinor in time, initially in state \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and hence an eigenstate of \( J_z = \frac{\hbar}{2} \sigma_z \).

9.

(a) Let the normalized energy eigenkets be written as \( |E\rangle = |R\rangle <R|E\rangle + |L\rangle <L|E\rangle \).

Now \( H|E\rangle = E|E\rangle \) therefore \( \Delta (|L\rangle <R| + |R\rangle <L|) = E|E\rangle \) or \( \Delta (|L\rangle <R|E\rangle + |R\rangle <L|E\rangle) = E(|R\rangle <R|E\rangle + |L\rangle <L|E\rangle) \). Due to the linear independence of \( |L\rangle \) and \( |R\rangle \), we have \( \Delta <R|E\rangle = E<L|E\rangle \) and \( \Delta <L|E\rangle = E<R|E\rangle \). Now due to normalization condition \( <R|E\rangle^2 + <L|E\rangle^2 = 1 \), we have \( \Delta^2 = E^2 \) or \( \Delta = \pm E \) (these define the two level system eigenvalues).

Take \( \Delta = +E \), and \( <R|E\rangle = <L|E\rangle = 1/\sqrt{2} \), then \( |E\rangle = \frac{1}{\sqrt{2}} (|R\rangle + |L\rangle) \); for \( \Delta = -E \), take \( <R|E\rangle = -<L|E\rangle = 1/\sqrt{2} \) and \( |-E\rangle = \frac{1}{\sqrt{2}} (|R\rangle - |L\rangle) \).

(b) Suppose at \( t=0 \), \( |\alpha\rangle = |R\rangle <R|\alpha\rangle + |L\rangle <L|\alpha\rangle = |\alpha\rangle, t=t_0=0 \). The evolution of state vector \( |\alpha\rangle, t_0=0; t \) is such that \( e^{-iHt/\hbar} |\alpha\rangle = |\alpha\rangle, t=t_0=0; t \). From part (a) we have \( |R\rangle = \frac{1}{\sqrt{2}} (+E\rangle - E\rangle) \) and \( |L\rangle = \frac{1}{\sqrt{2}} (+E\rangle + E\rangle) \), therefore

\[
e^{-iHt/\hbar} |\alpha\rangle = e^{-iHt/\hbar} \langle R|\alpha\rangle |R\rangle + e^{-iHt/\hbar} \langle L|\alpha\rangle |L\rangle
\]

\[
= \frac{1}{\sqrt{2}} <R|\alpha\rangle e^{-iHt/\hbar} (+E\rangle - E\rangle) + \frac{1}{\sqrt{2}} <L|\alpha\rangle e^{-iHt/\hbar} (+E\rangle + E\rangle).
\]

But \( e^{-iHt/\hbar} |E\rangle = e^{-i\Delta t/\hbar} |E\rangle \), hence from (1) we have
\[ |α, t_0=0; t> = e^{-i\Delta t/\hbar}|α> + e^{i\Delta t/\hbar}|-\varepsilon>\]
\[ + \frac{1}{\sqrt{2}}<R|α>(e^{-i\Delta t/\hbar}|+\varepsilon> - e^{i\Delta t/\hbar}|-\varepsilon>).\]

Rearrange r.h.s. of (2) back to the \(|R>, |L>\) basis, we have
\[ |α, t_0=0; t> = (2R|α>cos\Delta t/\hbar - iL|α>sin\Delta t/\hbar)|R>\]
\[ + (2L|α>cos\Delta t/\hbar - iR|α>sin\Delta t/\hbar)|L>\]  
(3)

(c) Suppose at t=0, \(|α> = |R> with certainty, than from (3) we have \(<L|α> = 0\) and \(<R|α> = 1\) (normalization). We need the development of \(|L>\) as a function of time, this corresponds to \(|α, t_0=0; t> = cos\Delta t/\hbar|R> - isin\Delta t/\hbar|L>\) and \(<L|α, t_0=0; t> = -isin\Delta t/\hbar.\) The transition probability is \(|<L|α, t_0=0; t>|^2 = sin^2\Delta t/\hbar.\)

(d) In the Schrödinger picture the base kets \(|R>\) and \(|L>\) remain stationary in time and the state vector obeys \(i\hbar \partial \partial t|α, t_0=0; t> = i\hbar |α, t_0=0; t>.\) Write
\[ |α, t_0=0; t> = a_R(t)|R> + a_L(t)|L>\] and using \(H = Δ(|L>|R> + |R>|L>),\) the Schrödinger equation leads to coupled equations \(i\hbar da_R(t)/dt = Δa_L(t)\) and \(i\hbar da_L(t)/dt = Δa_R(t)\)

where \(a_R(t) = <R|α, t_0=0; t>\) and \(a_L(t) = <L|α, t_0=0; t>.\) Solutions of the coupled equations can be obtained by noting that \(d^2/dt^2[a_R, L(t)] + (Δ^2/\hbar^2)a_R, L(t) = 0,\) hence
\[ a_L(t) = A cos\Delta t/\hbar + B sin\Delta t/\hbar, a_R(t) = C cos\Delta t/\hbar + D sin\Delta t/\hbar\]  
(4)

At \(t = 0\) \(|α> = <R|α>|R> + <L|α>|L> = a_R(0)|R> + a_L(0)|L>,\) hence \(a_R(0) = C =<R|α>\) and \(a_L(0) = A = <L|α>.\) Next the normalization condition at \(t,\) with \(t_0=0\)
\[<α, t_0=0; t|α, t_0=0; t> = 1 \text{ give}\]
\[cos^2\Delta t/\hbar + (2R|α>^* D + D^*<R|α> + 2L|α>^* B + B^*<L|α>)cos\Delta t/\hbar sin\Delta t/\hbar\]
\[ + (|D|^2 + |B|^2)sin^2\Delta t/\hbar = 1.\]  
(5)

Solution of (5) is possible with \(D = -i<L|α>\) and \(B = -i<R|α>,\) hence (4) for \(a_L(t)\) and \(a_R(t)\) gives the coefficients of \(|L>\) and \(|R>\) in (3) of (b).

(e) The lack of Hermiticity here is same as in problem 2, replacing \(H = H_{12}|L><2|\)
by $H = \Delta |L> <R|$. We find again $H^n = 0$ for $n>1$, and $U(t, t_0 = 0) = 1 - i t \Delta / h |L> <R|$ even for a finite time interval. The initial state is $<R|a>|R> + <L|a>|L>$; at a later time $t$ we have $(1 - i t \Delta / h |L> <R|) (<R|a>|R> + <L|a>|L>)$, hence probability for being found in $|L>$ is $<L|a> - (i t \Delta / h) <R|a>|^2$ and in $|R>$ is $<R|a>|^2$, but $<L|a> - (i t \Delta / h) <R|a>|^2 + <R|a>|^2 > |L|a>|^2 + <R|a>|^2$. Thus probability conservation is violated.

$H = p^2/2m + k_x m a^2 x^2$ for the one dimensional simple harmonic oscillator.

(a) In the Heisenberg picture, the operators $x$ and $p$ obey the Heisenberg equations of motion: $dp/dt = (1/i\hbar) [p, H] = -m o^2 x$, $dx/dt = (1/i\hbar) [x, H] = p/m$. This implies $\dot{x} = -o^2 x$ and $\dot{p} = -o^2 p$ with the initial conditions $x(0) = x_0$ and $p(0) = p_0$. The solutions are $x(t) = x_0 \cos o t + (p_0 / m o) \sin o t$, $p(t) = p_0 \cos o t - m o x_0 \sin o t$ which give $H = p^2 / 2m + k_x m a^2 x^2$, i.e. $H$ is time independent. Dynamical variables $x$ and $p$ are time-dependent in the Heisenberg picture. At $t = 0$, the Heisenberg and Schrödinger pictures coincide, thus $x_H(0) = x_S(0) = x_0$ (with $x_S(t) = x_S(0)$ and $p_H(0) = p_S(0) = p_0$ (with $p_S(t) = p_S(0)$) and we note the time-independence of dynamical variables in the Schrödinger picture.

The relationship between the Heisenberg and Schrödinger pictures is $x_H(t) = e^{iHt/\hbar} x_S e^{-iHt/\hbar}$ with $x_S = x_0$ and $p_H(t) = e^{iHt/\hbar} p_S e^{-iHt/\hbar}$ with $p_S = p_0$. Using $(2.3.48)$ - $(2.3.50)$, one knows $x_H(t) = x_0 \cos o t + (p_0 / m o) \sin o t$. Also

$e^{iHt/\hbar} p_S e^{-iHt/\hbar} = p_0 + (i t /\hbar) [H, p_S] + (i^2 t^2 / 2! \hbar^2) [H, [H, p_S]] + \ldots = p_0 - \frac{t^2 m o^2}{2!} p_0 - t m o x_0 + \frac{t^3 m o^4 x_0}{3!} + \ldots$

where we have used $[H, p_S] = -i\hbar p_0 / m$, $[H, p_0] = i\hbar m o^2 x_0$. This implies that $p_H(t) = p_0 \cos o t - m o x_0 \sin o t$.

(b) At $t = 0$, the general state vectors for both pictures are equal: $|a>_H = |a>_S = \ldots$
time independent, while $|a, t\rangle_S = e^{-i\omega t/H} |a, t\rangle = e^{-i\omega (n^{1/2}) t} |n\rangle$ and is thus time dependent. (We have used $H = \pm \omega (n^{1/2})$ which is time-independent in both pictures). We can recast $|a, t\rangle_S$ as $|a, t\rangle_S = \Gamma_n c_n(t) |n\rangle$ with $c_n(t) = c_n(0) e^{-i\omega (n^{1/2}) t}$. Also note $i\hbar \partial / \partial t |a, t\rangle_S = H |a, t\rangle_S$ which is the Schrödinger equation for the Schrödinger state vector. \textbf{Remarks:} $c_n(t)$ can be determined in the two pictures by (a) $c_n(t) = \langle n | a, t \rangle_S = c_n(0) e^{-i\omega (n^{1/2}) t}$, the Schrödinger picture with base kets $|n\rangle$ time independent, and (b) $c_n(t) = \langle n, t | a, t \rangle_H = \langle n | e^{-iHt/\hbar} | a, t \rangle_H = c_n(0) e^{-i\omega (n^{1/2}) t}$, the Heisenberg picture with base kets $|n, t\rangle = e^{i\hbar t/\hbar} |n\rangle$ which are time-dependent.

11. For a one-dimensional SHO potential $H = p^2/2m + \frac{i}{\hbar} \omega a^2x^2$, hence $\dot{x} = (1/i\hbar)[x, H] = p/m$, and $\dot{p} = (1/i\hbar)[p, H] = (1/i\hbar)(\omega a^2/2)[p, x^2] = (\omega a^2/2i\hbar)[-2i\hbar x] = -\omega^2 x$. Hence $\ddot{x} + \omega^2 x = 0$, and solution is $x(t) = A\cos \omega t + B\sin \omega t$. At $t=0$, $x(0) = A$ while $\dot{x}(0) = -A\omega \cos \omega t + B\omega \sin \omega t = \omega B$ and thus $p(0) = \omega B$. Thus in the Heisenberg picture $x(t) = x(0) \cos \omega t + (p(0)/\omega) \sin \omega t$.

Our state vector $|a\rangle = e^{-ipa/\hbar} |0\rangle$ at $t=0$; for $t>0$ we have in the Heisenberg picture $\langle x(t) \rangle = \langle a | x(t) | a \rangle$. We note that

$$e^{ip(0)a/\hbar} x(0) e^{-ip(0)a/\hbar} = e^{ip(0)a/\hbar} \left( [x(0), e^{-ip(0)a/\hbar}] + e^{-ip(0)a/\hbar} x(0) \right)$$

$$= x(0) + a,$$

while $e^{ip(0)a/\hbar} p(0) e^{-ip(0)a/\hbar} = p(0)$. Hence

$$\langle x(t) \rangle = \langle a | x(t) | a \rangle = \langle 0 | e^{ipa/\hbar} x(t) e^{-ipa/\hbar} | 0 \rangle$$

$$= \langle 0 | e^{ip(0)a/\hbar} [x(0) \cos \omega t + (p(0)/\omega) \sin \omega t] e^{-ipa/\hbar} | 0 \rangle.$$ 

Since $\langle 0 | x(0) | 0 \rangle = \langle 0 | p(0) | 0 \rangle = 0$, we obtain for $\langle x(t) \rangle = A\cos \omega t$.

12. (a) The wave function in problem 11 takes form $\langle x' | a \rangle = \langle x' | e^{-ipa/\hbar} | 0 \rangle$. Since $e^{-ipa/\hbar} | x' \rangle = | x' - a \rangle$ (hence $\langle x' | e^{-ipa/\hbar} = \langle x' - a \rangle$), we have $\langle x' | a \rangle = \langle x' - a | 0 \rangle$.

Hence $\langle x' | a \rangle = \frac{1}{\sqrt{2\omega}} \exp\left(-\frac{(x' - a)^2}{2\omega^2}\right)$. 

2a) Using the standard Pauli matrices, we see

\[
\hat{H} \rightarrow \mu \left( \begin{array}{cc}
b_0 & b_1 e^{-i\omega t} \\
b_1 e^{i\omega t} & -b_0
\end{array} \right)
\]

\[
= \mu \left( \begin{array}{cc}
b_0 & b_1 e^{-i\omega t} \\
b_1 e^{i\omega t} & -b_0
\end{array} \right)
\]

2b) By definition, \( \mathbf{i} \hbar \partial_t \Phi = \hat{H} \Phi \)

Define \( \Phi(t) = e^{\mathbf{i} \omega t \hat{\sigma}_z} \Phi(0) \)

Then, \( i \hbar \partial_t (e^{-\mathbf{i} \omega t \hat{\sigma}_z} \Phi(t)) = \hat{H} e^{-\mathbf{i} \omega t \hat{\sigma}_z} \Phi(t) \)

\[
\Rightarrow \partial_t \left( -\frac{\hbar \omega}{2} e^{-\mathbf{i} \omega t \hat{\sigma}_z} \Phi_0 + e^{-\mathbf{i} \omega t \hat{\sigma}_z} \partial_t \Phi_0 \right) = \hat{H} e^{-\mathbf{i} \omega t \hat{\sigma}_z} \Phi_0
\]

\[
\Rightarrow \partial_t \Phi_0 = \left[ e^{\mathbf{i} \omega t \hat{\sigma}_z} \hat{H} e^{-\mathbf{i} \omega t \hat{\sigma}_z} - \frac{\hbar \omega}{2} \hat{\sigma}_z \right] \Phi_0
\]

\[
= \hat{H}_0 \Phi_0
\]

Since \( e^{\mathbf{i} \omega t \hat{\sigma}_z} \) is unitary. Working in the basis where \( \hat{\sigma}_z \) is diagonal,

\[
\hat{H}_0 = \mu \left( \begin{array}{cc}
e^{-\mathbf{i} \omega t / 2} & 0 \\
0 & e^{\mathbf{i} \omega t / 2}
\end{array} \right) \left( \begin{array}{cc}
b_0 & b_1 e^{-i\omega t} \\
b_1 e^{i\omega t} & -b_0
\end{array} \right) \left( \begin{array}{cc}
e^{-\mathbf{i} \omega t / 2} & 0 \\
0 & e^{\mathbf{i} \omega t / 2}
\end{array} \right)
\]

\[
= \mu \left( \begin{array}{cc}
b_0 - \frac{\hbar \omega}{2} & b_1 \\
b_1 & b_0 + \frac{\hbar \omega}{2}
\end{array} \right)
\]

\( \square \)
We note \( \hat{U}_0(t) = e^{-i\hat{H}_0 t/\hbar} \) since \( \hat{H}_0 \) is time independent. Then, since \( \Phi_0(t=0) = \Phi(t=0) \)

\[ \Phi_0(t) = \hat{U}_0(t) \Phi_0(t=0) \]

\[ \Rightarrow \Phi(t) = e^{-i\omega t \hbar/2} \hat{U}_0(t) \Phi(t=0) \]

Since \( \hat{U}_0(t) = 1 - \frac{t\hat{H}_0}{\hbar} + \cdots \) we define

\[ b_0' = b_0 - \frac{\hbar \omega}{2} \]

and \( \hbar \omega_0^2 = \mu^2 (b_0^2 + b_1^2) \)

so that \( \hat{H}_0 = \hbar \omega_0^2 \hat{I} \) Noting that

\[ \hat{U}_0(t) = \left[ 1 - \frac{t^2 \omega_0^2 \hat{I}}{2 \hbar} + \cdots \right] - \frac{i\hat{H}_0}{\hbar \omega_0} \left[ \omega_0 t - \frac{t^2}{6} \omega_0^2 \hat{I} + \cdots \right] \]

Thus

\[ \Phi_0(t) \rightarrow \begin{pmatrix} \cos(\omega_0 t) & -\frac{ib_0'}{\hbar \omega_0} \sin(\omega_0 t) & -\frac{ib_1}{\hbar \omega_0} \sin(\omega_0 t) \\ -\frac{ib_0'}{\hbar \omega_0} \sin(\omega_0 t) & \cos(\omega_0 t) + \frac{ib_0'}{\hbar \omega_0} \sin(\omega_0 t) \end{pmatrix} \]

noting that \( e^{-i\omega t \hbar/2} \rightarrow \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} \)

and denoting \( \alpha = \frac{\mu b_0'}{\hbar \omega_0} \beta = \frac{\mu b_1}{\hbar \omega_0} \)

\[ \alpha^2 + \beta^2 = 1 \]

we see

\[ \hat{U}(t) = e^{-i\omega t \hbar/2} \begin{pmatrix} \cos(\omega_0 t) & -i\alpha \sin(\omega_0 t) & -i\alpha \sin(\omega_0 t) \\ i\alpha \sin(\omega_0 t) & e^{i\omega t} \cos(\omega_0 t) + i\alpha \sin(\omega_0 t) \end{pmatrix} \]

\[ 2\text{c, ctd) det}(\hat{U}(t)) = \text{det}(e^{-i\omega t \hbar/2}) \text{det}(e^{-i\omega t \hbar/2}) \]

\[ = e^{-\frac{i\omega t}{2} \text{Tr}[\omega_0^2]} e^{-\frac{i\omega t}{2} \text{Tr}[\omega_0^2]} \]

\[ = 1 \]
2d) we want $\langle \downarrow | \hat{U}(t) | \uparrow \rangle$

or, as matrices

$$(0 \ 1) \hat{U}(t) (1 \ 0)$$

$$= e^{i\omega t/2} (-i a_1 \sin(\omega t))$$

The probability is $|\hat{U}(t)|^2$

on $|\psi(t)\rangle = a_1^2 \sin^2(\omega t)$

note $a_1 = \mu b_1 \left[ (b_0 - \frac{\omega}{2})^2 + b_1^2 \right]^{-1}$

2c) If $|\psi(t=0)\rangle \rightarrow \frac{1}{\sqrt{2}} (\uparrow \downarrow)$ in this basis, then

$$\Psi(t) = \hat{U}(t) \Psi(t=0)$$

$$= \frac{1}{\sqrt{2}} e^{-i\omega t/2} \begin{pmatrix} \cos(\omega t) - i(a_0 + a_1) \sin(\omega t) \\ e^{i\omega t/2} \left( \cos(\omega t) + i(a_0 - a_1) \sin(\omega t) \right) \end{pmatrix}$$

3) We see $\hat{U}(t) = \exp \left( \frac{i\hbar}{\Delta} \hat{S}z(\uparrow \downarrow) \hat{S}_x \hat{S}_y \right)$

so that $\hat{U}(t\Phi) = \exp \left( \frac{-i\pi}{\Delta} \hat{S}_x \hat{S}_y \right)$

Also, $(\hat{S}_x^2 \hat{S}_y^2)^2 = \hat{I}$, so $\hat{U}(t\Phi) = 1 - \frac{i\pi}{\Delta} \hat{S}_x^2 \hat{S}_y^2 - \frac{\pi^2}{\Delta^2} \hat{I} + \ldots$

Expanding $(\begin{pmatrix} a \\ b \end{pmatrix}) \otimes (\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} a^2 & ab \\ ba & b^2 \end{pmatrix}$ and $\hat{S}_x^2 \otimes \hat{S}_y^2 \rightarrow \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$

we see $|\psi(t\Phi)\rangle = \hat{U}(t\Phi) |\psi(t=0)\rangle$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} a \\ b \end{pmatrix}$$