Chapter 2

Ph 215A Soln #3

1. Hamiltonian $H = \omega S_z$. The Heisenberg equations of motion are:

$$\dot{S}_x = (1/i\hbar)[S_x, H] = (\omega/i\hbar)[S_x, S_z] = -\omega S_y$$

$$\dot{S}_y = (1/i\hbar)[S_y, H] = (\omega/i\hbar)[S_y, S_z] = +\omega S_x$$

$$\dot{S}_z = 0.$$

Hence $\dot{S}_x + i\dot{S}_y = -\omega S_y + i\omega S_x = i\omega(S_x + iS_y)$ and $\dot{S}_x - i\dot{S}_y = -\omega S_y - i\omega S_x = -i\omega(S_x - iS_y)$, so $(S_x \pm iS_y)_t = (S_x \pm iS_y)_{t=0}\, e^{\pm i\omega t}$ and we have finally $S_x(t) = S_x(0)\cos\omega t - S_y(0)\sin\omega t$, $S_y(t) = S_y(0)\cos\omega t + S_x(0)\sin\omega t$, $S_z(t) = S_z(0)$.

2. The Hamiltonian is obviously not Hermitian. Physically, the particle can go from state 2 to state 1 but not from state 1 to state 2. Because H is not Hermitian the time evolution operator is not unitary. Since unitarity is important for probability conservation, we suspect that probability conservation is violated.

To illustrate this point, set $H_{11} = H_{22} = 0$ for simplicity. For the time evolution operator we get, as usual, $U(t,t_o=0) = \lim_{N\to\infty} (1 - itH/\hbar N)^N$ where U is actually not unitary. But $H^2 = H_{12}^2\, |1\rangle\langle 2|1\rangle\langle 2| = 0$, hence $H^n = 0$ for $n > 1$. This means that $U(t,t_o=0) = 1 - (itH_{12}/\hbar)|1\rangle\langle 2|$ even for a finite time interval. Now the most general initial state is $c_1|1\rangle + c_2|2\rangle$. At a later time we have $[1 - (itH_{12}/\hbar)|1\rangle\langle 2|](c_1|1\rangle + c_2|2\rangle) = c_1|1\rangle + c_2|2\rangle - (itH_{12}/\hbar)c_2|1\rangle$. Hence the probability for being found in $|1\rangle$ is $|c_1 - (itH_{12}/\hbar)c_2|^2$ and the probability for being found in $|2\rangle$ is $|c_2|^2$. But the total probability is $|c_1|^2 - 2\,\mathrm{Im}(c_1c_2^*)H_{12}t/\hbar + |c_2|^2H_{12}^2t^2/\hbar^2 + |c_2|^2 \neq |c_1|^2 + |c_2|^2$ in general, and in fact $\langle\alpha, t_o=0\,|\alpha, t_o=0\rangle \neq \langle\alpha, t_o=0; t\,|\alpha, t_o=0; t\rangle$, so probability conservation is violated!

3. At time $t = 0$, $\hat{n} = \sin\beta\,\hat{x} + \cos\beta\,\hat{z}$, and $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, and $\vec{S}\cdot\hat{n} = \frac{\hbar}{2}(\sin\beta\,\sigma_x + \cos\beta\,\sigma_z)$. The eigenvalue equation at $t = 0$ $\vec{S}\cdot\hat{n}|\psi\rangle = \frac{\hbar}{2}|\psi\rangle$ where $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ leads to

$a\cos\beta + b\sin\beta = a$, and a normalized eigenstate of form

$$\frac{(1+\cos\beta)^{1/2}}{2^{1/2}}\begin{pmatrix} 1 \\ \sin\beta/(1+\cos\beta) \end{pmatrix}. \tag{1}$$

The Hamiltonian $H = -\vec{\mu}_s\cdot\vec{B} = (g_s\mu_B/2)\sigma_z B$ is that under consideration.

(a) The time dependence of $\psi(t)$ is governed by $H|\psi\rangle = i\hbar\partial/\partial t|\psi\rangle$ or

$$-i\omega\begin{pmatrix} A(t) \\ -B(t) \end{pmatrix} = \partial/\partial t\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \tag{2}$$

where $\omega = g_s\mu_B B/2\hbar$. This leads to two equations $-i\omega A(t) = \partial/\partial t[A(t)]$ and $+i\omega B(t) = \partial/\partial t\,[B(t)]$, thus $A(t) = A(0)e^{-i\omega t}$ and $B(t) = B(0)e^{+i\omega t}$. Compare with (1) above, we have

$$\psi(t) = \begin{pmatrix} [(1+\cos\beta)^{1/2}/2^{1/2}]e^{-i\omega t} \\ [\sin\beta/2^{1/2}(1+\cos\beta)^{1/2}]e^{+i\omega t} \end{pmatrix}. \tag{3}$$

Next we express $|\psi(t)\rangle$ in the $|s_x;\pm\rangle$ basis as $\alpha_1|s_x;+\rangle + \alpha_2|s_x;-\rangle$ where $|s_x;\pm\rangle$ are given explicitly by (1.4.17a) and $\alpha_1 = \frac{1}{\sqrt{2}}(1,1)\begin{pmatrix} Ae^{-i\omega t} \\ Be^{+i\omega t}\end{pmatrix} = (1/\sqrt{2})Ae^{-i\omega t} + (1/\sqrt{2})Be^{i\omega t}$ and $\alpha_2 = \frac{1}{\sqrt{2}}(1,-1)\begin{pmatrix} Ae^{-i\omega t} \\ Be^{+i\omega t}\end{pmatrix} = (1/\sqrt{2})Ae^{-i\omega t} - (1/\sqrt{2})Be^{+i\omega t}$ (for short we have written $A(0) = A$ and $B(0) = B$). Hence probability of finding the electron in $s_x = \hbar/2$ state is $\alpha_1^*\alpha_1 = \frac{1}{2}[A^2+B^2+AB(e^{2i\omega t} + e^{-2i\omega t})] = \frac{1}{2}(1+\sin\beta\cos 2\omega t)$.

(b) $\langle s_x\rangle = \langle\psi(t)|s_x|\psi(t)\rangle = (A^*(t),\ B^*(t))\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} A(t) \\ B(t)\end{pmatrix} = \frac{\hbar}{2}(A^*(t)B(t) + B^*(t)A(t)) = (\hbar/2)\sin\beta\cos 2\omega t$.

(c) In case (i) $\beta\to 0$, $\alpha_1^*\alpha_1 = \frac{1}{2}$ and $\langle s_x\rangle = 0$; in case (ii) $\alpha_1^*\alpha_1 = \frac{1}{2}(1+\cos 2\omega t) = \cos^2\omega t$ while $\langle s_x\rangle \to \hbar(\cos^2\omega t - \frac{1}{2})$. These answers are eminently sensible since for $\beta = 0$ $\hat{n}$ is along the z-axis, hence there is equal probability of being found in $|s_x;+\rangle$ (i.e. $\alpha_1^*\alpha_1$) and in $|s_x;-\rangle$ (i.e. $\alpha_2^*\alpha_2$) - both being $\frac{1}{2}$. Yet $\langle s_x\rangle$

$= 0$ as the classical analogue would also be reasonable for an electron pointed spin-wise in the z-direction. For $\beta = \pi/2$ (i.e. $\hat{n}$ along OX), at $t=0$ $c_1^* c_1 = 1$, and $\langle s_x \rangle = \hbar/2$ are entirely reasonable in terms of initial state requirements.

4. First work out $x(t)$ and $p(t)$ in the Heisenberg picture. Evidently $\dot{x} = (1/i\hbar)[x, p^2/2m] = p/m$, and $\dot{p} = (1/i\hbar)[p, p^2/2m] = 0$. So $p(t) = p(0)$ and is independent of time, while $x(t) = x(0) + (p(0)/m)t$. Hence $[x(t), x(0)] = (t/m)[x(0), p(0)] = i\hbar t/m$.

5. $[H, x] = [p^2/2m + V(x), x] = -i\hbar p/m$, therefore $[[H,x], x] = -\hbar^2/m$. Take the expectation value of $[[H,x], x]$ w.r.t. an energy eigenket $|a''\rangle$, we have

$$\langle a''|Hxx|a''\rangle - 2\langle a''|xHx|a''\rangle + \langle a''|xxH|a''\rangle = -\hbar^2/m. \tag{1}$$

Use next $H|a''\rangle = E_{a''}|a''\rangle$ and $\langle a''|H = E_{a''}\langle a''|$, (1) becomes

$$E_{a''}\langle a''|xx|a''\rangle - 2\langle a''|xHx|a''\rangle + E_{a''}\langle a''|xx|a''\rangle = -\hbar^2/m \tag{2a}$$

$$\text{or} \quad -E_{a''}\langle a''|xx|a''\rangle + \langle a''|xHx|a''\rangle = \hbar^2/2m \tag{2b}$$

Now using closure property, we have $\langle a''|xHx|a''\rangle = \sum_{a'} \langle a''|xH|a'\rangle\langle a'|x|a''\rangle = \sum_{a'} E_{a'}\langle a''|x|a'\rangle\langle a'|x|a''\rangle = \sum_{a'} E_{a'}|\langle a''|x|a'\rangle|^2$, and $\langle a''|xx|a''\rangle = \sum_{a'}\langle a''|x|a'\rangle\langle a'|x|a''\rangle = \sum_{a'}|\langle a''|x|a'\rangle|^2$. Equation (2b) becomes

$$\sum_{a'}|\langle a''|x|a'\rangle|^2 (E_{a'} - E_{a''}) = \hbar^2/2m. \tag{3}$$

6. Let $H = \vec{p}^2/2m + V(\vec{x})$, and we compute $[\vec{x}\cdot\vec{p}, H]$ through the following steps.

$[\vec{x}\cdot\vec{p}, H] = [\vec{x}\cdot\vec{p}, \vec{p}^2/2m + V(\vec{x})] = (1/2m)[\vec{x}\cdot\vec{p}, \vec{p}^2] + [\vec{x}\cdot\vec{p}, V(\vec{x})] = (1/2m)\sum_{i,j}$ $[x_i p_i, p_j p_j] + \sum_i [x_i p_i, V(\vec{x})] = (1/2m)\sum_{i,j}(x_i[p_i, p_j p_j] + [x_i, p_j p_j]p_i) +$ $\sum_i (x_i[p_i, V(\vec{x})] + [x_i, V(\vec{x})]p_i) = (1/2m)\sum_{i,j}[x_i, p_j p_j]p_i + \sum_i x_i[p_i, V(\vec{x})]$ $= (1/2m)\sum_{i,j}([x_i, p_j]p_j p_i + p_j[x_i, p_j]p_i) + \sum_i x_i[p_i, V(\vec{x})] = (1/2m)\sum_{i,j}(i\hbar\delta_{ij}p_j p_i$ $+ p_j(i\hbar\delta_{ij})p_i) + \sum_i x_i(-i\hbar\partial V/\partial x_i) = i\hbar[\vec{p}^2/m - \vec{x}\cdot\vec{\nabla}V(\vec{x})]$. Hence $\langle[\vec{x}\cdot\vec{p}, H]\rangle =$

reads $\begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = i\hbar d/dt \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$ or $\delta B(t) = i\hbar dA(t)/dt$ and $\delta A(t) = i\hbar dB(t)/dt$. Thus $A(t) = -(\hbar/\delta)^2 d^2A(t)/dt^2$ and $B(t) = -(\hbar/\delta)^2 d^2B(t)/dt^2$, and $A(t) = A_1\cos\omega t + A_2\sin\omega t$, $B(t) = B_1\cos\omega t + B_2\sin\omega t$ are the simple harmonic solutions with $\omega = \delta/\hbar$. It is evident that $|a'\rangle = |\psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ hence $A_1 = 1$, $B_1 = 0$ and from normalization $B_2 = 1$, $A_2 = 0$. So $|\psi(t)\rangle = \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}$.

(c) We need to evaluate $|\langle a''|\psi(t)\rangle|^2$ where $\langle a''| = (0,1)$. Evidently probability is $\sin^2\omega t$.

(d) The Hamiltonian $H = \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = J_x$ for a spin ½ system if $\delta = \hbar/2$, hence $|\psi(t)\rangle = \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}$ describes the evolution of a spinor in time, initially in state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and hence an eigenstate of $J_z = \frac{\hbar}{2}\sigma_z$.

9. (a) Let the normalized energy eigenkets be written as $|E\rangle = |R\rangle\langle R|E\rangle + |L\rangle\langle L|E\rangle$. Now $H|E\rangle = E|E\rangle$ therefore $\Delta(|L\rangle\langle R| + |R\rangle\langle L|)|E\rangle = E|E\rangle$ or $\Delta(|L\rangle\langle R|E\rangle + |R\rangle\langle L|E\rangle) = E(|R\rangle\langle R|E\rangle + |L\rangle\langle L|E\rangle)$. Due to the linear independence of $|L\rangle$ and $|R\rangle$, we have $\Delta\langle R|E\rangle = E\langle L|E\rangle$ and $\Delta\langle L|E\rangle = E\langle R|E\rangle$. Now due to normalization condition $|\langle R|E\rangle|^2 + |\langle L|E\rangle|^2 = 1$, we have $\Delta^2 = E^2$ or $\Delta = \pm E$ (these define the two level system eigenvalues). Take $\Delta = +E$, and $\langle R|E\rangle = \langle L|E\rangle = 1/\sqrt{2}$, than $|+E\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$; for $\Delta = -E$, take $\langle R|E\rangle = -\langle L|E\rangle = 1/\sqrt{2}$ and $|-E\rangle = \frac{1}{\sqrt{2}}(|R\rangle - |L\rangle)$.

(b) Suppose at $t=0$, $|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle \equiv |\alpha, t=t_o=0\rangle$. The evolution of state vector $|\alpha, t_o=0; t\rangle$ is such that $e^{-iHt/\hbar}|\alpha\rangle = |\alpha, t_o=0; t\rangle$. From part (a) we have $|R\rangle = \frac{1}{\sqrt{2}}(|+E\rangle + |-E\rangle)$ and $|L\rangle = \frac{1}{\sqrt{2}}(|+E\rangle - |-E\rangle)$, therefore

$$e^{-iHt/\hbar}|\alpha\rangle = e^{-iHt/\hbar}(\langle R|\alpha\rangle|R\rangle + \langle L|\alpha\rangle|L\rangle)$$

$$= \frac{1}{\sqrt{2}}\langle R|\alpha\rangle e^{-iHt/\hbar}(|+E\rangle+|-E\rangle) + \frac{1}{\sqrt{2}}\langle L|\alpha\rangle\, e^{-iHt/\hbar}(|+E\rangle-|-E\rangle). \qquad (1)$$

But $e^{-iHt/\hbar}|\pm E\rangle = e^{\mp i\Delta t/\hbar}|\pm E\rangle$, hence from (1) we have

$$|\alpha,\ t_o=0;\ t\rangle = e^{-iHt/\hbar}|\alpha\rangle = \frac{1}{\sqrt{2}}\langle R|\alpha\rangle(e^{-i\Delta t/\hbar}|+E\rangle + e^{i\Delta t/\hbar}|-E\rangle) + \frac{1}{\sqrt{2}}\langle L|\alpha\rangle(e^{-i\Delta t/\hbar}|+E\rangle - e^{i\Delta t/\hbar}|-E\rangle). \quad (2)$$

Rearrange r.h.s. of (2) back to the $\{|R\rangle, |L\rangle\}$ basis, we have

$$|\alpha,\ t_o=0;\ t\rangle = (\langle R|\alpha\rangle\cos\Delta t/\hbar - i\langle L|\alpha\rangle\sin\Delta t/\hbar)|R\rangle + (\langle L|\alpha\rangle\cos\Delta t/\hbar - i\langle R|\alpha\rangle\sin\Delta t/\hbar)|L\rangle \quad (3)$$

(c) Suppose at t=0, $|\alpha\rangle = |R\rangle$ with certainty, than from (3) we have $\langle L|\alpha\rangle = 0$ and $\langle R|\alpha\rangle = 1$ (normalization). We need the development of $|L\rangle$ as a function of time, this corresponds to $|\alpha,t_o=0;t\rangle = \cos\Delta t/\hbar|R\rangle - i\sin\Delta t/\hbar|L\rangle$ and $\langle L|\alpha,t_o=0;t\rangle = -i\sin\Delta t/\hbar$. The transition probability is $|\langle L|\alpha,t_o=0;t\rangle|^2 = \sin^2\Delta t/\hbar$.

(d) In the Schrödinger picture the base kets $|R\rangle$ and $|L\rangle$ remain stationary in time and the state vector obeys $i\hbar\,\partial/\partial t|\alpha,t_o=0;\ t\rangle = H|\alpha,t_o=0;\ t\rangle$. Write $|\alpha,t_o=0;\ t\rangle = \alpha_R(t)|R\rangle + \alpha_L(t)|L\rangle$ and using $H = \Delta(|L\rangle\langle R| + |R\rangle\langle L|)$, the Schrödinger equation leads to coupled equations $i\hbar d\alpha_R(t)/dt = \Delta\alpha_L(t)$ and $i\hbar d\alpha_L(t)/dt = \Delta\alpha_R(t)$ where $\alpha_R(t) = \langle R|\alpha,t_o=0;\ t\rangle$ and $\alpha_L(t) = \langle L|\alpha,t_o=0;\ t\rangle$. Solutions of the coupled equations can be obtained by noting that $d^2/dt^2[\alpha_{R,L}(t)] + (\Delta^2/\hbar^2)\alpha_{R,L}(t) = 0$, hence

$$\alpha_L(t) = A\cos\Delta t/\hbar + B\sin\Delta t/\hbar,\quad \alpha_R(t) = C\cos\Delta t/\hbar + D\sin\Delta t/\hbar \quad (4)$$

At $t = 0$ $|\alpha\rangle = \langle R|\alpha\rangle|R\rangle + \langle L|\alpha\rangle|L\rangle = \alpha_R(0)|R\rangle + \alpha_L(0)|L\rangle$, hence $\alpha_R(0) = C = \langle R|\alpha\rangle$ and $\alpha_L(0) = A = \langle L|\alpha\rangle$. Next the normalization condition at t, with $t_o=0$ $\langle\alpha,t_o=0;\ t|\alpha,t_o=0;\ t\rangle = 1$ give

$$\cos^2\Delta t/\hbar + (\langle R|\alpha\rangle^* D + D^*\langle R|\alpha\rangle + \langle L|\alpha\rangle^* B + B^*\langle L|\alpha\rangle)\cos\Delta t/\hbar\,\sin\Delta t/\hbar + (|D|^2 + |B|^2)\sin^2\Delta t/\hbar = 1. \quad (5)$$

Solution of (5) is possible with $D = -i\langle L|\alpha\rangle$ and $B = -i\langle R|\alpha\rangle$, hence (4) for $\alpha_L(t)$ and $\alpha_R(t)$ gives the coefficients of $|L\rangle$ and $|R\rangle$ in (3) of (b).

(e) The lack of Hermiticity here is same as in problem 2, replacing $H = H_{12}|1\rangle\langle 2|$

by $H = \Delta|L\rangle\langle R|$. We find again $H^n = 0$ for $n>1$, and $U(t,t_o=0) = 1 - it\Delta/\hbar\ |L\rangle\langle R|$ even for a finite time interval. The initial state is $\langle R|\alpha\rangle|R\rangle + \langle L|\alpha\rangle|L\rangle$; at a later time t we have $(1 - it\Delta/\hbar\ |L\rangle\langle R|)(\langle R|\alpha\rangle|R\rangle + \langle L|\alpha\rangle|L\rangle)$, hence probability for being found in $|L\rangle$ is $|\langle L|\alpha\rangle - (it\Delta/\hbar)\langle R|\alpha\rangle|^2$ and in $|R\rangle$ is $|\langle R|\alpha\rangle|^2$, but $|\langle L|\alpha\rangle - (it\Delta/\hbar)\langle R|\alpha\rangle|^2 + |\langle R|\alpha\rangle|^2 \neq |\langle L|\alpha\rangle|^2 + |\langle R|\alpha\rangle|^2$. Thus probability conservation is violated.

10. $H = p^2/2m + \frac{1}{2}m\omega^2x^2$ for the one dimensional simple harmonic oscillator.

(a) In the Heisenberg picture, the operators x and p obey the Heisenberg equations of motion: $dp/dt = (1/i\hbar)[p, H] = -m\omega^2x$, $dx/dt = (1/i\hbar)[x, H] = p/m$. This implies $\ddot{x} = -\omega^2x$ and $\ddot{p} = -\omega^2p$ with the initial conditions $x(0) = x_o$ and $p(0) = p_o$ $\dot{x}(0) = p_o/m$ and $\dot{p}(0) = -m\omega^2x_o$. The solutions are $x(t) = x_o\cos\omega t + (p_o/m\omega)\sin\omega t$, $p(t) = p_o\cos\omega t - m\omega x_o\sin\omega t$ which give $H = p^2(t)/2m + \frac{1}{2}m\omega^2x^2(t) = p_o^2/2m + \frac{1}{2}m\omega^2x_o^2$, i.e. H is time independent. Dynamical variables x and p are time-dependent in the Heisenberg picture. At $t = 0$, the Heisenberg and Schrödinger pictures coincide, thus $x_H(0) = x_S(0) = x_o$ (with $x_S(t) = x_S(0)$) and $p_H(0) = p_S(0) = p_o$ (with $p_S(t) = p_S(0)$) and we note the time-independence of dynamical variables in the Schrödinger picture.

The relationship between the Heisenberg and Schrödinger pictures is $x_H(t) = e^{iHt/\hbar}x_Se^{-iHt/\hbar}$ with $x_S = x_o$ and $p_H(t) = e^{iHt/\hbar}p_Se^{-iHt/\hbar}$ with $p_S = p_o$. Using (2.3.48) - (2.3.50), one knows $x_H(t) = x_o\cos\omega t + (p_o/m\omega)\sin\omega t$. Also

$$e^{iHt/\hbar}p_oe^{-iHt/\hbar} = p_o + (it/\hbar)[H,p_o] + (i^2t^2/2!\hbar^2)[H,[H,p_o]]$$
$$+ (i^3t^3/3!\hbar^3)[H,[H,[H,p_o]]] + \ldots = p_o - \frac{(t^2\omega^2)}{2!}p_o - tm\omega^2x_o + \frac{t^3m\omega^4x_o}{3!} + \ldots.$$

where we have used $[H, x_o] = -i\hbar p_o/m$, $[H, p_o] = i\hbar m\omega^2x_o$. This implies that $p_H(t) = p_o\cos\omega t - m\omega x_o\sin\omega t$.

(b) At t=0, the general state vectors for both pictures are equal: $|\alpha\rangle_H = |\alpha\rangle_S =$

$|\alpha,t=0\rangle$, e.g. $|\alpha,t=0\rangle = \Sigma_n c_n(0)|n\rangle$. At $t\neq 0$, $|\alpha,t\rangle_H = |\alpha,t=0\rangle = \Sigma_n c_n(0)|n\rangle$ i.e. time independent, while $|\alpha,t\rangle_S = e^{-iHt/\hbar}|\alpha,t=0\rangle = \Sigma_n c_n(0)e^{-i\omega(n+\frac{1}{2})t}|n\rangle$ and is thus time dependent. (We have used $H = \hbar\omega(N+\frac{1}{2})$ which is time-independent in both pictures). We can recast $|\alpha,t\rangle_S$ as $|\alpha,t\rangle_S = \Sigma_n c_n(t)|n\rangle$ with $c_n(t) = c_n(0)e^{-i\omega(n+\frac{1}{2})t}$. Also note $i\hbar\partial/\partial t|\alpha,t\rangle_S = H|\alpha,t\rangle_S$ which is the Schrödinger equation for the Schrödinger state vector. Remarks: $c_n(t)$ can be determined in the two pictures by (a) $c_n(t) = \langle n|\alpha,t\rangle_S = c_n(0)e^{-i\omega(n+\frac{1}{2})t}$, the Schrödinger picture with base kets $|n\rangle$ time independent, and (b) $c_n(t) = \langle n,t|\alpha,t\rangle_H = \langle n|e^{-iHt/\hbar}|\alpha,t\rangle_H = c_n(0)e^{-i\omega(n+\frac{1}{2})t}$, the Heisenberg picture with base kets $|n,t\rangle = e^{iHt/\hbar}|n\rangle$ which are time-dependent.

11. For a one-dimensional SHO potential $H = p^2/2m + \frac{1}{2}m\omega^2x^2$, hence $\dot{x} = (1/i\hbar)[x,H] = p/m$, and $\dot{p} = (1/i\hbar)[p,H] = (1/i\hbar)(m\omega^2/2)[p,x^2] = (m\omega^2/2i\hbar)[-2i\hbar x] = -m\omega^2x$. Hence $\ddot{x} + \omega^2x = 0$, and solution is $x(t) = A\cos\omega t + B\sin\omega t$. At $t=0$, $x(0) = A$ while $\dot{x}(t) = -A\omega\cos\omega t + B\omega\cos\omega t$ leads to $\dot{x}(0) = B\omega$ and thus $p(0) = m\omega B$. Thus in the Heisenberg picture $x(t) = x(0)\cos\omega t + (p(0)/m\omega)\sin\omega t$.

Our state vector $|\alpha\rangle = e^{-ipa/\hbar}|0\rangle$ at $t=0$; for $t>0$ we have in the Heisenberg picture $\langle x(t)\rangle = \langle\alpha|x(t)|\alpha\rangle$. We note that

$$e^{ip(0)a/\hbar}x(0)e^{-ip(0)a/\hbar} = e^{ip(0)a/\hbar}([x(0),e^{-ip(0)a/\hbar}] + e^{-ip(0)a/\hbar}x(0)) = x(0) + a,$$

while $e^{ip(0)a/\hbar}p(0)e^{-ip(0)a/\hbar} = p(0)$. Hence

$$\langle x(t)\rangle = \langle\alpha|x(t)|\alpha\rangle = \langle 0|e^{ipa/\hbar}x(t)e^{-ipa/\hbar}|0\rangle = \langle 0|e^{ip(0)a/\hbar}[x(0)\cos\omega t + (p(0)/m\omega)\sin\omega t]e^{-ipa/\hbar}|0\rangle.$$

Since $\langle 0|x(0)|0\rangle = \langle 0|p(0)|0\rangle = 0$, we obtain for $\langle x(t)\rangle = a\cos\omega t$.

12. (a) The wave function in problem 11 takes form $\langle x'|\alpha\rangle = \langle x'|e^{-ipa/\hbar}|0\rangle$. Since $e^{ipa/\hbar}|x'\rangle = |x'-a\rangle$ (hence $\langle x'|e^{-ipa/\hbar} = \langle x'-a|$), we have $\langle x'|\alpha\rangle = \langle x'-a|0\rangle$. Hence $\langle x'|\alpha\rangle = \pi^{-\frac{1}{4}}x_0^{-\frac{1}{2}}\exp[-(x'-a)^2/2x_0^2]$.

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2a) Using the standard Pauli matrices, we see

$$\hat{H} \to \mu \begin{pmatrix} b_0 & b_1 \cos\omega t - i b_1 \sin\omega t \\ b_1 \cos\omega t + i b_1 \sin\omega t & -b_0 \end{pmatrix}$$

$$\boxed{= \mu \begin{pmatrix} b_0 & b_1 e^{-i\omega t} \\ b_1 e^{i\omega t} & -b_0 \end{pmatrix}}$$

2b) By definition, $i\hbar \partial_t \Psi = \hat{H}\Psi$

Define $\Psi_0(t) = e^{i\omega t \hat{\sigma}_z/2} \Psi(t)$

Then, $i\hbar \partial_t \left(e^{-i\omega t \hat{\sigma}_z/2} \Psi_0(t) \right) = \hat{H} e^{-i\omega t \hat{\sigma}_z/2} \Psi_0(t)$

$$\Rightarrow i\hbar \left(\frac{-i\omega \hat{\sigma}_z}{2} e^{-i\omega t \hat{\sigma}_z/2} \Psi_0 + e^{-i\omega t \hat{\sigma}_z/2} \partial_t \Psi_0 \right)$$

$$= \hat{H} e^{-i\omega t \hat{\sigma}_z/2} \Psi_0$$

$$\Rightarrow i\hbar \partial_t \Psi_0 = \left[e^{i\omega t \hat{\sigma}_z/2} \hat{H} e^{-i\omega t \hat{\sigma}_z/2} - \frac{\hbar\omega}{2} \hat{\sigma}_z \right] \Psi_0$$

$$= \hat{H}_0 \Psi_0$$

since $e^{i\omega t \hat{\sigma}_z/2}$ is unitary. Working in the basis where $\hat{\sigma}_z$ is diagonal,

$$\hat{H}_0 = \mu \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix} \begin{pmatrix} b_0 & b_1 e^{-i\omega t} \\ b_1 e^{i\omega t} & -b_0 \end{pmatrix} \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}$$

$$- \begin{pmatrix} \hbar\omega/2 & 0 \\ 0 & -\hbar\omega/2 \end{pmatrix}$$

$$= \mu \begin{pmatrix} b_0 - \hbar\omega/2 & b_1 \\ b_1 & -b_0 + \hbar\omega/2 \end{pmatrix}$$

cfd

Phys 215A soln #3, ctd

2b ctd) We note $\hat{U}_0(t) = e^{-i\hat{H}_0 t/\hbar}$ since $\hat{H}_0$ is time independent. Then, since $\Psi_0(t=0) = \Psi(t=0)$

$$\Psi_0(t) = \hat{U}_0(t)\,\Psi_0(t=0)$$

$$\Rightarrow \quad \Psi(t) = \underbrace{e^{-i\omega t\hat{\sigma}_z/2}\,\hat{U}_0(t)}_{\hat{U}(t)}\,\Psi(t=0)$$

Since $\hat{U}_0(t) = 1 - \frac{it}{\hbar}\hat{H}_0 + \cdots$ we define $b_0' = b_0 - \frac{\hbar\omega}{2}$ and $\hbar^2\omega_0^2 = \mu^2(b_0'^2 + b_1^2)$

so that $\hat{H}_0^2 = \hbar^2\omega_0^2\hat{I}$ Noting that

$$\hat{U}_0(t) = \left[1 - \frac{t^2}{2\hbar^2}\hat{H}_0^2 + \cdots\right] - \frac{i\hat{H}_0}{\hbar\omega_0}\left[\omega_0 t - \frac{1}{6}\frac{t^3}{\hbar^2}\hat{H}_0^2 + \cdots\right]$$

thus

$$\hat{U}_0(t) \to \begin{pmatrix} \cos(\omega_0 t) - \frac{i\mu b_0'}{\hbar\omega_0}\sin(\omega_0 t) & -\frac{i\mu b_1}{\hbar\omega_0}\sin(\omega_0 t) \\ -\frac{i\mu b_1}{\hbar\omega}\sin(\omega_0 t) & \cos(\omega_0 t) + \frac{i\mu b_0'}{\hbar\omega_0}\sin(\omega_0 t) \end{pmatrix}$$

noting that $e^{-i\omega t\hat{\sigma}_z/2} \to \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}$

and denoting $\alpha = \frac{\mu b_0'}{\hbar\omega_0}$ $\beta = \frac{\mu b_1}{\hbar\omega_0}$ $(\alpha^2 + \beta^2 = 1!)$

we see

$$\hat{U}(t) = e^{-\frac{i\omega t}{2}} \begin{pmatrix} \cos(\omega_0 t) - i\alpha\sin(\omega_0 t) & -i\beta\sin(\omega_0 t) \\ e^{i\omega t}\left(-i\beta\sin(\omega_0 t)\right) & e^{i\omega t}\left(\cos(\omega_0 t) + i\alpha\sin(\omega_0 t)\right) \end{pmatrix}$$

2c, ctd) $\det(\hat{U}(t)) = \det\left(e^{-i\omega t\hat{\sigma}_z/2}\right)\det\left(e^{-it\hat{H}_0/\hbar}\right)$

$$= e^{-\frac{i\omega t}{2}\mathrm{Tr}[\hat{\sigma}_z]}\, e^{-\frac{it}{\hbar}\mathrm{Tr}[\hat{H}_0]}$$

$$= \boxed{1}$$

ctd

Phys 215A, Soln #3 ctd

2d) We want $\langle \downarrow | \hat{u}(t) | \uparrow \rangle$

or, as matrices $(0\ 1)\ \hat{u}(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= e^{i\omega t/2} \begin{pmatrix} -i a_1 \sin(\omega_0 t) \end{pmatrix}$

The probability is $|\langle \downarrow | u(\pi) \rangle|^2$

or $P_{flip}(t) = \boxed{a_1^2 \sin^2(\omega_0 t)}$

note $a_1 = \mu b_1 \left[(b_0 - \hbar\omega/2)^2 + b_1^2 \right]^{-1}$

2c) If $\Psi(t=0) \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in this basis, then

$\Psi(t) = \hat{u}(t)\, \Psi(t=0)$

$$\boxed{= \frac{1}{\sqrt{2}} e^{-i\omega t/2} \begin{pmatrix} \cos(\omega_0 t) - i(a_0 + a_1)\sin(\omega_0 t) \\ e^{i\omega t}\left(\cos(\omega_0 t) + i(a_0 - a_1)\sin(\omega_0 t)\right) \end{pmatrix}}$$

3) We see $\hat{u}(t) = \exp\left(\frac{-i}{\hbar} \int_0^t \delta\pi J(\tau)\, \hat{\sigma}_1^z \hat{\sigma}_2^y \right)$

so that $\hat{u}(t_f) = \exp\left(-\frac{i\pi}{4} \hat{\sigma}_1^z \hat{\sigma}_2^y \right)$

Also, $\left(\hat{\sigma}_1^z \hat{\sigma}_2^y \right)^2 = \hat{I}$, so $\hat{u}(t_f) = 1 - \frac{i\pi}{4} \hat{\sigma}_1^z \hat{\sigma}_2^y - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 \hat{I} + \cdots$

$= \cos\left(\frac{\pi}{4}\right) \hat{I} - i \sin\left(\frac{\pi}{4}\right) \hat{\sigma}_1^z \hat{\sigma}_2^y$

Expanding $\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}$ and $\hat{\sigma}_1^z \otimes \hat{\sigma}_2^y \rightsquigarrow \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$

we see $|\Psi(t_f)\rangle = \hat{u}(t_f)\, |\Psi(t=0)\rangle$

$$= \boxed{\frac{1}{\sqrt{2}} \begin{pmatrix} a \\ a \\ b \\ -b \end{pmatrix}}$$