Physics 215c: Problem Set 3

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Please let me know if you encounter any typos in the solutions.

**Problem 1 (15)**

We consider the quantum Ising model on a hypercubic D-dimensional lattice in a transverse field with Hamiltonian

\[ \hat{H} = -J \sum_{\langle ij \rangle} \hat{\sigma}^z_i \hat{\sigma}^z_j - h \sum_i \hat{\sigma}^x_i. \]  

(a)

Consider the paramagnetic phase where \( J = 0 \). Let \(|i\rangle = |+_1+_2_-i_3\ldots+N\rangle\) in the x-basis and let \( \mathcal{P} = \sum |i\rangle \langle i| \). Note that

\[
\begin{align*}
\sigma^+_i |j\rangle &= (1 - 2\delta_{i,j})|j\rangle \\
\sigma^-_i \sigma^+_i |j\rangle &= \frac{\sigma^+_{\bar{i}} - \sigma^-_{\bar{i}} - \sigma^+_{\bar{i}} - \sigma^-_{\bar{i}}}{i} |j\rangle = -\sigma^+_{\bar{i}} - \sigma^-_{\bar{i}} [(1 - \delta_{i,j})(i,j) - \delta_{i,j}0)] \\
&= -[(1 - \delta_{i,k})(1 - \delta_{i,j})(i,j,k) - \delta_{k,j}(1 - \delta_{i,j})|i\rangle - \delta_{i,j}|k\rangle] \\
\end{align*}
\]

where \( \sigma^\pm_i = 1/2(\sigma^y_i \pm i\sigma^z_i) \). Then we can write the full Hamiltonian as

\[ \hat{H}' = \mathcal{P} \hat{H} \mathcal{P} \]

\[
\begin{align*}
&= \sum_m |m\rangle \langle m| \left( -J \sum_{\langle ij \rangle} \hat{\sigma}^z_i \hat{\sigma}^z_j - h \sum_i \hat{\sigma}^x_i \right) |n\rangle \langle n| \\
&= J \sum_{\langle ij, m, n \rangle} |m\rangle \langle m| [-\delta_{j,n} |i\rangle - \delta_{i,n} |j\rangle] \langle n| - h \sum_{i,n} (1 - 2\delta_{i,n}) |n\rangle \langle n| \\
&= -J \sum_{\langle ij \rangle} (|i\rangle \langle j| + |j\rangle \langle i|) - h(N - 2) \mathcal{P} \\
&= \frac{1}{2}(1 - 2\delta_{i,j}) (|i\rangle \langle j| + |j\rangle \langle i|) - h(N - 2) \mathcal{P} \\
&= \frac{1}{2}(1 - 2\delta_{i,j}) (|i\rangle \langle i| + |j\rangle \langle j|) - h(N - 2) \mathcal{P} \\
&= \frac{1}{2} (|i\rangle \langle i| + |j\rangle \langle j| - 2\delta_{i,j} |i\rangle \langle j| - 2\delta_{j,i} |j\rangle \langle i|) - h(N - 2) \mathcal{P} \\
&= \frac{1}{2} |i\rangle \langle i| - \frac{1}{2} \delta_{i,j} |j\rangle \langle j| - h(N - 2) \mathcal{P} \\
&= \frac{1}{2} |i\rangle \langle i| - \frac{1}{2} \delta_{i,j} |j\rangle \langle j| - h(N - 2) \mathcal{P} \\
\end{align*}
\]
Let
\[ |k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{i k \cdot x_j} |j\rangle \]  
(4)

Let \( e_k \) be a basis of unit lattice displacement vectors and, by a horrendous abuse of notation, let \( |i + e_k\rangle \) be the basis state with spin-up at \( x_i + e_k \). Then
\[
\hat{H}'|k\rangle = -J \sum_i \sum_{e_k} (|i\rangle\langle i + e_k|) |k\rangle - h(N - 2) \mathcal{P} |k\rangle
\]
\[
= -J \sum_i \sum_{e_k} (|i\rangle\langle i + e_k|) |k\rangle - h(N - 2) |k\rangle
\]
\[
= -J \sum_{i,e_k} e^{i k \cdot (x_i + e_k)} |i\rangle - h(N - 2) |k\rangle
\]
\[
= \left( -J \sum_{e_k} e^{i k \cdot e_k} - h(N - 2) \right) |k\rangle
\]
(5)

We thus find the energy spectrum is given by
\[
\epsilon_k = -2J \sum_{d=1}^{D} \cos(k \cdot e_d) - h(N - 2)
\]
(6)

where \( \{e_d\} \) are the subbasis of positive lattice displacement vectors.

(c)

Now consider the ferromagnetic phase with \( h = 0 \) and a ground state \( |0+\rangle_F \) with all spins aligned in the \( +\hat{z} \) direction. The ground state is actually degenerate, with another state \( |0-\rangle_F \) with all spins aligned in the \( -\hat{z} \) direction. We label the first excited states with only the \( i \)-th spin flipped to \( -\hat{z} \) by \( |i\rangle_F \). The energies of the first two degenerate energy levels are \( E_{0,\pm} = -JN D \) and \( E_{1,i} = -J(N - 4) D \).

The perturbing Hamiltonian is
\[
\hat{H}_h = -h \sum_i \hat{\sigma}_i^z
\]
(7)

Note first that \( \mathcal{H} \) does not have matrix elements that mix degenerate ground states, so we can proceed as in non-degenerate perturbation theory.

Next note that
\[
\hat{\sigma}_i^z = \hat{\sigma}^+ - \hat{\sigma}^-
\]
(8)

so that the perturbing Hamiltonian only mixes the ground state with the first excited states.
Thus we have, at first order

\[ E_{0+}^{(1)} = F \langle 0 + | \mathcal{H}_h | 0 + \rangle_F \]
\[ = F \langle 0 + | - \hbar \sum_i (\hat{\sigma}^+ - \hat{\sigma}^-) | 0 + \rangle_F \]
\[ = -\hbar \sum_i F \langle 0 + | i \rangle_F = 0 \]

(9)

At second order we have

\[ E_{0+}^{(2)} = \sum_i | F \langle i | \mathcal{H}_h | 0 + \rangle_F |^2 \]
\[ = \sum_i \frac{\hbar^2}{-4JD} = \frac{-Nh^2}{4JD} \]

(10)

(d)

The effective Hamiltonian on the degenerate first excited states is given by

\[ H_{ij}^{\text{eff}} = \sum_n \frac{F \langle i | \mathcal{H}_h | n \rangle_F F \langle n | \mathcal{H}_h | j \rangle_F}{\epsilon_1 - \epsilon_n} \]

(11)

where \( \epsilon_n \) is the energy above the vacuum at 0th order and \( | n \rangle \) are unperturbed energy eigenstates of different energy to the first excited states.

Again, since the perturbing Hamiltonian only flips one spin at a time, we need only consider matrix elements with the vacuum and the second excited states. The excited states have two spins flipped and we denote them by \( | i, j \rangle_F \). These have energy \( \epsilon_2 = 8JD - 4J \) when they are nearest neighbours and energy \( \epsilon_3 = 8JD \) otherwise.

We will consider the matrix elements in cases.

Let \( i \neq j \) not be nearest neighbours. Then

\[ H_{ij}^{\text{eff}} = \frac{F \langle i | \mathcal{H}_h | 0 + \rangle_F F \langle 0 + | \mathcal{H}_h | j \rangle_F}{4JD} + \sum_{k,l \text{ not n.n.}} \frac{F \langle i | \mathcal{H}_h | k,l \rangle_F F \langle k,l | \mathcal{H}_h | j \rangle_F}{-4JD} \]
\[ = \hbar^2 \left( \frac{1}{4DJ} + \frac{1}{-4DJ} \right) \]
\[ = 0 \]

(12)
Let $i \neq j$ be nearest neighbours. Then

$$H_{ij}^{\text{eff}} = \frac{F \langle i | \mathcal{H}_h | 0+ \rangle F}{4JD} + \sum_{<k,l>} \frac{F \langle i | \mathcal{H}_h | k, l \rangle F F \langle k, l | \mathcal{H}_h | j \rangle F}{-4JD + 4}$$

$$= \frac{h^2}{4J} \left( \frac{1}{4JD} + \frac{1}{-4DJ + 4J} \right)$$

$$= \frac{h^2}{4J} \left( \frac{1}{D(1-D)} \right)$$

(13)

Let $i = j$. Then

$$H_{ii}^{\text{eff}} = \frac{F \langle i | \mathcal{H}_h | 0+ \rangle F}{4JD} + \sum_{<k,i>} \frac{F \langle i | \mathcal{H}_h | k, i \rangle F F \langle k, i | \mathcal{H}_h | i \rangle F}{-4JD + 4} + \sum_{k \text{ not n.n. to } i} \frac{F \langle i | \mathcal{H}_h | k, i \rangle F F \langle k, i | \mathcal{H}_h | i \rangle F}{-4JD}$$

$$= \frac{h^2}{4J} \left( \frac{2D}{1-D} - \frac{N - 2(D + 1)}{D} \right)$$

(14)

We do indeed see that all of the non-vanishing matrix elements diverge like $1/(1 - D)$.

(e)

The full effective Hamiltonian, projected onto the degenerate subspace is given by

$$\hat{H} = \sum_i |i\rangle_F F \langle i | \mathcal{H}_h | i \rangle F F \langle i | + \sum_{ij} |i\rangle_F H_{ij}^{\text{eff}} F \langle j | .$$

(15)

Then its action on the plane wave state $|\mathbf{k}\rangle_F$ is given by

$$\hat{H} |\mathbf{k}\rangle_F = -JD(N-4) \sum_i F \langle i | e^{i\mathbf{k} \cdot \mathbf{x}_i} \frac{e^{i\mathbf{k} \cdot \mathbf{x}_i}}{\sqrt{N}} | i\rangle_F F \langle i | H_{ij}^{\text{eff}} F \langle i | + \sum_{ij} \frac{e^{i\mathbf{k} \cdot \mathbf{x}_j}}{\sqrt{N}} | i\rangle_F$$

$$= -JD(N-4) |\mathbf{k}\rangle + \frac{h^2}{4J} \left( \frac{2D}{1-D} - \frac{N - 2(D + 1)}{D} \right) |\mathbf{k}\rangle + \frac{h^2}{4J} \left( \frac{1}{D(1-D)} \right) \sum_{<i,j>} e^{i\mathbf{k} \cdot \mathbf{x}_j} | i\rangle_F$$

$$= -JD(N-4) |\mathbf{k}\rangle + \frac{h^2}{4J} \left( \frac{2D}{1-D} - \frac{N - 2(D + 1)}{D} \right) |\mathbf{k}\rangle + \frac{h^2}{4J} \left( \frac{2}{D(1-D)} \right) \sum_{d=1}^{D} \cos(\mathbf{k} \cdot \mathbf{e}_d) |\mathbf{k}\rangle .$$

(16)

Thus we have a spectrum

$$\epsilon_k = -JD(N-4) + \frac{h^2}{4JD(1-D)} \left( 2D^2 - N(1-D) - 2(1-D)^2 + \sum_{d=1}^{D} \cos(\mathbf{k} \cdot \mathbf{e}_d) \right)$$

(17)
Problem 2 (15)

We have the Quantum XY Hamiltonian

\[ \hat{H}_{XY} = -J \sum_{\langle ij \rangle} (\hat{\sigma}^z_i \hat{\sigma}^z_j + \hat{\sigma}^y_i \hat{\sigma}^y_j) - \hbar \sum_i \hat{\sigma}^z_i. \]  

(a)

Define

\[ \hat{U} = e^{i\phi \sum_i \sigma^z_i}. \]  

We wish to show the commutator of \( \hat{U} \) with the Hamiltonian vanishes. We prove the simpler sufficient condition, that the commutator with \( \sigma^z_{tot} \) vanishes:

\[ \left[ \hat{H}_{XY}, \sigma^z_{tot} \right] = -J \sum_{\langle ij \rangle} (\hat{\sigma}^z_i \hat{\sigma}^z_j + \hat{\sigma}^y_i \hat{\sigma}^y_j), \sum_k \hat{\sigma}^z_k \]  

\[ = -J \sum_{\langle ij \rangle} \sum_k \left[ (\hat{\sigma}^z_i \hat{\sigma}^z_j + \hat{\sigma}^y_i \hat{\sigma}^y_j), \hat{\sigma}^z_k \right] \]  

For any site, we have

\[ [A\sigma^x + B\sigma^y, \sigma^z] = 2i(-A\sigma^y + B\sigma^x) \]  

so that we have

\[ \left[ \hat{H}_{XY}, \sigma^z_{tot} \right] = -2iJ \sum_{\langle ij \rangle} \sum_k \left[ \delta_{k,i} (-\sigma^y_i \sigma^z_j + \sigma^z_i \sigma^y_j) + \delta_{k,j} (-\sigma^y_i \sigma^z_j + \sigma^z_i \sigma^y_j) \right] \]  

\[ = -2iJ \sum_{\langle ij \rangle} \left[ (-\sigma^y_i \sigma^z_j + \sigma^z_i \sigma^y_j) + (-\sigma^z_i \sigma^y_j + \sigma^y_i \sigma^z_j) \right] \]  

\[ = 0 \]  

(b)

We wish to diagonalize the Hamiltonian projected onto the degenerate subspace.

Consider the paramagnetic phase where \( J = 0 \). Let \( |i\rangle = |+_{1+2\ldots} -_{1+2\ldots-N}\rangle \) in the \( z \)-basis and let
\( \mathcal{P} = \sum |i\rangle \langle i| \). Note that, like before,

\[
\sigma_z^i |j\rangle = (1 - 2\delta_{i,j}) |j\rangle
\]

\[
\sigma_x^i |j\rangle = \frac{\sigma_x^i - \sigma_y^i}{\sqrt{2}} |j\rangle
\]

\[
\sigma_y^i |j\rangle = \frac{\sigma_x^i + \sigma_y^i}{\sqrt{2}} |j\rangle
\]

Then we can write the projection of the full Hamiltonian exactly as in the first question, but having simply doubled the contribution from the \( J \) term:

\[
\hat{H}_{XY}' = \mathcal{P} \hat{H}_{XY} \mathcal{P} = -2J \sum_{\langle ij \rangle} |i\rangle \langle j| + |j\rangle \langle i| - \hbar (N - 2) \mathcal{P}
\]

We find our calculation of the spectrum from question 1 carries forward, but with \( J \to 2J \):

\[
\epsilon_k = -4J \sum_{d=1}^{D} \cos(k \cdot e_d) - \hbar (N - 2)
\]

for eigenstates \( |k\rangle \).

\(\text{(c)}\)

Define \( \hat{\sigma}_z^i = (\hat{\sigma}_z^i \pm i \hat{\sigma}_y^i) / 2 \). Then substituting and expanding gives

\[
\hat{H}_{XY} = -J \sum_{\langle ij \rangle} (\hat{\sigma}_z^i \hat{\sigma}_z^j + \hat{\sigma}_y^i \hat{\sigma}_y^j) - \hbar \sum_{i} \hat{\sigma}_z^i
\]

\[
= -2J \sum_{\langle ij \rangle} (\hat{\sigma}_z^i \hat{\sigma}_z^j + \hat{\sigma}_y^i \hat{\sigma}_y^j) - \hbar \sum_{i} \hat{\sigma}_z^i
\]

\(\text{(d)}\)

We work in 1 dimension and let

\[
\hat{\sigma}_z^i = \prod_{j < i} (1 - 2\hat{c}_j^\dagger \hat{c}_j) \hat{c}_i, \quad \hat{\sigma}_i^z = \prod_{j < i} (1 - 2\hat{c}_j^\dagger \hat{c}_j) \hat{c}_i^\dagger, \quad \hat{\sigma}_z^i = (1 - 2\hat{c}_i^\dagger \hat{c}_i)
\]

6
for fermionic operators $\hat{c}_i$. Note that $N_i = \hat{c}_i^\dagger \hat{c}_i$ is just the number operator for the fermionic operator, is bosonic, and has commutation relations

$$[\hat{N}_i, \hat{c}_j] = -\delta_{i,j} \hat{c}_j, \quad [\hat{N}_i, \hat{c}_j^\dagger] = \delta_{i,j} \hat{c}_j^\dagger, \quad [\hat{N}_i, N_j] = 0$$
(28)

We check the commutation relations of the Pauli matrices. First,

$$[\hat{\sigma}_i^z, \hat{\sigma}_j^z] = 4 [N_i, N_j] = 0$$
(29)

Next,

$$[\hat{\sigma}_i^+, \hat{\sigma}_j^z] = \prod_{k<i} (1 - 2N_k)\hat{c}_i (1 - 2N_j) = -2 \prod_{k<i} (1 - 2N_k)\hat{c}_i N_j = -2\delta_{i,j} \prod_{k<i} (1 - 2N_k) c_i$$
$$= -2\delta_{i,j} \hat{\sigma}_i^+$$
(30)

and, by conjugation,

$$[\hat{\sigma}_i^-, \hat{\sigma}_j^z] = -[\hat{\sigma}_i^+, \hat{\sigma}_j^z]^\dagger = 2\delta_{i,j} \hat{\sigma}_i^-$$
(31)

and lastly, making use of the fact that $(1 - 2N_k)^2 = 1$ (FYI: $1 - 2N_k = (-1)^{N_k}$),

$$[\hat{\sigma}_i^+, \hat{\sigma}_j^-] = \prod_{k<i} (1 - 2N_k)\hat{c}_i \prod_{l<j} (1 - 2N_l)\hat{c}_j^\dagger$$
$$= \delta_{i,j} \left[ \hat{c}_i \hat{c}_j^\dagger + \Theta(i - j) \left( 1 - 2N_j \right) \hat{c}_i \hat{c}_j^\dagger + \Theta(j - i) \left( 1 - 2N_i \right) \hat{c}_i \hat{c}_j^\dagger \right]$$
$$= \delta_{i,j} \left( 1 - 2\hat{c}_i \hat{c}_j^\dagger \right) + \Theta(i - j) \left( 1 - 2N_j \right) \{ \hat{c}_i, \hat{c}_j^\dagger \} + \Theta(j - i) \left( 1 - 2N_i \right) \{ \hat{c}_i, \hat{c}_j^\dagger \}$$
$$= \delta_{i,j} \hat{\sigma}_i^z$$
(32)
and

\[
[\hat{\sigma}^+_i, \hat{\sigma}^+_j] = \left[ \prod_{k<i} (1 - 2N_k) \hat{c}_i, \prod_{l<j} (1 - 2N_l) \hat{c}_j \right]
= \delta_{i,j} [\hat{c}_i, \hat{c}_j] + \Theta(i-j) [(1 - 2N_j) \hat{c}_i, \hat{c}_j] + \Theta(j-i) [\hat{c}_i, (1 - 2N_i) \hat{c}_j]
= \Theta(i-j)(1 - 2N_j) [\hat{c}_i, \hat{c}_j] + \Theta(j-i)(1 - 2N_i) [\hat{c}_i, \hat{c}_j]
= 0
\]

(33)

and, again by conjugation,

\[
[\hat{\sigma}^-_i, \hat{\sigma}^-_j] = 0
\]

(34)

Thus we have reproduced the correct Pauli matrix algebra.

\[\text{(e)}\]

We have, in one dimension, assuming a periodic lattice,

\[
\hat{H}_{XY} = -2J \sum_{\langle ij \rangle} (\hat{\sigma}^+_i \hat{\sigma}^-_j + \hat{\sigma}^-_i \hat{\sigma}^+_j) - \hbar \sum_i \hat{\sigma}^z_i
= -2J \sum_{i=1}^N (\hat{\sigma}^+_i \hat{\sigma}^-_{i+1} + \hat{\sigma}^-_i \hat{\sigma}^+_{i+1}) - \hbar \sum_i \hat{\sigma}^z_i
= -2J \sum_{i=1}^N \left( \hat{c}_i (-1)^{N_i} \hat{c}^\dagger_{i+1} + \hat{c}^\dagger_i (-1)^{N_i} \hat{c}_{i+1} \right) - \hbar \sum_i (-1)^{N_i}
= -2J \sum_{i=1}^N \left( \hat{c}^\dagger_i \hat{c}^\dagger_{i+1} + \hat{c}^\dagger_{i+1} \hat{c}_i \right) (-1)^{N_i} - \hbar \sum_i (-1)^{N_i}
= -2J \sum_{i=1}^N \left( -\hat{c}_i \hat{c}^\dagger_{i+1} + \hat{c}^\dagger_{i+1} \hat{c}_i \right) - \hbar \sum_i (-1)^{N_i}
\]

(35)

where in the last line we used the fact that \(\hat{c}_i\) is only non-vanishing on states where \((-1)^{N_i} = -1\) and \(\hat{c}^\dagger_i\) is only non-vanishing on states where \((-1)^{N_i} = 1\)
We have
\[ \{ \hat{c}_k, \hat{c}^\dagger_{k'} \} = \frac{1}{N} \sum_{x_i, x_j} e^{i(kx_i - k'x_j)} \{ \hat{c}_i, \hat{c}^\dagger_j \} \]
\[ = \frac{1}{N} \sum_{x_i=1}^N e^{ix_i(k-k')} \]
\[ = \delta_{k,k'} \]
(36)

and
\[ \{ \hat{c}_k, \hat{c}_{k'}^\dagger \} = \frac{1}{N} \sum_{x_i, x_j} e^{i(kx_i + k'x_j)} \{ \hat{c}_i, \hat{c}^\dagger_j \} \]
\[ = 0 \]
(37)

and
\[ \{ \hat{c}^\dagger_k, \hat{c}^\dagger_{k'} \} = \frac{1}{N} \sum_{x_i, x_j} e^{-i(kx_i + k'x_j)} \{ \hat{c}^\dagger_i, \hat{c}^\dagger_j \} \]
\[ = 0 \]
(38)

We have
\[
\mathcal{H}_{XY} = -2J \sum_{i=1}^N \left( \hat{c}^\dagger_{i+1} \hat{c}_i + \hat{c}^\dagger_i \hat{c}_{i+1} \right) - \hbar \sum_i \left( -1 \right)^{N_i} \\
= -2J \sum_{k, k'} \sum_{i=1}^N \left( \frac{1}{N} e^{i(k-k')x_i} e^{-ik'\hat{c}^\dagger_k \hat{c}_k} + e^{-i(k-k')x_i} e^{ik'\hat{c}^\dagger_k \hat{c}_k} \right) - \hbar(N - 2 \sum_k \frac{1}{N} \sum_i e^{i(k-k')x_i} \hat{c}^\dagger_k \hat{c}_k) \\
= -2J \sum_k \left( e^{-ik\hat{c}^\dagger_k \hat{c}_k} + e^{ik'\hat{c}^\dagger_k \hat{c}_k} \right) - \hbar(N - 2 \sum_k \hat{c}_k^\dagger \hat{c}_k) \\
= \sum_k \left( -4J \cos(k) + 2\hbar \right) N_k \\
(39)
\]

where we have removed an arbitrary constant.

The single particle dispersion is then
\[ \epsilon_k = -4J \cos(k) + 2\hbar \]
(40)
which agrees with part (b) for a single excitation (again up to a choice of arbitrary constant).

(h)

We consider the state
\[ |G\rangle = \prod_{k|\epsilon_k<0} \hat{c}_k^\dagger |\text{vac}\rangle \]  
and evaluate
\[
M_z = \frac{1}{N} \sum_i \langle G|\sigma_i^z|G\rangle
\]
\[
= \frac{1}{N} \sum_i \langle G|(-1)^{N_i}|G\rangle
= \frac{1}{N} \sum_k \langle G|(-1)^{N_k}|G\rangle
= \frac{N_> - N_<}{N} = 1 - 2 \frac{N_<}{N}
\]
\[(42)\]

where \(N_>\) is the number of fermions with energy above zero that are not excited and \(N_<\) is the number of excited fermions with energy below 0.

The excited modes are those such that
\[
\cos(k) > \frac{h}{2J} \Rightarrow \left| \frac{2\pi n}{N} \right| < \arccos\left(\frac{h}{2J}\right) \Rightarrow |n| < \frac{N}{2\pi} \arccos\left(\frac{h}{2J}\right)
\]
\[(43)\]

The number of such modes is then exactly
\[
N_< = \Theta(1 - h/2J) \left(2 \left\lfloor \frac{N}{2\pi} \arccos\left(\frac{h}{2J}\right) \right\rfloor + 1 \right)
\]
\[(44)\]

We see that \(N_< = 0\) and \(M_z = 1\) when \(h \geq h_c\) for \(h_c = 2J\).

Analogously, the magnetization is \(-1\) when \(h < -2j\) and there is a second critical value at \(h_{c,-} = 2J\).

(i)

Working in the continuous approximation, between the two critical values we have
\[
\frac{\partial M_z}{\partial h} = \frac{\partial 2N_</N}{\partial h} = \frac{\partial}{\partial h} \left( \frac{2}{\pi} \arccos\left(\frac{h}{2J}\right) \right) = -\frac{2}{\pi} \left( h_c^2 - h^2 \right)^{-1/2}
\]
\[(45)\]

(When we are strictly beyond the critical values the susceptibility vanishes.) Near \(h = h_c\), but below the critical value, we can write this as
\[
\frac{\partial M_z}{\partial h} \approx -\sqrt{\frac{2}{\pi^2 h_c^2}} (h_c - h)^{-1/2}
\]
\[(46)\]

giving \(\gamma = 1/2\).