1.) PATH INTEGRALS FOR 1D OSCILLATOR

Consider a 1d Harmonic oscillator with Hamiltonian,
\[ H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \]  

The quantum mechanical propagator
\[ U(x_2,t_2;x_1,t_1) = \langle x_2 | e^{-iH(t_2-t_1)/\hbar} | x_1 \rangle, \]

is given by
\[ U(x_2,t_2;x_1,t_1) = A(t_2-t_1)e^{iS_{cl}}, \]
where \( S_{cl} \) is the action for the classical path propagating from \( x_1 \) to \( x_2 \) in time \( t_2 - t_1 \). The prefactor is given by
\[ A(t) = \left\{ \frac{m\omega}{2\pi i\hbar \sin(\omega t)} \right\}^{1/2}. \]

A.) By evaluating \( S_{cl} \), obtain an explicit expression for the oscillator propogator.

B.) Using your result from part (A), obtain the imaginary time propogator for the oscillator, defined as,
\[ U_I(x_2,x_1;\tau) = \langle x_2 | e^{-H\tau/\hbar} | x_1 \rangle. \]

C.) The imaginary time propogator can be formally expanded in terms of the eigenstates and energies of \( H \) as
\[ U_I(x_2,x_1;\tau) = \sum_n \phi_n(x_2)\phi_n^*(x_1)e^{-E_n\tau/\hbar}. \]

Using your explicit expression for \( U_I \) obtained in (B) above, obtain the eigen energies and associated eigen wavefunctions of the two lowest oscillator levels.

D.) Use your expression for \( U_I \) from part (B) to evaluate the harmonic oscillator partition function, \( Z = Tr \exp(-\beta H) \), and compare it to the sum,
\[ \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}. \]

2.) GRADSHTEYN/RYZHNIK FOR PATH INTEGRALS

As quoted by R. Shankar (comedic author of the textbook on Quantum Mechanics) - “The table of path integrals has but one entry”. So let’s write the book. Consider a single quantum particle moving in one-spatial dimension with a general quadratic Lagrangian of the form,
\[ L = \frac{1}{2}m\dot{x}^2 - [a + bx + cx^2 + dx + ex\dot{x}]. \]

Of interest is computing the path (or functional) integral for the propagator,
\[ U(x_f,t_f;x_i,t_i) = \int_{x(t_i)=x_i}^{x(t_f)=x_f} Dx(t)\exp(iS/\hbar), \]
with the action \( S = \int_{t_i}^{t_f} dt \mathcal{L} \), and the integration measure as discussed in class.

A.) From the Euler-Lagrange equations, obtain an explicit expression for the classical trajectory, \( x_{cl}(t) \), satisfying the boundary conditions, \( x_{cl}(t_i) = x_i \) and \( x_{cl}(t_f) = x_f \).

Now we expand about the classical path by writing the general trajectory as,

\[
x(t) = x_{cl}(t) + y(t) \tag{10}
\]

which defines a path \( y(t) \) which satisfies the boundary conditions \( y(t_i) = y(t_f) = 0 \).

B.) Re-express the functional integral in Eq.(9) above as a functional integral over the trajectories \( y(t) \), essentially changing variables in the (multi-dimensional) path integral (convince yourself that the Jacobian is unity) to show that it can be re-cast in the form,

\[
U(x_f, t_f; x_i, t_i) = U_\omega(t_{fi}) \exp(iS_{cl}/\hbar) \tag{11}
\]

where \( S_{cl} \) is the action for your classical trajectory, \( x_{cl}(t) \), and \( t_{fi} = t_f - t_i \). Here \( U_\omega(t_{fi}) \) is the propagator for a harmonic oscillator with frequency \( \omega = \sqrt{2c/m} \) and Lagrangian \( \mathcal{L}_{osc} = m(\dot{y}^2 - \omega^2 y^2)/2 \) to propagate from \( y = 0 \) back to \( y = 0 \) in a time \( t_{fi} \):

\[
U_\omega(t_{fi}) = \int_{y(0)=0}^{y(t_{fi})=0} Dy(t) \exp(\frac{i}{\hbar} \int_0^{t_{fi}} dt \mathcal{L}_{osc}) \tag{12}
\]

In class we already obtained the result for the free particle propagator,

\[
U_0(t_{fi}) = \sqrt{\frac{m}{2\pi \hbar t_{fi}}} \tag{13}
\]

So we only need now to evaluate the frequency dependence. To this end we perform another functional change of variables by introducing a Fourier decomposition of the path \( y(t) \):

\[
y(t) = \sum_{n=1}^\infty y_n \sin(n \pi t/t_{fi}) \tag{14}
\]

C.) By inserting this expansion into Eq.(12) above, show that \( U_\omega(t_{fi}) \) can be re-expressed as,

\[
U_\omega(t_{fi}) = C \prod_{n=1}^\infty \int_{-\infty}^\infty dy_n \exp(i a_n y_n^2) \tag{15}
\]

where the constant out front is independent of \( \omega \). Extract the specific form for \( a_n \).

D.) Finally, perform the Gaussian integrals over \( y_n \) to extract the frequency dependence of \( U_\omega \), and show that it equals,

\[
U_\omega(t_{fi}) = \sqrt{\frac{m\omega}{2\pi \hbar \sin(\omega t_{fi})}} \tag{16}
\]

Hint: A useful identity is

\[
\sin(x) = x \prod_{n=1}^\infty \left[ 1 - \frac{x^2}{(n\pi)^2} \right]. \tag{17}
\]

Collect your results to obtain a final expression for the original propagator, \( U(x_f, t_f; x_i, t_i) \).
3.) LOCALIZATION

A problem of much interest is wave propagation in a random background potential, which arises in many different contexts such as the scattering of sound waves propagating through the earth after an earthquake or produced by an oil company “thumper” truck, or the propagation of microwaves from your cell phone through a random “landscape” of city buildings or the propagation of electron waves through a metal or semiconductor scattering off crystalline imperfections. Let’s focus on the latter example, where the wave equation is a quantum wave equation (ie Schrodinger equation). So, consider an electron moving through a solid scattering with Hamiltonian,

\[ H = \frac{1}{2m} |\vec{\rho} - \frac{e}{c} \vec{A}(\vec{r})|^2 + V(\vec{r}), \]

where \(\vec{r}\) and \(\vec{\rho}\) are the electrons position and momentum operators, respectively, and \(V(\vec{r})\) is an unspecified “random” potential of position. An external magnetic field has been applied with \(\vec{B} = \nabla \times \vec{A}\).

A.) Following Chapter 21 in Shankar’s book (or another reference for path integrals), derive the phase-space path integral representation of the electron propagator,

\[ U(\vec{r}_f, t_f; \vec{r}_i, t_i) = \langle \vec{r}_f | \exp(-iH(t_f - t_i)/\hbar) | \vec{r}_i \rangle, \]

as a functional path integral over paths \(\vec{r}(t)\) and \(\vec{p}(t)\). To avoid subtleties associated with operator ordering, assume the gauge \(\nabla \cdot \vec{A}\), so that \(\vec{p} \cdot \vec{A}(\vec{r}) = \vec{A}(\vec{r}) \cdot \vec{p}\).

B.) Explicitly perform the path integral over \(\vec{p}(t)\) to obtain a configuration space path integral representation of the electron propagator.

The probability for the electron to propagate from \(\vec{r}_i\) to \(\vec{r}_f\) is proportional to the squared modulus of the quantum amplitude, \(P(\vec{r}_f, \vec{r}_i, t_f, t_i) = |U|^2\), which can be expressed as a double functional integral over two sets of paths, \(\vec{r}_1(t)\) and \(\vec{r}_2(t)\). Since the potential is random, the paths that contributes in the functional integral will pick up a “random” phase from the exponential of the potential term in the action, and will generally destructively interfere with one another. It turns out that dropping all such interference terms, keeping only contributions with \(\vec{r}_1(t) = \vec{r}_2(t)\), essentially leads to classical diffusive dynamics for the electron probability. But consider the “return probability” - the probability for the electron to return to it’s initial location - \(P(\vec{r}, \vec{r}, t)\), where all the paths are closed “loops” with \(\vec{r}_1(t_i) = \vec{r}_1(t_f) = \vec{r}_2(t_i) = \vec{r}_2(t_f)\). In the quantum amplitude for this process there are very special pairs of “time reversed” paths which satisfy, \(\vec{r}_1(t_i + t) = \vec{r}_2(t_f - t)\), for \(0 \leq t \leq t_f - t_i\).

C.) Use your path integral expression for the propagator to show that provided the magnetic field vanishes, the complex amplitudes for these two paths, \(A_1 = \exp(iS_1/\hbar)\) and \(A_2 = \exp(iS_2/\hbar)\) are equal - that is, they have equal phases and will interfere constructively upon squaring to obtain the return probability, \(P(\vec{r}, \vec{r}, t)\).

Such constructive interference which is only present in the return probability, enhances the probability that the electron returns to the origin (relative to the classical return probability following from the diffusion equation). These processes ultimately can lead to a phenomena called “localization”, where the eigenstates of the electron moving in the random potential can become spatially confined - for example becoming “bound” to various minima in the potential \(V(\vec{r})\). Localization of electron waves in the conduction band of a metal or semiconductor, destroys the metallic conduction leading to an “insulator” with electrical resistance which diverges upon cooling towards zero temperature. But now consider the effects of an applied magnetic field.

D.) Show that once a magnetic field is present the amplitude for the two time reversed paths now pick up a relative phase, \(A_1 = \exp(2i\phi)A_2\). Obtain an explicit expression for the phase, \(\phi\), as a line integral of the vector potential along the path \(\vec{r}_1(t)\), and show that for a simple path consisting of a closed circular loop that \(\phi = 2\pi\Phi/\Phi_0\) with \(\Phi\) the magnetic flux threading through the loop and \(\Phi_0 = hc/e\) equaling the magnetic flux quantum. In a uniform magnetic field of one Tesla applied perpendicular to the circular path, for what radius does the accumulated phase \(\phi = \pi\)?

E.) Due to the relative phase factor, the time reversed paths no longer constructively interfere when a magnetic field is applied. Based on this, do you think that a magnetic field when applied to such a “localized insulator” will tend to increase the electrical resistance (positive magnetoresistance) or decrease it (negative magnetoresistance)?
4. SCATTERING OF IDENTICAL PARTICLES

Consider two spinless particles with the same mass which interact via a potential \( v(|r_1 - r_2|) \). Suppose they collide with the center of mass at rest in the lab frame, incident along the z-axis, say. Given the scattering amplitude \( f(\theta) \) for the case where the particles are distinguishable, find \( f_{B/F}(\theta) \) in the case that the two particles are indistinguishable bosons (B) or fermions (F). Does the total cross section depend on the statistics? For what values of \( \theta \) does \( f_F(\theta) \) vanish? Explain why in terms of particle worldlines.

BONUS: HANBURY-BROWN TWISS EXPERIMENT

Consider the Hanbury-Brown Twiss experiment which measures the probability of simultaneous detection of two photons from a star in two detectors a distance \( d \) apart. This experiment can be used to measure the angular diameter of stars. Ignore photon polarization and assume the quantum amplitude at point \( r \) of a photon emitted from a point source at \( R_j \) is proportional to

\[
e^{i\theta} e^{ik|r-R_j|}/|r-R_j|
\]

with \( \theta \) a random phase, different for each photon emitted. Model the star as two such photon sources with \( R_1 \) on the left edge of the star and \( R_2 \) on the right edge. Assume that the vector connecting the two detectors is parallel to the vector connecting the right and left edges of the star.

Find out how the coincidence probability for two photons to arrive simultaneously one at each detector varies with the detector separation \( d \). First, recall that the quantum amplitude for the propagation of two distinguishable (and independent) particles is the product of the quantum amplitudes for each one separately. Next, consider three processes involving the propagation of two photons from an initial to a final state: (A) Two photons emitted from the source at \( R_1 \) arrive at the two detectors; (B) Two photons emitted from source at \( R_2 \) arrive at the two detectors; (C) One photon emitted from each source arrive at the two detectors. Even though the final states for each process is the same, the initial states are not. Being distinguishable, one must add the probabilities for each of the three processes. However, to obtain each probability requires first summing the quantum amplitudes for all two-photon “trajectories” connecting the initial and final state for that process, and then squaring. Make approximations appropriate to the case \( d << D << L \), where \( D = 10^9 \text{ m} \) is the diameter of the star and \( L = 10 \) light years is the distance to the star. Assuming the photon wavelength is \( 1 \mu \text{m} \), what would be a good separation for the detectors in order to measure this stars angular diameter (ie. \( D/L \))?