1.) QUANTUM ISING MODEL IN A TRANSVERSE FIELD

Consider the quantum Ising model in a transverse field defined on the sites of a hypercubic $d$-dimensional lattice as in class;

$$
\hat{H} = -J \sum_{\langle ij \rangle} \hat{\sigma}^z_i \hat{\sigma}^z_j - h \sum_i \hat{\sigma}^x_i.
$$

(a) Consider first the paramagnetic phase with $J = 0$ where the ground state consists of all spins pointing in the $\hat{x}$-direction. Let $|i\rangle$ denote a state with a single spin, at site $i$, flipped to point in the $-\hat{x}$-direction. With $J = 0$ this is an eigenstate with energy independent of the location of the flipped spin. Define a projection operator into this degenerate manifold of states;

$$
P = \sum_i |i\rangle \langle i|.
$$

Obtain an explicit expression for the full Hamiltonian when projected into this degenerate manifold, $\hat{H}' = P\hat{H}P$.

(b) In order to split the degeneracy of the spin-flipped states to leading order in $J \ll h$ (using first order degenerate perturbation theory), requires diagonalizing the perturbation in the degenerate manifold. Using your projected Hamiltonian show that the plane wave state,

$$
|k\rangle = \frac{1}{\sqrt{N}} \sum_i e^{ik \cdot x_i} |i\rangle,
$$

is in fact an eigenstate, $\hat{H}'|k\rangle = \epsilon_k |k\rangle$. Deduce the energy spectrum of the excited states, $\epsilon_k$.

(c) Next consider the Ferromagnetic state with $h = 0$ where a ground state consists of all spins aligned along the (plus, say) $z$-direction. Using perturbation theory compute the shift in the ground state energy to second order in the transverse field $h$.

(d) When $h = 0$ an exact degenerate manifold of excited eigenstates can be obtained by starting with the fully polarized Ferromagnetic state and flipping one spin at site $i$, which we will again denote as $|i\rangle$ (despite the “degenerate notation”, do not confuse this spin-flipped state in the FM with the spin-flipped states in the PM). To understand how this degeneracy will be split by small $h$, one can use second order degenerate perturbation theory. With the first order term vanishing, $\langle i|\hat{H}_h|j\rangle = 0$, the second order shift can be obtained by diagonalizing the effective Hamiltonian;

$$
H_{ij}^{eff} = \sum_n' \frac{\langle i|\hat{H}_h|n\rangle \langle n|\hat{H}_h|j\rangle}{\epsilon_0 - E_n},
$$

where $\epsilon_0$ is the energy of the degenerate manifold (relative to the ground state) and the primed summation is over a complete set of unperturbed eigenstates (with energy $E_n$) excluding the states in the degenerate manifold. By computing the matrix elements and performing the summation, obtain an explicit expression for $H_{ij}^{eff}$ in general dimension $d$. Verify that the non-vanishing matrix elements are actually infinite in $d = 1$.

(e) The full effective Hamiltonian projected into the degenerate manifold is,

$$
\hat{H} = \sum_i |i\rangle \langle i|\hat{H}_J|i\rangle + \sum_{ij} |i\rangle H_{ij}^{eff} |j\rangle.
$$
Demonstrate explicitly that a plane wave state (as in Eq. (3) for the paramagnetic case) is an exact eigenstate of \( \hat{H} \), and compute the corresponding eigenenergy as a function of momentum \( \mathbf{k} \). Noting the shift in the ground state energy that you computed in (c) above, extract finally the excitation energy, \( \epsilon_k \) of the spin-wave excitation in the Ferromagnetic state.

2) QUANTUM XY MODEL

Another quantum spin model which arises in various contexts is the so-called XY model, with Hamiltonian,

\[
\hat{H}_{\text{XY}} = -J \sum_{<ij>} (\hat{\sigma}_i^x \hat{\sigma}_j^x + \hat{\sigma}_i^y \hat{\sigma}_j^y) - \hbar \sum_i \hat{\sigma}_i^z,
\]

where as before \( \hat{\sigma}_i^\alpha \) is a vector of Pauli operators, one at each site of a hyper-cubic lattice, and the summation in the first term is over near-neighbor sites only.

(a) Consider the unitary operator,

\[
\hat{U} = \prod_i e^{i \phi \hat{\sigma}_i^z / 2} = e^{i \phi \hat{\sigma}_{\text{tot}}^z / 2},
\]

with \( \hat{\sigma}_{\text{tot}}^z = \sum_i \hat{\sigma}_i^z \) the \( z \)-component of the total spin, which rotates all of the spins by an angle \( \phi \) around the \( z \)-axis. Show that the Hamiltonian \( \hat{H}_{\text{XY}} \) commutes with \( \hat{U} \). This means that it is possible to simultaneously diagonalize the Hamiltonian and the \( z \)-component of the total spin.

(b) Consider the limit \( J = 0 \) where the exact ground state consists of all spins aligned along the \( z \)-axis. As in Problem (1) consider a degenerate manifold of excited states consisting of a single flipped spin. Using degenerate perturbation theory, compute the splitting to leading order in \( J \), and extract the energy-momentum dispersion relation of the “particle” excitation (in arbitrary dimension, \( d \)).

(c) It is convenient to define spin raising and lowering operators, \( \hat{\sigma}_i^\pm = (\hat{\sigma}_i^x \pm i \hat{\sigma}_i^y) / 2 \). Re-express the Hamiltonian \( \hat{H}_{\text{XY}} \) in terms of \( \hat{\sigma}_i^\pm \) and \( \hat{\sigma}_i^z \).

(d) For the remainder of this problem we will specialize to a one-dimensional lattice. It will be convenient to introduce a Jordan-Wigner transformation ( \( \text{https://en.wikipedia.org/wiki/JordanWigner\_transformation} \)), a mapping in one-dimension between spin-1/2 operators and Fermions;

\[
\hat{\sigma}_i^+ = \prod_{j<i} (1 - 2 \hat{c}_j^\dagger \hat{c}_j) \hat{c}_i; \quad \hat{\sigma}_i^- = \prod_{j<i} (1 - 2 \hat{c}_j^\dagger \hat{c}_j) \hat{c}_i^\dagger; \quad \hat{\sigma}_i^z = (1 - 2 \hat{c}_i^\dagger \hat{c}_i).
\]

Here, \( \hat{c}_i^\dagger, \hat{c}_i \) are Fermions, satisfying the canonical anti-commutation relations,

\[
\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{ij}; \quad \{\hat{c}_i, \hat{c}_j\} = \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0.
\]

Using the expressions in Eq. (8) relating the spins to Fermions show that the spin operators as defined satisfy canonical commutation relations; \( [\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta] = 2i \delta_{ij} \epsilon_{\alpha\beta\gamma} \hat{\sigma}_i^\gamma \).

(e) Using the Jordan-Wigner transformation re-write the Hamiltonian \( \hat{H}_{\text{XY}} \) in terms of the Fermion operators.

(f) It is convenient to introduce the Fourier (and inverse Fourier) transform of the Fermion operators as,

\[
\hat{c}_k = \frac{1}{\sqrt{N}} \sum_{i=1}^N \hat{c}_i e^{ikx_i}; \quad \hat{c}_i = \frac{1}{\sqrt{N}} \sum_k \hat{c}_k e^{-ikx_i},
\]

where \( x_i = i, 1, 2, ..., N \) runs over the \( N \) sites (assumed even) of the 1d lattice with periodic boundary conditions, and the momentum \( k = 2\pi n / N \) with integer \( n = -N/2 + 1, -N/2 + 2, ..., N/2 \). Here, \( \hat{c}_k^\dagger \) is the creation operator for a Fermion in a momentum eigenstate \( k \). Using the anticommutation relations in Eq. (9), show that,

\[
\{\hat{c}_k, \hat{c}_{k'}^\dagger\} = \delta_{kk'}; \quad \{\hat{c}_k, \hat{c}_k\} = \{\hat{c}_k^\dagger, \hat{c}_k^\dagger\} = 0.
\]

(g) Show that the Hamiltonian, when re-expressed in terms of \( \hat{c}_k, \hat{c}_k^\dagger \) can be put into a diagonal form;

\[
\hat{H}_{\text{XY}} = \sum_k E_k \hat{c}_k^\dagger \hat{c}_k.
\]
with \( |k| \leq \pi \). Obtain an expression for the “particle” dispersion \( E_k \). Check that in \( d = 1 \) and for \( h >> J \) this dispersion coincides with that obtained using perturbation theory in part (b).

The ground state of this free Fermion Hamiltonian corresponds to filling up all momentum states with negative energy, \( E_k < 0 \):

\[
|G\rangle = \prod_k \hat{c}_k^\dagger |\text{vac}\rangle,
\]

where the prime denotes the restricted product over negative energy states only, and \( |\text{vac}\rangle \) is a vacuum of Fermions that satisfies, \( \hat{c}_k |\text{vac}\rangle = 0 \).

(h) The magnetization of the spins along the \( z \)-axis is given by \( M_z = (1/N) \sum_i \langle G | \hat{\sigma}_z^i | G \rangle \). By re-expressing \( M_z \) in terms of Fermions using Jordan-Wigner, compute the magnetization as a function of \( h/J \). What is the critical value of the field, \( h_c \), above which the magnetization is fully saturated at plus one (or minus one if \( h \) is negative)?

(i) The magnetic susceptibility is defined as \( \chi = \partial M_z / \partial h \). Obtain an expression for the susceptibility as a function of \( h \), and show that the susceptibility diverges upon approaching the critical field from below as, \( \chi(h) \sim (h_c - h)^{-\gamma} \). What is the value of the critical exponent \( \gamma \)?