Due: Friday, December 6, 2019 by 5pm

Put homework in mailbox labelled 217A on 1st floor of Broida (by elevators).

1.) RENORMALIZATION GROUP FOR SCALAR FIELD THEORY

Consider the partition function expressed as a functional integral over a scalar field, $\phi(x)$, in $D$-space/time dimensions:

$$Z = \int [D\phi] e^{-[S_0 + S_{int}]} ,$$

with a Gaussian part of the action given by,

$$S_0 = \frac{1}{2} \int_0^\Lambda \frac{d^Dk}{(2\pi)^D} [r + k^2] |\phi(k)|^2 .$$

We first consider the Gaussian theory with interactions set to zero, $S_{int} = 0$.

a.) Upon splitting the field into slow and fast modes with rescaling parameter $b > 1$,

$$\phi_<(k) = \phi(k), \quad k < \Lambda/b; \quad \phi_>(k) = \phi(k), \quad \Lambda/b < k < \Lambda,$$

implement an RG transformation by first integrating over the fast modes and then rescaling the momenta and fields. Show that you can choose the field rescaling appropriately, so that the Gaussian action $S_0$ with $r = 0$ is a fixed point of the RG transformation, $S_0 \to S_0' = S_0$. Verify that the “mass” term $r\phi^2$ is a relevant perturbation at the Gaussian fixed point under your RG with $r' = b^2 r$ independent of the dimension $D$.

b.) Now consider interaction terms involving even powers of the field $\phi$,

$$S_{int}^n = \sum_{n=2}^\infty u_{2n} \int d^Dx \ [\phi(x)]^{2n} .$$

Within your RG scheme above, show that the zeroth order flows (0-loop) of the coupling constants take the form,

$$\frac{du_{2n}}{d\ell} = \lambda_{2n} u_{2n} ,$$

(with $b = 1 + d\ell$, that is $\ln b = d\ell$) and extract an expression for the RG eigenvalues $\lambda_{2n}$ at the Gaussian fixed point. [Hint: It might be easier to consider the rescaling of the field $\phi(x)$ in position - rather than momentum - space.] Verify that for $D > 3$ all of the couplings are irrelevant at the Gaussian fixed point except the term $u_4 \phi^4$.

c.) Next consider interaction terms which involve spatial gradients, such as,

$$S_{int}^b = \sum_{n=1}^\infty g_{2n} \int d^Dx \ (\nabla^{2n}\phi)^2 + \sum_{m=1}^\infty v_{2m} \int d^Dx \ (\nabla\phi)^2 \phi^{2m} .$$
Extract the RG eigenvalues of $g_{2m}, v_{2m}$ to lowest order (0-loop) at the Gaussian fixed point, and show that once again for $D > 3$ all terms are irrelevant.

d.) Henceforth we drop all of the irrelevant interaction terms except $u\phi^4$, which at 0-loop level is a relevant perturbation for $D = 4 - \epsilon$ dimensions. Working perturbatively to first order in $u$, draw the one-loop Feynmann diagram that renormalizes the “mass” terms $\delta r \sim O(u)$, paying attention to the sign and combinatoric factor. Upon expanding the integrand for small $r$ up to linear order, evaluate the momentum integration over the fast modes ($\Lambda/b < k < \Lambda$) in $D = 4$ dimensions and extract the dependence on the parameter $b$. Finally, rescale the momenta and field $\phi$ to complete the RG transformation, and extract the RG flow equation, $dr/d\ell$, up to linear order in $u$.

e.) Next, at order $u^2$, draw the one-loop diagram that renormalizes the interaction ($\delta u \sim O(u^2)$). Evaluate the integration over fast modes in $D = 4$ (and at $r = 0$) and obtain the RG flow equation for $du/d\ell$.

f.) Show that your RG flow equations for $r$ and $u$ have a non-trivial fixed point with $r^*, u^* = O(\epsilon)$. Linearizing around the fixed point, obtain the two RG eigenvalues and corresponding eigenvectors.

g.) Finally, using scaling, deduce the order $\epsilon$ correction to the correlation length exponent - ie. deduce the universal dimensionless number, $a_1$, where: $\nu^{-1} = 2 + a_1\epsilon + O(\epsilon^2)$.

Bonus: For the $O(N)$ model introduced in Problem Set 3, rederive the RG flow equations as above (in sections (d) and (e)), and obtain the $N$—dependence of the fixed point and critical exponent, $\nu^{-1}$. For $N \rightarrow \infty$, compare your first order in $\epsilon$ expression, $\nu^{-1} = 2 + a_N\epsilon$, to the exact expression for $\nu^{-1}$ that you obtained in Problem Set 3.

2.) SECOND QUANTIZATION

a.) Using the definition of the Boson field operator,

$$\hat{\psi}(r) = \frac{1}{\sqrt{L^d}} \sum_k \hat{a}_k e^{ik \cdot r}, \quad (7)$$

and the commutation relations, $[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta_{k, k'}$ show that $[\hat{\psi}(r), \hat{\psi}^\dagger(r')] = \delta(r - r').$

b.) The state $|k\rangle = \hat{a}_k^\dagger |0\rangle$ is a state with one boson at momentum $k$ and $|r\rangle = \hat{\psi}^\dagger(r)|0\rangle$ is a state with one boson at position $r$. Compute $\langle r|k\rangle$.

c.) Re-express the following operators in terms of the boson field operators $\hat{\psi}$ and $\hat{\psi}^\dagger$:

$$\hat{N} = \sum_k \hat{a}_k^\dagger \hat{a}_k \quad (8)$$

$$\hat{P} = \sum_k k \hat{a}_k^\dagger \hat{a}_k \quad (9)$$

$$\hat{K} = \sum_k \frac{k^2}{2m} \hat{a}_k^\dagger \hat{a}_k \quad (10)$$
\[ \hat{U} = \frac{u}{L^d} \sum_{k_1,k_2,k_3} \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \hat{a}_{k_3} \hat{a}_{k_1+k_2-k_3}. \] (11)

Describe briefly the physical meaning of each operator.