1.) QUANTUM ISING MODEL IN A TRANSVERSE FIELD

Consider the quantum Ising model in a transverse field defined on the sites of a hypercubic d-dimensional lattice as in class:

\[ \hat{H} = -J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - h \sum_i \hat{\sigma}_i^x. \]  

(a) Consider first the paramagnetic phase with \( J = 0 \) where the ground state consists of all spins pointing in the \( \hat{x} \)-direction. Let \( |i\rangle \) denote a state with a single spin, at site \( i \), flipped to point in the \( -\hat{x} \)-direction. With \( J = 0 \) this is an eigenstate with energy independent of the location of the flipped spin. Define a projection operator into this degenerate manifold of states:

\[ P = \sum_i |i\rangle \langle i|. \]  

Obtain an explicit expression for the full Hamiltonian when projected into this degenerate manifold, \( \hat{H}' = P \hat{H} P \).

(b) In order to split the degeneracy of the spin-flipped states to leading order in \( J \ll h \) (using first order degenerate perturbation theory), requires diagonalizing the perturbation in the degenerate manifold. Using your projected Hamiltonian show that the plane wave state, 

\[ |k\rangle = \frac{1}{\sqrt{N}} \sum_i e^{ik \cdot x_i} |i\rangle, \]  

is in fact an eigenstate, \( \hat{H}' |k\rangle = \epsilon_k |k\rangle \). Deduce the energy spectrum of the excited states, \( \epsilon_k \).

(c) Next consider the Ferromagnetic state with \( h = 0 \) where a ground state consists of all spins aligned along the (plus, say) \( z \)-direction. Using perturbation theory compute the shift in the ground state energy to second order in the transverse field \( h \).

(d) When \( h = 0 \) an exact degenerate manifold of excited eigenstates can be obtained by starting with the fully polarized Ferromagnetic state and flipping one spin at site \( i \), which we will again denote as \( |i\rangle \) (despite the “degenerate notation”, do not confuse this spin-flipped state in the FM with the spin-flipped states in the PM). To understand how this degeneracy will be split by small \( h \), one can use second order degenerate perturbation theory. With the first order term vanishing, \( \langle i|\hat{H}_h|j\rangle = 0 \), the second order shift can be obtained by diagonalizing the effective Hamiltonian:

\[ H_{ij}^{\text{eff}} = \sum_n \frac{\langle i|\hat{H}_h|n\rangle \langle n|\hat{H}_h|j\rangle}{\epsilon_0 - E_n}, \]  

where \( \epsilon_0 \) is the energy of the degenerate manifold (relative to the ground state) and the primed summation is over a complete set of unperturbed eigenstates (with energy \( E_n \)) excluding the states in the degenerate manifold. By computing the matrix elements and performing the summation, obtain an explicit expression for \( H_{ij}^{\text{eff}} \) in general dimension \( d \). Verify that the non-vanishing matrix elements are actually infinite in \( d = 1 \).

(e) The full effective Hamiltonian projected into the degenerate manifold is,

\[ \hat{H} = \sum_i |i\rangle \langle i| \hat{H}_J |i\rangle + \sum_{ij} |i\rangle H_{ij}^{\text{eff}} \langle j|. \]
Demonstrate explicitly that a plane wave state (as in Eq. (3) for the paramagnetic case) is an exact eigenstate of $\hat{H}$, and compute the corresponding eigenenergy as a function of momentum $k$. Noting the shift in the ground state energy that you computed in (c) above, extract finally the excitation energy, $\epsilon_k$ of the spin-wave excitation in the Ferromagnetic state.

2) QUANTUM XY MODEL

Another quantum spin model which arises in various contexts is the so-called XY model, with Hamiltonian,

$$
\hat{H}_{XY} = -J \sum_{<ij>} (\hat{\sigma}_i^x \hat{\sigma}_j^x + \hat{\sigma}_i^y \hat{\sigma}_j^y) - h \sum_i \hat{\sigma}_i^z,
$$

(6)

where as before $\hat{\sigma}_i^\alpha$ is a vector of Pauli operators, one at each site of a hyper-cubic lattice, and the summation in the first term is over near-neighbor sites only.

(a) Consider the unitary operator,

$$
\hat{U} = \prod_i e^{i\phi \hat{\sigma}_i^z/2} = e^{i\phi \hat{\sigma}_{\text{tot}}^z/2},
$$

(7)

with $\hat{\sigma}_i = \hat{\sigma}_i^z$ the $z-$component of the total spin, which rotates all of the spins by an angle $\phi$ around the $z-$axis. Show that the Hamiltonian $\hat{H}_{XY}$ commutes with $\hat{U}$. This means that it is possible to simultaneously diagonalize the Hamiltonian and the $z-$component of the total spin.

(b) Consider the limit $J = 0$ where the exact ground state consists of all spins aligned along the $z$-axis. As in Problem (1) consider a degenerate manifold of excited states consisting of a single flipped spin. Using degenerate perturbation theory, compute the splitting to leading order in $J$, and extract the energy-momentum dispersion relation of the “particle” excitation (in arbitrary dimension, $d$).

(c) It is convenient to define spin raising and lowering operators, $\hat{\sigma}_i^\pm = (\hat{\sigma}_i^x \pm i\hat{\sigma}_i^y)/2$. Re-express the Hamiltonian $\hat{H}_{XY}$ in terms of $\hat{\sigma}_i^\pm$ and $\hat{\sigma}_i^z$.

(d) For the remainder of this problem we will specialize to a one-dimensional lattice. It will be convenient to introduce a Jordan-Wigner transformation (https://en.wikipedia.org/wiki/JordanWigner_transformation), a mapping in one-dimension between spin-1/2 operators and Fermions;

$$
\hat{\sigma}_i^+ = \prod_{j<i} (1 - 2\hat{c}_j^\dagger \hat{c}_j); \quad \hat{\sigma}_i^- = \prod_{j<i} (1 - 2\hat{c}_j^\dagger \hat{c}_j) \hat{c}_i^\dagger; \quad \hat{\sigma}_i^z = (1 - 2\hat{c}_i^\dagger \hat{c}_i).
$$

(8)

Here, $\hat{c}_i^\dagger, \hat{c}_i$ are Fermions, satisfying the canonical anti-commutation relations,

$$
\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{ij}; \quad \{\hat{c}_i, \hat{c}_j\} = \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0.
$$

(9)

Using the expressions in Eq. (8) relating the spins to Fermions show that the spin operators as defined satisfy canonical commutation relations: $[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta] = 2i\delta_{ij}\epsilon_{\alpha\beta\gamma}\hat{\sigma}_i^\gamma$.

(e) Using the Jordan-Wigner transformation re-write the Hamiltonian $\hat{H}_{XY}$ in terms of the Fermion operators.

(f) It is convenient to introduce the Fourier (and inverse Fourier) transform of the Fermion operators as,

$$
\hat{c}_k = \frac{1}{\sqrt{N}} \sum_{i=1}^N \hat{c}_i e^{ikx_i}; \quad \hat{c}_i = \frac{1}{\sqrt{N}} \sum_k \hat{c}_k e^{-ikx_i},
$$

(10)

where $x_i = i = 1, 2, ..., N$ runs over the $N$ sites (assumed even) of the 1d lattice with periodic boundary conditions, and the momentum $k = 2\pi n/N$ with integer $n = -N/2 + 1, -N/2 + 2, ..., N/2$. Here, $\hat{c}_k^\dagger$ is the creation operator for a Fermion in a momentum eigenstate $k$. Using the anticommutation relations in Eq. (9), show that,

$$
\{\hat{c}_k, \hat{c}_{k'}^\dagger\} = \delta_{kk'}; \quad \{\hat{c}_k, \hat{c}_k\} = \{\hat{c}_k^\dagger, \hat{c}_k^\dagger\} = 0.
$$

(11)

(g) Show that the Hamiltonian, when re-expressed in terms of $\hat{c}_k, \hat{c}_k^\dagger$ can be put into a diagonal form;

$$
\hat{H}_{XY} = \sum_k E_k \hat{c}_k^\dagger \hat{c}_k,
$$

(12)
with $|k| \leq \pi$. Obtain an expression for the “particle” dispersion $E_k$. Check that in $d = 1$ and for $h >> J$ this dispersion coincides with that obtained using perturbation theory in part (b).

The ground state of this free Fermion Hamiltonian corresponds to filling up all momentum states with negative energy, $E_k < 0$:

$$|G\rangle = \prod_k c_k^\dagger |\text{vac}\rangle,$$

where the prime denotes the restricted product over negative energy states only, and $|\text{vac}\rangle$ is a vacuum of Fermions that satisfies, $\hat{c}_k|\text{vac}\rangle = 0$.

\[ (h) \] The magnetization of the spins along the $z$–axis is given by $M_z = (1/N) \sum_i \langle G|\hat{\sigma}_z^2|G\rangle$. By re-expressing $M_z$ in terms of Fermions using Jordan-Wigner, compute the magnetization as a function of $h/J$. What is the critical value of the field, $h_c$, above which the magnetization is fully saturated at plus one (or minus one if $h$ is negative)?

\[ (i) \] The magnetic susceptibility is defined as $\chi = \partial M_z/\partial h$. Obtain an expression for the susceptibility as a function of $h$, and show that the susceptibility diverges upon approaching the critical field from below as, $\chi(h) \sim (h_c - h)^{-\gamma}$. What is the value of the critical exponent $\gamma$?

3) BONUS: MAJORANA ZERO MODES IN 1D SUPERCONDUCTING WIRES

Decoherence is a major impediment in building a scalable quantum computer. Quantum computing requires two seemingly contradictory requirements. Firstly, one needs to engineer a quantum system with q-bits (quantum degrees of freedom) which can be readily and reliably manipulated (i.e. controlled) by the (classical) experimenter. But “external” degrees of freedom which can be used for controlling the quantum state in a quantum computer will invariably entangle with the state, causing decoherence of the quantum state in the computer. With too much decoherence a quantum computer will cease to operate, and will not be able to out perform a classical computer.

“Topological quantum computing” provides one possible way to avoid such decoherence. In certain two-dimensional systems of interacting quantum particles (say electrons), the quantum particles “condense” into a topologically ordered state. These topologically ordered states can support gapped “quasi-particle” excitations - excitations which behave like particles but can have fractional anyon statistics (neither bosonic nor fermionic) or even “non-Abelian” statistics. Quantum computation can then in principle be performed by braiding such quasiparticles around one another. Since the quantum statistics of these quasiparticles is encoded “non-locally” in terms of the original electrons, their braiding properties are protected from (local) disturbances which would otherwise lead to decoherence.

During the past 3-4 years it has been realized that one should be able to engineer one-dimensional “wires” which can support non-Abelian quasiparticles at the two ends of the wire - or at the boundary between two segments of a wire which are in a different state. Building wire networks out of such wires would enable one to move and braid these non-Abelian quasiparticles and implement topological quantum computing.

A semiconducting quantum wire with strong spin-orbit scattering which is resting on top of a superconductor, when placed in an external magnetic field, can enter a topological state which supports Majorana-zero mode at the two ends of the wire. These Majorana zero-modes behave as non-Abelian quasiparticles when braided. Here we consider a toy model for such a 1d wire system which will enable us to demonstrate that there are Majorana zero-modes bound to each end. A journal article that discusses the physics of such wires can be found at: http://www.kitp.ucsb.edu/sites/default/files/users/mpaf/p152.pdf

The appropriate Hamiltonian for such a wire takes the form,

$$\hat{H} = - \sum_{j=1}^{N-1} \left( t\hat{c}_j^\dagger \hat{c}_{j+1} + t\hat{c}_{j+1}^\dagger \hat{c}_j + \Delta \hat{c}_j \hat{c}_{j+1} + \Delta \hat{c}_{j+1}^\dagger \hat{c}_j^\dagger \right) - \mu \sum_{j=1}^{N} \hat{c}_j^\dagger \hat{c}_j,$$

where the electron creation/annihilation operators at site $j$, denoted $\hat{c}_j^\dagger$, $\hat{c}_j$ satisfy canonical Fermion anti-commutation relations,

$$\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{ij}; \quad \{\hat{c}_i, \hat{c}_j\} = 0 \quad \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0.$$

Notice that these electron operators do not have a spin-label (due to the spin-orbit interaction and the external magnetic field the electron spin is not conserved, and the spin can be ignored). It is important to emphasize that this finite length 1d wire has open ends (not periodic boundary conditions).
The first 2 terms in the Hamiltonian describe the hopping motion of electrons along the wire, the third and fourth terms represent the coupling of the wire to a (bulk) superconductor (Δ represents the amplitude for tunneling Cooper pairs between the superconductor and the wire) and the last term controls the number of electrons on the wire (μ is the chemical potential). We will now explore 2 special cases for the parameters, t, Δ, μ, for which the Hamiltonian can be easily solved.

First consider the case with \( t = \Delta = 0 \) and \( \mu < 0 \),

\[
\hat{H}_\mu \equiv \hat{H}(t = \Delta = 0, \mu < 0) = +|\mu| \sum_{j=1}^{N} \hat{c}_j \hat{c}_j^\dagger.
\]  

(16)

Let \(|\text{vac}\rangle_c\) denote a state with no electrons present, that satisfies, \(\hat{c}_j |\text{vac}\rangle_c = 0\) for all \(j\). Consider a set of states of the form,

\[
|n_1, n_2, ..., n_N\rangle_c \equiv (\hat{c}_1^\dagger)^{n_1} (\hat{c}_2^\dagger)^{n_2} ... (\hat{c}_N^\dagger)^{n_N} |\text{vac}\rangle_c = \prod_{j=1}^{N} (\hat{c}_j^\dagger)^{n_j} |\text{vac}\rangle_c,
\]  

(17)

where \(n_j = 0, 1\) denotes the number of electrons present on each site \(j\).

(a) Using the anti-commutation relations of the \(\hat{c}_j, \hat{c}_j^\dagger\) show that these states form an orthonormal set,

\[
c\langle n_1, n_2, ..., n_N|n'_1, n'_2, ..., n'_N\rangle_c = \prod_{j=1}^{N} \delta_{n_j, n'_j}.
\]  

(18)

There are \(2^N\) such states, so that this orthonormal set spans the full Hilbert space.

(b) Moreover, show that the states \(|n_1, n_2, ..., n_N\rangle_c\) are eigenstates of \(\hat{H}_\mu\), and obtain an expression for the eigenvalues, \(E(\{n_j\})\).

(c) Show that the ground state of \(\hat{H}_\mu\) is non-degenerate. What is the energy gap between the ground state and the lowest energy excited state(s)? What is the degeneracy of this lowest energy excited state multiplet? Since the ground state is unique, there are no zero-energy quasiparticles living at the ends of the wires (cf to discussion below).

Next consider the special case with \( t = \Delta > 0, \mu = 0 \),

\[
\hat{H}_t \equiv \hat{H}(t = \Delta, \mu = 0) = -t \sum_{j=1}^{N-1} (\hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + \hat{c}_j \hat{c}_{j+1} + \hat{c}_j^\dagger \hat{c}_{j+1} + \hat{c}_{j+1}^\dagger \hat{c}_{j+1}).
\]  

(19)

In order to solve this Hamiltonian it is useful to define two sets of Majorana (real) Fermion operators,

\[
\hat{\gamma}_{A,j} = i(\hat{c}_j - \hat{c}_j^\dagger); \quad \hat{\gamma}_{B,j} = (\hat{c}_j + \hat{c}_j^\dagger).
\]  

(20)

Notice that these Majorana operators are self-adjoint (real), satisfying \(\hat{\gamma}_{A,j}^\dagger = \hat{\gamma}_{A,j}\) and \(\hat{\gamma}_{B,j}^\dagger = \hat{\gamma}_{B,j}\).

(d) Show that these Majorana operators satisfy the following anti-commutation relations,

\[
\{\hat{\gamma}_{A,j}, \hat{\gamma}_{A,j'}\} = \{\hat{\gamma}_{B,j}, \hat{\gamma}_{B,j'}\} = 2\delta_{jj'}; \quad \{\hat{\gamma}_{A,j}, \hat{\gamma}_{B,j'}\} = 0.
\]  

(21)

(e) Re-express the Hamiltonian \(\hat{H}_t\) in terms of the Majorana operators, \(\hat{\gamma}_{A,j}\) and \(\hat{\gamma}_{B,j}\).

It is convenient to define a new set of (complex) Fermion operators, \(\hat{d}_{j}, \hat{\bar{d}}_j\), via,

\[
\hat{d}_j = (\hat{\gamma}_{A,j+1} + i\hat{\gamma}_{B,j})/2
\]  

(22)

for \(j = 1, 2, ..., N - 1\) and

\[
\hat{d}_{\text{end}} = (\hat{\gamma}_{A,1} + i\hat{\gamma}_{B,N})/2.
\]  

(23)

Notice that the complex Fermion operator \(\hat{d}_{\text{end}}\) is very “non-local”, being defined as a linear combination of two real operators on opposite ends of the wire.
(f) Verify that these new complex $d$-Fermion operators satisfy the same canonical anticommutation relations as the complex $c$-Fermions, ie
\[
\{ \hat{d}_i, \hat{d}^\dagger_j \} = \delta_{ij}; \quad \{ \hat{d}_{\text{end}}, \hat{d}^\dagger_{\text{end}} \} = 1; \quad \{ \hat{d}_{\text{end}}, \hat{d}^\dagger_j \} = 0; \quad \text{etc.} \quad (24)
\]

(g) Show that the Hamiltonian $\hat{H}_t$, when re-expressed in terms of the $d$--Fermion operators, takes a simple “site-local” form,
\[
\hat{H}_t = t \sum_{j=1}^{N-1} \hat{d}^\dagger_j \hat{d}_j,
\]
and is independent of the operator $\hat{d}_{\text{end}}$.

(h) Next consider an orthonormal complete set of states built from the $d$--operators,
\[
|0; n_1, n_2, ..., n_{N-1}\rangle_d = \prod_{j=1}^{N-1} (d^\dagger_j)^{n_j} |\text{vac}\rangle_d; \quad |1; n_1, n_2, ..., n_{N-1}\rangle_d = \hat{d}^\dagger_{\text{end}} |0; n_1, n_2, ..., n_{N-1}\rangle_d,
\]
where $n_j = 0, 1$ for each $j = 1, 2, ..., N - 1$. The state $|\text{vac}\rangle_d$, which corresponds to a vacuum of $d$--Fermions (no $d$--Fermions present), is annihilated by the destruction operators $\hat{d}_j |\text{vac}\rangle_d = \hat{d}_{\text{end}} |\text{vac}\rangle_d = 0$. Show that these states are eigenstates of $\hat{H}_t$, and find the corresponding eigenvalues, $E(0; \{n_j\})$ and $E(1; \{n_j\})$.

(i) For $t > 0$, show that the ground state of $\hat{H}_t$ is 2-fold degenerate, and that the 2-states, denoted $|0_g\rangle, |1_g\rangle$ satisfy,
\[
|1_g\rangle = \hat{d}_{\text{end}}^\dagger |0_g\rangle; \quad \hat{d}_{\text{end}} |0_g\rangle = 0.
\]
What is the energy gap between these 2-degenerate ground states and the lowest energy excited states?

Remarkably, the complex Fermion operator that toggles between the 2-ground states, $\hat{d}_{\text{end}}$, is built from two Majorana operators that are on opposite ends of the wire, since $\hat{d}_{\text{end}} = (\hat{\gamma}_{A,1} + i\hat{\gamma}_{B,N})/2$. [The operators $\hat{\gamma}_{A,1}$ and $\hat{\gamma}_{B,N}$ are sometimes referred to as “Majorana zero modes”.] The robustness of the 2-fold degenerate ground state manifold against decoherence can be understood by considering the set of “local” operators, denoted $\hat{O}_j$, which are operators that can be built from electron operators $\hat{c}_{j'}$ with $j' \approx j$. One can show that any local operator, when projected into the 2-fold degenerate ground state manifold, is proportional to the identity,
\[
\langle g_0 ^{|\text{O}_j |0_g\rangle = \langle 1_g ^{|\text{O}_j |1_g\rangle; \quad \langle 0_g ^{|\text{O}_j |1_g\rangle = \langle 1_g ^{|\text{O}_j |0_g\rangle = 0. \quad (28)
\]

This implies that the two ground states cannot be distinguished by making local measurements and that local measurements cannot cause transitions between the two states. The 2-fold degenerate ground state manifold is robust against local decoherence.