Problem Set 5 223B Condensed Matter Physics, Winter 2017

Due: Tuesday, March 21, 2017 by 11pm
Put homework in mailbox labelled 223B on 1st floor of Broida (by elevators).

1.) THE COOPER PROBLEM

Ashcroft and Mermin Chapter 34 problem 4.

2.) BCS THEORY

A.) For the BCS superconducting ground state, obtain an expression for the mean number of electrons with momentum $\mathbf{k}$,

$$\langle n_{\mathbf{k}\uparrow} \rangle = \langle c^{\dagger}_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} \rangle, \tag{1}$$

as a function of the electron dispersion $\epsilon_{\mathbf{k}} = (\hbar^2 \mathbf{k}^2/2m) - \mu$ and the energy gap $\Delta_{\mathbf{k}}$. Near the Fermi surface one can linearize the dispersion, $\epsilon_{\mathbf{k}} = v_F(|\mathbf{k}| - k_F)$. Sketch your result for the momentum distribution function versus $|\mathbf{k}|$ (assuming a constant Δ for simplicity), and constrast the behavior with the momentum distribution function of a (non-interacting) metal.

B.) Show that the ground state energy of the superconducting state can be expressed as:

$$E_s = \sum_{\mathbf{k},\alpha} \epsilon_{\mathbf{k}} \langle n_{\mathbf{k}\alpha} \rangle - \sum_{\mathbf{k}} \frac{\Delta^2_{\mathbf{k}}}{2E_{\mathbf{k}}}, \tag{2}$$

where $E^2_{\mathbf{k}} = \epsilon^2_{\mathbf{k}} + \Delta^2_{\mathbf{k}}$. (Be careful about retaining all additive contributions to the energy when implementing the BCS factorization.) Interpret your result and explain what trade-offs in energy are required to stabalize the superconducting state.

C.) Assuming that $\Delta_{\mathbf{k}}$ is a constant for $|\epsilon_{\mathbf{k}}| < \omega_D$, and zero otherwise (where $\omega_D << E_F$ is a low frequency scale such as the deBye energy), show that the energy of the BCS ground state lies below that of the normal state (ie. the kinetic energy of the non-interacting Fermi sea, E_n)) by

$$E_s - E_n = (-\frac{1}{2} N(E_F)\Delta^2)(volume), \tag{3}$$

where $N(E_F)$ is the normal metal density of states at the Fermi-energy. For a critical field of $H_c = 100G$, estimate Δ in degrees Kelvin.

D.) Consider a process of electron tunnelling into a superconductor (eg from a metal through an insulating barrier). The density of available states in the superconductor at energy E (measured with respect to the Fermi energy) is given by

$$N_s(E) = \frac{1}{V} \sum_{\mathbf{k}} (1 - \langle n_{\mathbf{k}\uparrow} \rangle) \delta(E - E_{\mathbf{k}}), \tag{4}$$

with $E_{\mathbf{k}} = \sqrt{\epsilon^2_{\mathbf{k}} + \Delta^2}$ the quasiparticle excitation energy. The δ function insures conservation of energy. Interpret the term in brackets involving $\langle n_{\mathbf{k}\uparrow} \rangle$. Using your expression for $\langle n_{\mathbf{k}\uparrow} \rangle$ from part (A), show that the superconducting tunnelling density of states can be expressed as:

$$N_s(E) = \frac{N(E_F)E}{\sqrt{E^2 - \Delta^2}}, \tag{5}$$

for $E > \Delta$, and zero for $E < \Delta$. What has happened to the electronic states below the gap, $0 < E < \Delta$?