Problem Set 5 215B- Quantum Mechanics - Winter 2013

Due: Friday, March 15, 2013 by 5pm
Put homework in mailbox labelled 215B on 1st floor of Broida (by elevators).

1.) S-MATRIX SYMMETRY FOR 1D SCATTERING

Consider a particle of mass $m$ moving in one-dimension, and incident on a potential barrier, $V(x)$, (non-zero near the origin, $x = 0$), at energy $E = \hbar^2 k^2 / 2m$. For this 1d problem the S-matrix is a 2 by 2 matrix, as defined in class.

A.) Show that conservation of probability implies that the S-matrix is unitary. How many (real) parameters are needed to parameterize the most general unitary (2 by 2) S-matrix?

B.) Use unitarity of the S-matrix to show that the transmission probability for a particle incident at energy $E$ from the left (large negative x) is the same as for a particle incident from the right (large positive x).

C.) Show that time reversal invariance implies that the S-matrix is symmetric. How many parameters are needed to describe the most general unitary, symmetric (2 by 2) S-matrix?

D.) Assume that the potential is also parity invariant: $V(x) = V(-x)$. Derive any additional constraints on the form of the S-matrix. What is the most general form of the S-matrix in this case, and how many parameters are needed to parameterize it?

2.) CONDUCTANCE OF A ONE-CHANNEL QUANTUM WIRE

Consider a single-channel quantum wire, modelled in terms of a 1d system of (non-interacting) electrons, with states filled up to the Fermi energy, $E_F = \hbar^2 k^2_F / 2m$. Impurity scattering in the wire is described by a potential $V(x)$.

A.) For a single delta-function scattering potential,

$$ V(x) = V_0 \delta(x), \quad (1) $$

calculate the S-matrix as a function of the incident energy, $E$ (or wave vector $k$). Verify that your result is consistent with the general form obtained in Problem 1D.

B.) Employing the Landauer formula, and your results from part (A), compute the conductance, $G$, of the quantum wire for a small bias voltage (linear response).

C.) Consider a double barrier structure in the wire, modelled via two delta-functions:

$$ V(x) = V_0 \delta(x + a) + V_0 \delta(x - a). \quad (2) $$

Compute the S-matrix as a function of energy, $E$. Here, you can perhaps use the general form from problem 1D to reduce the labor.

D.) Obtain the conductance, $G$, for this double barrier structure, again using the Landauer formula, and plot it (or sketch it) as a function of the Fermi energy $E_F$. Discuss any structure that appears in $G(E_F)$. Is the resistance of the double barrier structure ($R = 1/G$) twice as large as the single barrier resistance (that is, do the resistances “add in series”)? Why?

3.) INELASTIC SCATTERING FROM A HARMONIC OSCILLATOR

Consider a particle with mass $m$ confined in a 3d Harmonic potential, with Hamiltonian

$$ H_{osc} = \frac{1}{2m} \vec{p}^2 + \frac{1}{2} m\omega_0^2 \vec{X}^2. \quad (3) $$
A particle with mass $M$ (and position operator $\vec{R}$) is incident with wave vector $\vec{k}$, and interacts with the harmonically confined particle via a potential $V(\vec{X} - \vec{R})$ with

$$V(\vec{r}) = V_0 \exp(-\vec{r}^2 / 2a^2). \tag{4}$$

A.) Assuming the harmonically confined particle is initially in its ground state, use the Born approximation to compute the differential cross section, $d^2\sigma/d\Omega d\epsilon$, for the mass $M$ particle to scatter off the confined (mass $m$) particle, as a function of the energy and momentum transfer.

[Two useful identities: (i) For any two operators $A$ and $B$ whose commutator $[A, B]$ is a c-number,

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \tag{5}.$$]

(ii) For any operator $A$ linear in the position and momentum operators of a 1d oscillator,

$$\langle 0 | e^A | 0 \rangle = e^{\langle 0 | A^2 | 0 \rangle} \tag{6},$$

where $|0\rangle$ is the harmonic oscillator ground state.]

B.) After the scattering process, what’s the highest excited state the oscillator can be in?