1.) QUANTUM ISING MODEL IN A TRANSVERSE FIELD

Consider the quantum Ising model in a transverse field defined on the sites of a hypercubic d-dimensional lattice as in class:

\[ \hat{H} = -J \sum_{\langle ij \rangle} \hat{\sigma}^z_i \hat{\sigma}^z_j - h \sum_i \hat{\sigma}^x_i. \]  

(a) Consider first the paramagnetic phase with $J = 0$ where the ground state consists of all spins pointing in the $\hat{x}$-direction. Let $|i\rangle$ denote a state with a single spin, at site $i$, flipped to point in the $-\hat{x}$-direction. With $J = 0$ this is an eigenstate with energy independent of the location of the flipped spin. Define a projection operator into this degenerate manifold of states:

\[ P = \sum_i |i\rangle \langle i|. \]  

Obtain an explicit expression for the full Hamiltonian when projected into this degenerate manifold, $\hat{H}' = P \hat{H} P$.

(b) In order to split the degeneracy of the spin-flipped states to leading order in $J << h$ (using first order degenerate perturbation theory), requires diagonalizing the perturbation in the degenerate manifold. Using your projected Hamiltonian show that the plane wave state,

\[ |k\rangle = \frac{1}{\sqrt{N}} \sum_i e^{i k \cdot x_i} |i\rangle, \]

is in fact an eigenstate, $\hat{H}'|k\rangle = \epsilon_k |k\rangle$. Deduce the energy spectrum of the excited states, $\epsilon_k$.

(c) Next consider the Ferromagnetic state with $h = 0$ where a ground state consists of all spins aligned along the (plus, say) z-direction. Using perturbation theory compute the shift in the ground state energy to second order in the transverse field $h$.

(d) When $h = 0$ an exact degenerate manifold of excited eigenstates can be obtained by starting with the fully polarized Ferromagnetic state and flipping one spin at site $i$, which we will again denote as $|i\rangle$ (despite the “degenerate notation”, do not confuse this spin-flipped state in the FM with the spin-flipped states in the PM). To understand how this degeneracy will be split by small $h$, one can use second order degenerate perturbation theory. With the first order term vanishing, $\langle i|\hat{H}_h|j\rangle = 0$, the second order shift can be obtained by diagonalizing the effective Hamiltonian:

\[ H_{ij}^{eff} = \sum_n \frac{\langle i|\hat{H}_h|n\rangle \langle n|\hat{H}_h|j\rangle}{\epsilon_0 - E_n}, \]

where $\epsilon_0$ is the energy of the degenerate manifold (relative to the ground state) and the primed summation is over a complete set of unperturbed eigenstates (with energy $E_n$) excluding the states in the degenerate manifold. By computing the matrix elements and performing the summation, obtain an explicit expression for $H_{ij}^{eff}$ in general dimension $d$. Verify that the non-vanishing matrix elements are actually infinite in $d = 1$.

(e) The full effective Hamiltonian projected into the degenerate manifold is,

\[ \hat{H} = \sum_i |i\rangle \langle i|\hat{H}_d|i\rangle + \sum_{ij} |i\rangle H_{ij}^{eff} \langle j|. \]
Demonstrate explicitly that a plane wave state (as in Eq. (3) for the paramagnetic case) is an exact eigenstate of $\hat{H}$, and compute the corresponding eigenenergy as a function of momentum $k$. Noting the shift in the ground state energy that you computed in (c) above, extract finally the excitation energy, $\epsilon_k$ of the spin-wave excitation in the Ferromagnetic state.

2) QUANTUM XY MODEL

Another quantum spin model which arises in various contexts is the so-called $XY$ model, with Hamiltonian,

$$\hat{H}_{XY} = -J \sum_{\langle ij \rangle} (\hat{\sigma}_i^x \hat{\sigma}_j^x + \hat{\sigma}_i^y \hat{\sigma}_j^y) - \hbar \sum_i \hat{\sigma}_i^z,$$

where as before $\hat{\sigma}_i^\alpha$ is a vector of Pauli operators, one at each site of a hyper-cubic lattice, and the summation in the first term is over near-neighbor sites only.

(a) Consider the unitary operator,

$$\hat{U} = \prod_i e^{i \phi \hat{\sigma}_i^z/2} = e^{i \phi \hat{\sigma}_{\text{tot}}^z/2},$$

with $\hat{\sigma}_{\text{tot}}^z = \sum_i \hat{\sigma}_i^z$ the $z-$component of the total spin, which rotates all of the spins by an angle $\phi$ around the $z-$axis. Show that the Hamiltonian $\hat{H}_{XY}$ commutes with $\hat{U}$. This means that it is possible to simultaneously diagonalize the Hamiltonian and the $z-$component of the total spin.

(b) Consider the limit $J = 0$ where the exact ground state consists of all spins aligned along the $z$-axis. As in Problem (1) consider a degenerate manifold of excited states consisting of a single flipped spin. Using degenerate perturbation theory, compute the splitting to leading order in $J$, and extract the energy-momentum dispersion relation of the “particle” excitation (in arbitrary dimension, $d$).