Problem 1. QUANTUM ISING MODEL IN A TRANSVERSE FIELD

We consider the quantum Ising model on a hypercubic $D$-dimensional lattice in a transverse field with Hamiltonian

$$\hat{H} = -J \sum_{\langle ij \rangle} \hat{\sigma}^z_i \hat{\sigma}^z_j - h \sum_i \hat{\sigma}^x_i. \quad (1)$$

(a)

Consider the paramagnetic phase where $J = 0$. Let $|i\rangle = |+1, +2 \ldots -i, \ldots +N\rangle$ in the $x$-basis and let $P = \sum_i |i\rangle \langle i|$. We typically work in the $z$-basis, but of course nothing changes if we cyclically relabel axes $x \rightarrow y \rightarrow z \rightarrow x$. Therefore, $\hat{\sigma}^x |+\rangle = i |\rangle$ and $\hat{\sigma}^z |\rangle = -i |+, \rangle$ (this is what $\hat{\sigma}^y$ would do in the $z$-basis). We have, for $n \neq m$,

$$\begin{align*}
\hat{\sigma}^x_n |j\rangle &= (1 - 2\delta_{n,j}) |j\rangle \\
\hat{\sigma}^z_m \hat{\sigma}^z_n |j\rangle &= i \hat{\sigma}^z_m \left[ (1 - \delta_{n,j}) |n, j\rangle - \delta_{n,j} |0\rangle \right] \\
&= - \left[ (1 - \delta_{m,j})(1 - \delta_{n,j}) |m, n, j\rangle - \delta_{m,j}(1 - \delta_{n,j}) |n\rangle - \delta_{n,j} |m\rangle \right]
\end{align*} \quad (2)$$

Then we can write the full Hamiltonian as

$$\hat{H}' = P \hat{H} P$$

$$\begin{align*}
&= \sum_m |m\rangle \langle m| \left( -J \sum_{\langle ij \rangle} \hat{\sigma}^z_i \hat{\sigma}^z_j - h \sum_i \hat{\sigma}^x_i \right) \sum_n |n\rangle \langle n| \\
&= J \sum_{\langle ij, mn \rangle} |m\rangle \langle m| \left[ -(\delta_{j, n} - \delta_{i, n}) |i\rangle \langle j| - h \sum_{i, n} (1 - 2\delta_{i, n}) |n\rangle \langle n| \right] \sum_n |n\rangle \langle n| \\
&= -J \sum_{\langle ij \rangle} \left( |i\rangle \langle j| + |j\rangle \langle i| \right) - h(N - 2) P
\end{align*} \quad (3)$$

In words, the $h$ term in $\hat{H}$ simply gives $-h$ for every ‘$+$’ spin and $+h$ for every ‘$-$’ spin, yielding the second term in $\hat{H}'$ above. The $J$ term in $\hat{H}$ flips pairs of neighboring spins; this either takes us out of the relevant manifold of states, or hops the single down-spin to one of its neighbors, yielding the first term in $\hat{H}'$ above.

(b)

Let

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ik \cdot x_j} |j\rangle \quad (4)$$

Let $e_d$ be a basis of unit lattice displacement vectors and, by a horrendous abuse of notation, let $|i + e_d\rangle$ be the basis state with spin-up at $x_i + e_d$. Then

$$\begin{align*}
\hat{H}' |k\rangle &= -J \sum_{i=1}^D \left( |i\rangle \langle i + e_d| + |i\rangle \langle i - e_d| \right) |k\rangle - h(N - 2) P |k\rangle \\
&= -J \sum_{i=1}^D \frac{1}{\sqrt{N}} \left( e^{ik \cdot (x_i + e_d)} + e^{ik \cdot (x_i - e_d)} \right) |i\rangle - h(N - 2) |k\rangle \\
&= -J \sum_{d=1}^D \left( e^{ik \cdot e_d} + e^{-ik \cdot e_d} \right) |k\rangle - h(N - 2) |k\rangle
\end{align*} \quad (5)$$

We thus find the energy spectrum is given by

$$e_k = -h(N - 2) - 2J \sum_{d=1}^D \cos(k \cdot e_d) \quad (6)$$

where $\{e_d\}$ are a basis of unit lattice displacement vectors.
Now consider the ferromagnetic phase with \( h = 0 \) and a ground state \( |0_+\rangle \) with all spins aligned in the \(+z\) direction. The ground state is actually degenerate, with another state \( |0_-\rangle \) with all spins aligned in the \(-z\) direction. We label the first excited states with only the \( i \)-th spin flipped to \(-z\) by \( |i\rangle \). The energies of the first two degenerate energy levels are \( E_{0,\pm} = -JND \) and \( E_{1,i} = -J(N - 4)D \).

The perturbing Hamiltonian is 

\[
\mathcal{H}_h = -h \sum_i \hat{\sigma}_i^z
\]  

(7)

Note first that \( \mathcal{H} \) does not have matrix elements that mix degenerate ground states, so we can proceed as in non-degenerate perturbation theory.

Next note that, working in the \( z \)-basis,

\[
\hat{\sigma}_i^z = \hat{\sigma}_i^+ + \hat{\sigma}_i^-
\]  

(8)

so that the perturbing Hamiltonian only mixes the ground state with the first excited states.

Thus we have, at first order

\[
E_{0,\pm}^{(1)} = \langle 0_+ | \mathcal{H}_h | 0_+ \rangle = -h \sum_i \langle 0_+ | (\hat{\sigma}_i^+ + \hat{\sigma}_i^-) | 0_+ \rangle = 0
\]  

(9)

At second order we have

\[
E_{0,\pm}^{(2)} = \sum_i \left| \langle i | \mathcal{H}_h | 0_+ \rangle \right|^2 = \sum_i \frac{h^2}{-4JD} = -\frac{Nh^2}{4JD}
\]  

(10)

The effective Hamiltonian on the degenerate first excited states is given by

\[
\mathcal{H}_{ij}^{eff} = \sum_n \frac{\langle i | \mathcal{H}_h | n \rangle \langle n | \mathcal{H}_h | j \rangle}{\epsilon_1 - \epsilon_n}
\]  

(11)

where \( \epsilon_n \) is the energy above the ground state at 0th order and \( |n\rangle \) are unperturbed energy eigenstates of different energy to the first excited states.

Again, since the perturbing Hamiltonian only flips one spin at a time, we need only consider matrix elements with the vacuum and the second excited states. The excited states have two spins flipped and we denote them by \( |i, j\rangle \). These have energy \( \epsilon_2 = 8JD - 4J \) when the flipped spins are nearest neighbours and energy \( \epsilon_3 = 8JD \) otherwise.

We will consider the matrix elements in cases.

Let \( i \neq j \) not be nearest neighbours. Then

\[
\begin{align*}
\mathcal{H}_{ij}^{eff} &= \frac{\langle i | \mathcal{H}_h | 0_+ \rangle \langle 0_+ | \mathcal{H}_h | j \rangle}{4JD} + \sum_{k,l \ not \ n.n.} \frac{\langle i | \mathcal{H}_h | k, l \rangle \langle k, l | \mathcal{H}_h | j \rangle}{-4JD} \\
&= \frac{h^2}{4DJ} \left( \frac{1}{D} + \frac{1}{-4DJ} \right) \\
&= 0
\end{align*}
\]  

(12)

Let \( i \neq j \) be nearest neighbours. Then

\[
\begin{align*}
\mathcal{H}_{ij}^{eff} &= \frac{\langle i | \mathcal{H}_h | 0_+ \rangle \langle 0_+ | \mathcal{H}_h | j \rangle}{4JD} + \sum_{\langle k,l \rangle} \frac{\langle i | \mathcal{H}_h | k, l \rangle \langle k, l | \mathcal{H}_h | j \rangle}{-4JD + 4J} \\
&= \frac{h^2}{4DJ} \left( \frac{1}{D} + \frac{1}{-4DJ} \right) \\
&= -\frac{h^2}{4JD} \left( \frac{1}{D - 1} \right)
\end{align*}
\]  

(13)
Let \( i = j \). Then
\[
H^\text{eff}_{ii} = \frac{\langle i \vert \mathcal{H}_h \vert 0_+ \rangle \langle 0_+ \vert \mathcal{H}_h \vert i \rangle}{4JD} + \sum_{\langle k,i \rangle} \frac{\langle i \vert \mathcal{H}_h \vert k, i \rangle \langle k, i \vert \mathcal{H}_h \vert i \rangle}{-4JD + 4J} \sum_{k \text{ not n.n. to } i} \frac{\langle i \vert \mathcal{H}_h \vert k, i \rangle \langle k, i \vert \mathcal{H}_h \vert i \rangle}{-4JD} \]
\[
= \frac{\hbar^2}{4J} \left( \frac{1}{D} + \frac{2D}{1-D} - \frac{N - 2D - 1}{D} \right) \]
\[
= -\frac{\hbar^2}{4JD} \left( N + \frac{2}{D - 1} \right) \]
\[
= -\frac{\hbar^2}{4JD} \left( N + \frac{2}{D - 1} \right) . \tag{14}
\]

We do indeed see that all of the non-vanishing matrix elements diverge like \( 1/(1-D) \).

(e) The full effective Hamiltonian, projected onto the degenerate subspace is given by
\[
\hat{H} = \sum_i |i\rangle \langle i| \mathcal{H}_J |i\rangle \langle i| + \sum_{i,j} |i\rangle \langle i| H^\text{eff}_{ij} |j\rangle \langle j|. \tag{15}
\]

Then its action on the plane wave state \(|k\rangle\) is given by
\[
\hat{H} |k\rangle = -JD(N-4) \sum_i |i\rangle \langle i| e^{ik \cdot x_i} \sqrt{N} + \sum_{i,j} |i\rangle \langle i| H^\text{eff}_{ij} e^{ik \cdot x_j} \langle j| \]
\[
= -JD(N-4) |k\rangle - \frac{\hbar^2}{4JD} \left( N + \frac{2}{D - 1} \right) |k\rangle - \frac{\hbar^2}{4JD} \left( N + \frac{2}{D - 1} \right) \frac{1}{\sqrt{N}} \sum_{i,j} e^{ik \cdot x_j} |i\rangle \]
\[
= -JD(N-4) - \frac{\hbar^2}{4JD} \left( N + \frac{2}{D - 1} \right) + \frac{h^2}{4JD} \left( \frac{2}{D - 1} \right) \sum_{d=1}^{D} \cos(k \cdot e_d) |k\rangle . \tag{16}
\]

Thus the eigenenergy as a function of \( k \) is
\[
E_k = -JD(N-4) - \frac{\hbar^2}{4JD} \left[ N + \frac{2}{D - 1} \left( 1 + \sum_{d=1}^{D} \cos(k \cdot e_d) \right) \right] \tag{17}
\]

Subtracting the perturbed ground state energy from part (c), \( E_{0,+} = -JND - N\hbar^2/4JD \), we obtain the spin-wave excitation spectrum:
\[
e_k = 4JD - \frac{\hbar^2}{2JD(D-1)} \left[ 1 + \sum_{d=1}^{D} \cos(k \cdot e_d) \right] . \tag{18}
\]