Critical dynamics of thermal conductivity at the normal-superconducting transition

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We study the effect of thermal fluctuations on the critical dynamics of the three-dimensional disordered superconductor across the normal-superconducting transition. We employ a phenomenological hydrodynamic approach, and in particular find the thermal conductivity to be smooth and nonsingular at the transition.

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I. INTRODUCTION

Some 15 years after the discovery of the cuprate superconductors, many tantalizing puzzles remain, particularly with regards to ground-state and possible quantum critical properties with varied doping. It has been difficult to disentangle consistent analyses faithful to experiment from the zoo of microscopic theories and models. In contrast, there have been notable successes in understanding the phenomenology of the finite temperature transition into the superconducting phase, where quantum fluctuations are unimportant and progress can be made employing classical Landau-Ginzburg approaches. In particular, at optimal doping in YBCO, the observed critical behavior in the electrical conductivity, specific heat, penetration depth, and other quantities[1] appears to be consistent with theoretical expectations.

Recent attention has focused on thermal transport experiments,[2] which have shed light on the low-temperature transport of quasiparticles in the superconducting phase, and have revealed low-temperature violations of the Wiedemann-Franz law in the normal state of the electron-doped material suggestive of a non-Fermi liquid ground state.[3] Here, we revisit the theory of thermal conductivity, focusing on the critical behavior near the finite temperature superconducting transition where progress is possible without the need for a microscopic quantum model. Older works within a BCS framework generalizing the Aslamazov-Larkin calculations to thermal conductivity,[4] have predicted a diverging thermal conductivity upon cooling into the superconductor, reminiscent of the behavior of ^{4}He at the λ transition. This appears to be at odds with experiment in the cuprates, which typically show a finite and rather smooth thermal conductivity as one cools through T_c, with a large growth upon further cooling usually ascribed to quasiballistic-quasiparticle transport.

Our study focuses on the three-dimensional disordered superconductor, most appropriate to optimally doped YBCO, which is the least two-dimensional cuprate. We follow the phenomenological hydrodynamic approach to critical dynamics pioneered by Hohenburg and Halperin.[5] Indeed, one of the early successes of this dynamical scaling approach was the correct description of the diverging thermal conductivity near the λ transition in ^{4}He. Here, we modify this theory to account for impurities and long-ranged Coulomb forces appropriate to the superconductor. Our central conclusion is that rather than a divergent thermal conductivity as in ^{4}He, the thermal conductivity is predicted to be *finite* and exhibit *smooth* nonsingular behavior at the superconducting transition. Surprisingly, the critical singularities that dominate the electrical conductivity and other thermodynamic properties are found to be *completely* absent from the thermal conductivity. There is effectively a decoupling between the thermal and electrical-transport coefficients upon approaching the superconducting phase, in strong contrast to the universal Lorenz ratio relating these two transport coefficients in a conventional metal.

II. THERMAL CONDUCTIVITY

The thermal conductivity κ, relates the heat current $\vec{Q}$ to an applied thermal gradient $\vec{\nabla} T$ under the condition of no particle flow,

$$\vec{Q} = -\kappa \vec{\nabla} T, \quad \vec{j} = 0, \tag{1}$$

where $\vec{j}$ is the particle current. Within linear response one has an Einstein relation for the thermal conductivity,

$$\kappa = D_T C_V, \tag{2}$$

where D_T is the thermal diffusion constant, and C_V the specific heat at constant volume.

In systems with a condensed ground state such as superfluid ^{4}He, the situation is a little more involved since energy can couple to the order parameter in a nondissipative fashion. This leads to a ballistic wavelike propagation of heat—second sound—in the superfluid state, and an infinite thermal conductivity. In the normal state, while heat does propagate diffusively, its associated thermal conductivity diverges due to critical fluctuations upon approaching the transition temperature. As we shall discuss, in contrast to its sister system of superfluid ^{4}He, the impure superconductor can only support a diffusive rather than an oscillatory heat mode, and the thermal conductivity κ is finite at all temperatures.

In the high-T_c superconductors, the role of thermal fluctuations near the superconducting transition is drastically enhanced relative to their low-T_c counterparts. Being generally anisotropic, and strongly type II, the Ginzburg criterion shows that critical fluctuations are present over a relatively wide range of temperatures, perhaps as large as 5–10 K.[6]

Within this temperature window we can dispense with microscopic models, and appeal to a critical hydrodynamic approach.

III. SUPERFLUID HYDRODYNAMICS REVISITED

To identify the appropriate model near criticality, we follow Ref. 5, and first focus on the hydrodynamics in the normal state. Since the frequency ω of the ''slow'' hydrodynamic modes vanishes with wave vector $\vec{q}$, one need only study the dynamics of conserved densities that cannot relax rapidly on long length scales. We consider first an uncharged and pure fluid, such as ^{4}He. In this case there are five such conserved densities: three components of the particle current density $\vec{j}$ and the energy ϵ, in addition to the particle number density ρ. Associated with these five conserved densities are five hydrodynamic modes: propagating first sound that involves the density and longitudinal current density (counting as two modes with $\omega = \pm cq$) and three diffusive modes (with $\omega = iDq^2$)—energy and two transverse current density modes. To see this, consider the continuity equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0, \tag{3}$$

and the Navier Stokes equation for current density conservation linearized for small current density:

$$\frac{\partial \vec{j}}{\partial t} + \frac{1}{m}\vec{\nabla} p = \nu \nabla^2 \vec{j}, \tag{4}$$

where p is the pressure, m the mass of the constituent ^{4}He particle, and ν the kinematic viscosity. One can also define the fluid velocity field $\vec{v}$, which satisfies $\vec{j} = \rho \vec{v}$. Upon taking the divergence of Eq. (4), combining with the continuity equation and linearizing for small current density and density variations, one obtains first sound with velocity, $c = \sqrt{1/m \, \partial p / \partial \rho|_s}$. Equation (4) also implies two diffusive transverse current density modes. In addition, the energy density ϵ is conserved,

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot \vec{Q} = 0, \tag{5}$$

where $\vec{Q}$ is the heat current. Upon combining Eq. (5) with the expression that defines the thermal conductivity, one obtains the heat diffusion equation

$$\partial_t \epsilon = D_T \nabla^2 \epsilon, \tag{6}$$

where we have used $C_V = \partial \epsilon / \partial T$ and the Einstein relation relating κ and the thermal diffusion coefficient, D_T.

On approaching the λ transition into the superfluid, one must augment this hydrodynamics with the slow dynamical relaxation of the order parameter. This leads to six hydrodynamic modes in the superfluid phase: first sound, second sound (involving the order parameter and predominantly energy), and the two diffusive transverse current density modes. In the superfluid one adopts a ''two-fluid'' description, in which the total density is decomposed into a superfluid and a normal fluid component: $\rho = \rho_s + \rho_n$. Moreover, one introduces a second velocity field—the superfluid velocity $\vec{v}_s$, in addition to the ''normal'' fluid velocity denoted $\vec{v}_n$, and the total current density is decomposed as $\vec{j} = \rho_s \vec{v}_s + \rho_n \vec{v}_n$. The superfluid velocity field is taken to satisfy a Josephson relation

$$m \frac{\partial \vec{v}_s}{\partial t} + \vec{\nabla} \mu = 0, \tag{7}$$

where μ is the chemical potential.[7] Diffusion of energy in the normal fluid is replaced by second sound oscillations of normal and superfluid. As the superfluid carries no entropy, the oscillations involve an entropy wave associated with the normal fluid. This is easiest seen in the absence of dissipation $(\nu = 0, \kappa = 0)$, where conservation of entropy density s, maybe expressed in terms of the associated entropy current $s\vec{v}_n$ carried by the normal fluid as

$$\frac{\partial s}{\partial t} + \vec{\nabla} \cdot (s \vec{v}_n) = 0. \tag{8}$$

One can combine Eqs. (3), (4), (7), and (8), and use the fact that local changes in chemical potential are related to local thermal gradients via the thermodynamic relation

$$\rho d\mu = -s dT + dp. \tag{9}$$

One then finds the superfluid velocity and the entropy participating in second sound motion with velocity

$$c_s = \sqrt{\frac{1}{\rho m} \frac{\rho_s s^2}{\rho_n \left(\frac{\partial s}{\partial T} \right)}}.$$

First sound and the two diffusive transverse current density modes are also present in the superfluid phase, giving the anticipated six hydrodynamic modes.

Turning next to the critical dynamics at the λ transition, one needs to take into account purely those hydrodynamic modes that go soft. Coming from the superfluid side, the only such mode is the second-sound mode since its velocity vanishes along with the superfluid density. The first-sound velocity remains finite through the transition, so that first sound oscillates much more quickly than second sound at the same wave vector. One thereby argues that first sound should not enter into the low-frequency critical dynamics. Moreover, couplings of transverse current density modes are found to be irrelevant.[5] Thus for the critical dynamics, one can focus on the conserved heat density and the order parameter, which together comprise second sound in the superfluid. More precisely, the nondissipative coupling of energy and the order parameter below T_c, which results in second sound, is captured by the Poisson relationship

$$\{\psi, M\} \sim \psi, \tag{10}$$

where M is the net combination of the energy and particle number densities taking part in the oscillation, and ψ is the complex order parameter. The critical dynamics of such a system is described by a generalized time-dependent Ginzburg-Landau type model involving these two fields, and was denoted as model F in Ref. 5.

IV. COULOMB INTERACTIONS AND IMPURITIES

Turning to the case of the charged, pure fluid, which is appropriate in the normal phase for a pure metal, the presence of an electric field $\vec{E}$ gives rise to acceleration. Thus, the right-hand side of Eq. (4) acquires a term $e/m\rho\vec{E}$, and that of Eq. (7) acquires a term $e\vec{E}$, where "e" is the charge of the electron. In addition, distortions of the charged fluid itself give rise to an electric field described by Poisson's equation,

$$\vec{\nabla}\cdot\vec{E}=4\pi e\,\delta\rho, \tag{11}$$

where $\delta\rho$ corresponds to density distortions. Consequently, in three dimensions, the sound mode involving density and longitudinal current density becomes a high-energy plasmon with frequency $\omega_p=\sqrt{4\pi e^2\rho/m}$ in the limit $\vec{q}\to 0$, and thus drops out of the hydrodynamic description. The transverse diffusive current density modes remain. Importantly, even with the modifications associated with the presence of charge, one finds the diffusive heat mode above T_c.

Below T_c, in the pure three-dimensional superconductor, the low-frequency, long-wavelength second-sound mode survives because it involves no net density distortions. To see this, we note that due to the gapped plasmon the density is not a hydrodynamic variable, so one can effectively set $\delta\rho=0$. Upon linearizing the continuity equation Eq. (3) for small velocities, this implies that $\nabla\cdot\vec{v}_n=-(\rho_s/\rho_n)\nabla\cdot\vec{v}_s$. Upon inserting this into the linearized entropy conservation Eq. (8), and combining with the divergence of the Josephson relation Eq. (7), one finds that

$$\frac{\partial^2 s}{\partial t^2}+\frac{\rho_s s}{\rho_n m}\nabla^2\mu=0. \tag{12}$$

Finally, using the thermodynamic relation Eq. (9) one arrives at the wave equation for the entropy density propagating once more with velocity

$$c_s=\sqrt{\frac{1}{\rho m}\,\frac{\rho_s s^2}{\rho_n\left(\dfrac{\partial s}{\partial T}\right)}}.$$

In addition to second sound, one expects the two diffusive transverse current density modes to be present in the charged superfluid, just as in ^{4}He.

Thus, essentially the sole effect of Coulomb interactions on the superfluid hydrodynamics is the conversion of first sound into the nonhydrodynamic plasmon mode. Since first sound was argued in any event to decouple from the critical dynamics at the λ transition, the critical dynamics in the charged superfluid is expected to be described by the same theory—that is model F, except with the density M referring to pure energy.

Consider next the case of an uncharged fluid in the presence of impurities (e.g., ^{4}He absorbed in a porous medium). Impurities violate momentum conservation and this leads to a dramatic modification of the hydrodynamics. In particular, with only density and energy being conserved, one expects two hydrodynamic modes above T_c and three below. Absence of momentum conservation can be explicitly captured by

$$\frac{\partial\vec{j}_n}{\partial t}=-\frac{\vec{j}_n}{\tau}, \tag{13}$$

where $1/\tau$ is the decay rate of current density $\vec{j}_n$. Then, above T_c, the first-sound mode involving longitudinal current density and density cannot propagate at low frequencies. It is replaced by a damped current density mode ($\omega=-i/\tau$) and a diffusive density mode with diffusion constant $D_\rho=c^2\tau$, where "c" is the velocity of first sound in the pure case. Energy continues to be a diffusive mode in the presence of impurities.

Below T_c, the uncharged impure system once again has a superfluid component moving with velocity v_s. The superfluid can couple to density in a nondissipative fashion, but not to entropy, which is not even conserved, and is still mainly associated with the normal fluid that can no longer propagate ballistically. We can thus simply drop the normal fluid velocity from the continuity equation Eq. (3), which upon linearization and combination with the Josephson relation gives, $\partial_t^2\rho=(\rho_s/m)\nabla^2\mu$. This describes a fourth sound mode involving purely superfluid oscillations propagating with velocity

$$c_{imp}=\sqrt{\frac{\rho_s}{m\left(\dfrac{\partial\rho}{\partial\mu}\right)}}.$$

In addition, the diffusive energy mode persists below T_c.

To model the critical dynamics one needs to reinterpret the conserved density M in model F as the particle number density ρ and then augment the model with an additional conserved energy density that is diffusive both above and below T_c. The conserved energy density will be coupled to the order parameter ψ via a term in the free energy of the form $\epsilon|\psi|^2$. This will lead to a coupling in the resulting time-dependent equations of motion.

Finally, we arrive at the case of interest—the impure superconductor—that we model by the charged, impure superfluid. The hydrodynamic modes may be obtained by incorporating the modifications described above for charge and impurities to the linearized hydrodynamic equations of the ^{4}He system. The normal state describes an impure metal. Due to the long-ranged Coulomb interactions the density, although conserved, is gapped up at the plasma frequency and thus again drops out of the hydrodynamic description. Moreover, with impurities, the transverse current density modes are also damped. So above T_c, in this system of the impure metal, associated with the only remaining hydrodynamic variable one has a diffusive thermal mode.

Below T_c, superconducting order gives rise to a second mode. As in the case of the uncharged, impure fluid, the order parameter cannot couple nondissipatively to heat. But unlike in this case, it cannot even do so with density, which as in the pure charged case, is no longer a hydrodynamic variable since its distortions cost electrostatic energy. As a result, there exist two *diffusive* modes related to the energy and order parameter, and no oscillatory hydrodynamic modes.[13,8,9]

While our hydrodynamic treatment employed simple linearized equations appropriate for mode analysis, an exhaustive study along the lines of the case of ^{4}He (Ref. 10) may prove useful for the case of the impure superconductor.

V. CRITICALITY

We are now equipped to model the critical dynamics of the dirty superconductor. We have seen that above T_c, there only exists a diffusive energy mode. Below T_c, in striking contrast to superfluid ^{4}He, instead of a second-sound oscillatory mode, there exists one diffusive mode associated with conserved energy density, and one diffusive mode associated with the order parameter, which takes the form of a complex scalar. The simplest phenomenological model incorporating these ingredients, and all possible relevant couplings, is described by model *C* of Ref. 5, and is analyzed in detail for its critical dynamics in Ref. 11. It is defined by the following set of equations of motion involving the complex order parameter ψ and the conserved energy density:

$$\frac{\partial\psi}{\partial t} = -\Gamma_0 \frac{\delta F_0}{\delta\psi^*} + \eta, \tag{14}$$

$$\frac{\partial\epsilon}{\partial t} = -\kappa_0\nabla^2\left(\frac{\delta F_0}{\delta\epsilon} - \delta\right) + \zeta, \tag{15}$$

$$F_0 = \int d^dx\left(\frac{1}{2}r_0|\psi|^2 + u_0|\psi|^4 + \frac{1}{2}|\vec{\nabla}\psi|^2 + \gamma_0|\psi|^2\epsilon + \frac{1}{2}C_0^{-1}\epsilon^2\right). \tag{16}$$

Here Γ_0 and κ_0 are the bare transport coefficients for the order parameter and the energy, respectively. An external source field, δ has been included in Eq. (16), and η and ζ are Langevin noise sources appropriate for ψ (complex) and ϵ (real), respectively. The energy density and the superconducting order parameter are coupled together via the coupling constant γ_0.

In equilibrium, ψ and ϵ minimize the functional F_0. In a functional integral formulation applicable for the equilibrium distribution,[11] one can integrate over the energy field, ϵ. The resulting functional of the order parameter ψ displays the statics of three-dimensional 3D *XY* critical behavior, appropriate for an extreme type II superconductor (see, e.g., Refs. 6 and 1). The Harris criterion (see, e.g., Ref. 12) shows that this holds true even in the presence of disorder. Thus, the pure 3D *XY* model provides an appropriate description for critical static properties of the dirty superconductor.

We next turn to the critical dynamics. To obtain the thermal conductivity, one first defines the energy density linear response function $\chi(\vec{q},\omega)$, by the relation

$$\langle\epsilon(\vec{q},\omega)\rangle_\delta = \chi(\vec{q},\omega)\,\delta(\vec{q},\omega). \tag{17}$$

Quite generally, in such a diffusive system, one expects this response function at low frequencies and wave vectors to take the standard form

$$\chi(\vec{q},\omega) = \frac{\kappa q^2}{-i\omega + D_T q^2}, \tag{18}$$

where κ is the thermal conductivity and D_T the thermal diffusion constant. Thus, the thermal conductivity can be extracted as

$$\kappa = \lim_{\vec{q}\to 0} q^{-2}\left[\left.\frac{i\partial\chi^{-1}(\vec{q},\omega)}{\partial\omega}\right|_{\omega=0}\right]^{-1}. \tag{19}$$

In the absence of coupling to the order parameter, χ can be readily computed from the (linear) equation of motion for ϵ which gives

$$\chi_0^{-1}(\vec{q},\omega) = \frac{-i\omega}{\kappa_0 q^2} + C_0^{-1}, \tag{20}$$

which has the general diffusive form as in Eq. (18) with the identifications $\kappa = \kappa_0$ and $D_T = \kappa_0/C_0$.

The effect of the fluctuating order parameter on the thermal conductivity κ, can be studied by treating the couplings u_0 and γ_0 perturbatively. Specifically, one can set up a perturbative expansion for the "self-energy" Σ that modifies the thermal response function,

$$\chi^{-1}(\vec{q},\omega) = \chi_0^{-1}(\vec{q},\omega) + \Sigma(\vec{q},\omega). \tag{21}$$

As is clear from Eq. (19), a renormalization of the bare thermal conductivity κ_0 requires a contribution to the self-energy of the form $\Sigma \sim -i\omega/q^2$, which is divergent as $q\to 0$ for fixed non-zero ω. But as argued by Refs. 5 and 11, away from criticality, with $r_0 > 0$, the self-energy will be finite in the $q\to 0$ limit at all orders in perturbation theory, being protected in the infrared by r_0 and in the ultraviolet by a high momentum cutoff. Thus, these perturbations can generate no corrections to κ, and the thermal conductivity will be given exactly by the "bare" parameter κ_0.[11] Since κ_0 is a coupling constant that depends on short distance physics only, it will necessarily be a smooth function of temperature. This thereby establishes that the thermal conductivity should be nonsingular and smooth as one cools through the superconducting transition.

It is worth emphasizing that although the thermal conductivity itself is nonsingular at the transition, critical singularities driven by the order-parameter flucutations will generally enter into the dynamical relaxation of the energy. Smoothness of the thermal conductivity is intimately tied to the fact that it is a zero-frequency, zero-wave-vector quantity. Indeed, the dependence of κ on wave vector or frequency is expected to be singular at the critical point.

To recapitulate and emphasize, fluctuations in the superconducting order parameter will certainly affect the time-dependent relaxation of the energy. However, the thermal conductivity—the transport coefficient associated with energy propagation—remains smooth and finite across the transition.

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