Fractionalization in an easy-axis Kagome antiferromagnet

L. Balents,[1] M. P. A. Fisher,[2] and S. M. Girvin[2,3]
[1]*Department of Physics, University of California, Santa Barbara, California 93106-4030*
[2]*Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030*
[3]*Physics Department, Sloane Physics Laboratory, Yale University, New Haven, Connecticut 06520-8120*
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We study an antiferromagnetic spin-1/2 model with up to third nearest-neighbor couplings on the Kagome lattice in the easy-axis limit, and show that its low-energy dynamics are governed by a four-site XY ring exchange Hamiltonian. Simple "vortex pairing" arguments suggest that the model sustains a novel fractionalized phase, which we confirm by exactly solving a modification of the Hamiltonian including a further four-site interaction.In this limit, the system is a featureless "spin liquid," with gaps to all excitations, in particular: deconfined $S^z=1/2$ bosonic "spinons" and Ising vortices or "visons." We use an Ising duality transformation to express vison correlators as nonlocal strings in terms of the spin operators, and calculate the string correlators using the ground state wave function of the modified Hamiltonian. Remarkably, this wave function is exactly given by a kind of Gutzwiller projection of an XY ferromagnet.Finally, we show that the deconfined spin-liquid state persists over a finite range as the additional four-spin interaction is reduced, and study the effect of this reduction on the dynamics of spinons and visons.

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I. INTRODUCTION

It has now been almost 15 years since Anderson suggested that two-dimensional (2D) spin-1/2 antiferromagnets might condense into a featureless "spin-liquid" quantum ground state.[1] In close analogy with the one-dimensional Heisenberg antiferromagnetic chain, the 2D spin liquid was posited to support deconfined spinon excitations—"particles" carrying $s=1/2$ in stark contrast with the $s=1$ triplet excitations of more familiar non-magnetic phases such as the spin-Peierls state and with the $S^z=1$ magnon excitations of the 2D Neél state.[2] Early attempts to demonstrate the existence of the 2D spin-liquid focused on quantum dimer models[3] motivated directly by resonating valence bond (RVB) ideas,[4] slave-Fermion mean field theories[5] and large N generalizations[6] of the spin models. While the topological character of the spin liquid was mentioned in some of these pioneering studies,[7] generally the focus was on characterizing the spin liquid by an absence of spin ordering and spatial symmetry breaking. In the past few years, it has been emphasized that the precise way to characterize a 2D spin-liquid phase[8]—as with other 2D fractionalized phases—is in terms of "topological order," a notion introduced by Wen in the context of the fractional quantum Hall effect.[9] Central to the notion of topological order in 2D is the presence of vortex-like excitations with long-ranged statistical interactions.[7,10] In the simplest 2D spin liquid these pointlike excitations have been dubbed "visons" since they carry an Ising or Z_2 flux.[10] Upon transporting a spinon around a vison, the spinon's wave function acquires a minus sign. A theoretical description of this long-ranged statistical interaction is most readily incorporated in the context of a gauge theory with a discrete Ising symmetry, in which the visons carry the Z_2 flux and the spinons the Z_2 charge.[10,11] The Z_2 gauge theory can be dualized into a vortex representation, wherein the topological order follows from the notion of "vortex pairing."[12]

Efforts to identify microscopic spin Hamiltonians that might actually exhibit such topologically ordered phases have focussed on strongly frustrated 2D $s=1/2$ antiferromagnets. Due to the "sign problem" these efforts have been essentially limited to exact diagonalization studies on very small lattices. Nevertheless, such numerics do identify a few models which appear to be in a spin-liquid phase: the Kagome antiferromagnet with near neighbor interactions[13] and a triangular lattice model with 4-spin ring exchange terms.[14] The importance of multispin ring exchange processes in driving 2D fractionalization is also apparent within the Z_2 gauge theory formulation.[10] In an important recent development, Moessner and Sondhi[15] have compellingly argued that a particular quantum dimer model on the triangular lattice is in a featureless liquid phase, closely analogous to the desired "spin -liquid" phase of a spin Hamiltonian.

In this paper we revisit the $s=1/2$ Kagome antiferromagnet, in the presence of second and third neighbor exchange interactions. By passing to an easy-axis limit of this model, substantial analytic and numerical progress is possible both in establishing the presence of a fractionalized spin liquid and of directly analyzing its topological properties. Specifically, in the easy-axis limit we map the model exactly onto an XY Hamiltonian consisting solely of a local 4-spin ring exhange interaction. Since the sign of the ring exchange term is "bosonic"— opposite to the sign obtained upon cyclically permuting four underlying $s=1/2$ fermions (e.g., electrons)[14]—the Hamiltonian does *not* suffer from a sign problem and so should be amenable to quantum Monte Carlo. Furthermore, if the two levels of the spin-1/2 on each site of the Kagome lattice is reinterpreted as the presence or absence of a (quantum) dimer living on a bond of a triangular lattice, the model can be reinterpreted as a quantum dimer model which is very similar to that considered by Moessner and Sondhi,[15] the distinction being that three, rather than one, dimers emerge from each site. This realization allows us to exploit the important work of Rokhsar and Kivelson[3] who

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identified an exactly soluble point of a generalized square lattice quantum dimer model. With a similar generalization, our model also possesses an exact zero energy wave function: an equal weight superposition of all allowed spin configurations in the low-energy singlet sector. We show that this wave function can be viewed as an exact version of the popular variational state consisting of the Gutzwiller projection of a superfluid/superconductor.[16] Finally, we are able to implement an exact duality transformation which enables us to identify the operators which create both the spinon excitation and the topological vison excitation. Employing the exact wave function, we compute numerically the vison two-point correlation function, and show that it is exponentially decaying—the hallmark of a 2D fractionalized phase.[10] We thereby demonstrate that the (gapped) spinons are genuine deconfined particlelike excitations.

The paper is organized as follows. In Sec. II we introduce a generalized $s=1/2$ Kagome antiferromagnet and show how it can be mapped onto a bosonic ring model in the easy-axis limit. With a slight further generalization, we identify an exactly soluble point in Sec. III and obtain an exact spin-liquid ground state wave function. In Sec. IV we exploit an exact duality transformation which maps the Kagome spin model onto a Z_2 gauge theory living on the dual lattice to identify the spinon and vison excitations. The vison two-point correlation function is then evaluated numerically using the exact wave function in Sec. V, and we demonstrate that it is short-ranged thereby directly establishing the presence of fractionalization in the spin-liquid ground state. Finally, Sec. VI is devoted to a brief discussion of the implications of this finding.

II. MODEL

We consider a spin-1/2 Heisenberg antiferromagnet on a Kagome lattice with Hamiltonian

$$\mathcal{H}=\sum_{ij} J_{ij}\vec{S}_i\cdot\vec{S}_j . \tag{1}$$

Since the Kagome lattice consists of corner sharing triangles, the nearest-neighbor exchange interaction, denoted J_1, is strongly frustrating. Here we extend this standard near-neighbor model to incude further neighbor interactions, J_2,J_3, which act between pairs of sites on the hexagons in the Kagome lattice (Fig. 1). Specifically, two spins on the same hexagon separated by 120 degrees are coupled via J_2, and J_3 is the coupling between two spins diametrically across from one another on the hexagon.

Instead of the usual nearest-neighbor Kagome antiferromagnet (with $J_2=J_3=0$), we specialize instead to the case with equal exchange interactions, $J_1=J_2=J_3=J$. This generalized Kagome antiferromagnet can be cast into a simple form

$$\mathcal{H}=J\sum_{\hexagon} \vec{S}_{\hexagon}\cdot\vec{S}_{\hexagon} , \tag{2}$$

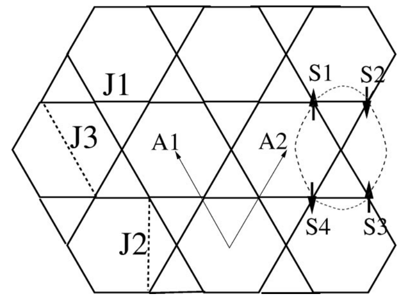

FIG. 1. Kagome lattice and interactions. Two primitive vectors $\vec{a}_1,\vec{a}_2$ are shown, as are the labels 1, . . . ,4 for the four sites on a bow tie. The ring term is generated both from the short-dashed and long-dashed virtual exchange processes.

where the summation is over all hexagons on the Kagome lattice and $\vec{S}_{\hexagon}=\Sigma_{i=1}^{6}\vec{S}_i$ is the sum of the six spins on each hexagon. A similar form was obtained by Palmer and Chalker[17] for a Heisenberg model on the "checkerboard" lattice, with the Hamiltonian expressed as a sum over the total spin living on elementary square plaquettes, squared.

As for the nearest-neighbor model, the generalized Kagome antiferromagnet described by Eq. (2) has a nontrivial classical limit. There is a thermodynamically large set of classical ground states, which includes any configuration for which the classical vector $\vec{S}_{\hexagon}=0$ for each hexagon. The breaking of this degeneracy by quantum fluctuations could give rise to "order-by-disorder." For the spin-1/2 case of interest, however, Eq. (2) is essentially intractable analytically. To make progress, we retain SU(2) spins with $S=1/2$ on each site, but generalize the Hamiltonian to allow for anisotropic exchange interactions. Specifically, we consider an "easy axis" limit, with the exchange interaction along the z axis in spin space larger than in the x-y plane: $J_z>J_\perp$. In the extreme easy-axis limit, one can first analyze the J_z terms alone, and then treat the remaining terms as a perturbation: $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1$ with

$$\mathcal{H}_0=J_z\sum_{\hexagon} (S^z_{\hexagon})^2 \tag{3}$$

and

$$\mathcal{H}_1=J_\perp\sum_{\hexagon} [(S^x_{\hexagon})^2+(S^y_{\hexagon})^2-3], \tag{4}$$

where the subtraction of 3 was included for convenience. In an eigenbasis of $S^z_i=S=\pm 1/2$, the Hamiltonian $\mathcal{H}_0$ describes a classical spin system. The classical ground state consists of all spin configurations which have zero (z-axis) magnetization on each and every hexagonal plaquette: $S^z_{\hexagon}=0$. There are many such configurations (note that unlike the nearest-neighbor model, the generalized Kagome antiferromagnet is unfrustrated in the easy-axis limit), with a ground state degeneracy that grows exponentially with system size, much like other fully frustrated classical spin models such as the triangular lattice Ising antiferromagnet. The full Hamiltonian, $\mathcal{H}$, lifts this huge degeneracy, splitting the classically degenerate ground states into a *low-energy manifold*, still characterized, however, by the good quantum numbers $S^z_{\hexagon}=0$.

Some properties of this easy-axis limit are immediately evident. For instance, all states in the low-energy manifold have $S^z_{\hexagon}=0$ for every hexagon, and there is a large gap of approximately J_z to states with any nonzero $S^z_{\hexagon}$. Hence the ground state has in this sense a "spin gap." Thus the easy-axis generalized Kagome antiferromagnet has no XY spin order, but translational symmetry breaking is not precluded. More subtle aspects of this model are less evident. In particular, we would like to ascertain the presence or absence of more subtle "topological" order, and the types of "singlet" (more precisely $S^z=0$) and spinful ($S^z\neq 0$) excitations.

To proceed, we treat $\mathcal{H}_1$ as a perturbation with $J_\perp \ll J_z$, and project back into this *low energy manifold* of degenerate classical ground states with $S^z_{\hexagon}=0$. (This procedure is very much analogous to the derivation of the Heisenberg model starting from the Hubbard model with $t\ll U$. Indeed, in the language of "hard core bosons" in which the boson number corresponds to $S^z_i+1/2$, the perturbing Hamiltonian $\mathcal{H}_1$ describes boson hopping amongst a pair of sites on the same hexagon.) Within second order degenerate perturbation theory (in $J_\perp$) for the low-energy manifold, there are two types of (virtual) processes which contribute, preserving the vanishing magnetization on every hexagon. In the first, two antiparallel spins within a single hexagon exchange and then exchange back again. This "diagonal" process leads (within the low-energy manifold) to a simple constant energy shift $E_0=-(9/2)N_{\hexagon}J_\perp^2/J_z$, where $N_{\hexagon}$ is the total number of hexagons. Because this trivial shift does not split the extensive degeneracy, we neglect it in what follows. More interesting are off-diagonal processes, in which two pairs of antiparallel spins on opposite sites of a five-site "bow-tie" plaquette exchange (see Fig. 1). This process involves spins on only four sites, and is an analog of electron exchange "ring" moves. One can readily verify that such "ring" moves on the bow tie leave invariant the (z axis) magnetization on every hexagon.

Up to second order in $J_\perp/J_z$, within the low-energy manifold, the full Kagome Heisenberg antiferromagnet is thereby reduced to the form $\mathcal{H}_0+\mathcal{H}_{\rm ring}$ with

$$\mathcal{H}_{\rm ring}=-J_{\rm ring}\sum_{\bowtie}(S_1^+S_2^-S_3^+S_4^-+\text{h.c.}),\qquad(5)$$

where the labels 1, . . . ,4 denote the four spins at the ends of each bow tie as labeled in Fig. 1. Here the ring exchange interaction $J_{\rm ring}=J_\perp^2/J_z$, and by assumption one has $J_\perp \ll J_z$. It is noteworthy that in this extreme easy-axis limit the frustrated Kagome magnet does *not* have a sign problem, and as such could be profitably attacked via quantum Monte Carlo.

III. SOLUBLE SPIN LIQUID

We now use $\mathcal{H}_{\rm ring}$ to address the nature of the spin-gapped state of the easy-axis generalized Kagome antiferromagnet. Several arguments point to a spin-liquid phase which supports fractionalized "spinon" excitations which carry spin $S^z=1/2$. Such a fractionalized state must also support vortexlike excitations, dubbed "visons," which carry no spin but have a long-ranged statistical interaction with spinons.

A first suggestion to this effect comes from viewing $\mathcal{H}_{\rm ring}$ as a lattice boson model, and a spin-liquid state thereby as a bosonic Mott insulator. Generally, such bosonic insulating states can be regarded as quantum-mechanical condensates of vortices.[18] To examine the vortex excitations, it is convenient to think of $S_i^\pm$ as lattice boson raising and lowering operators. Formally, one may then express $S_i^\pm=e^{\pm i\phi_i}$—fluctuations in the U(1) phases ϕ_i (conjugate to S_i^z) are induced by the constraint $S_i^z=\pm 1/2$. It is then illuminating to re-express the bosonic ring term as

$$\mathcal{H}_{\rm ring}=-2J_{\rm ring}\sum_{\bowtie}\cos(\phi_1-\phi_2+\phi_3-\phi_4).\qquad(6)$$

Consider now a vortex centered on some site (the "core"). Classically, for the four sites on the bow tie surrounding the vortex core, $\phi_j=(j/4)2\pi N_v$, where N_v denotes the number of vortices (vorticity) on this plaquette. The (core) energy of this vortex configuration is proportional to

$$E_{\rm vort}=2J_{\rm ring}(1-\cos(N_v\pi)).\qquad(7)$$

Notice that plaquettes with an odd number of vortices, N_v, cost an energy $4J_{\rm ring}$ relative to the even-N_v plaquettes. In particular, a single strength vortex is costly, but double-strength vortices are cheap. The same conclusion can be shown more formally using an exact duality transformation.

Typically single strength vortices condense, but one can also imagine insulating states which result from a condensation of composites made from N_v vortices.[12] Such insulators are necessarily fractionalized since they support deconfined (but gapped) charge excitations with "boson charge" $Q=S^z=1/N_v$. Based on the energetics of the ring term which tends to expel single vortices with double vortices being energetically cheaper, one expects that the insulating state for the Kagome ring model will have spin $S^z=1/2$ excitations—if it is fractionalized at all. If fractionalized, the "vison" can be understood as an unpaired vortex state in the vortex-pair condensate, a "dual" analog of a BCS quasiparticle.

Further evidence that the ground state of this model might be fractionalized comes from its formal equivalence to a particular quantum dimer model. Mapping to a dimer model is straightforward since the sites of the Kagome lattice can be viewed as the centers of the links of a triangular lattice. The two $S^z=+(-)1/2$ states on a site correspond to the presence (or absence) of a dimer on the associated link on the trian gular lattice. The ring term above corresponds directly to the elementary quantum dimer move on the triangular lattice considered recently by Sondhi and Moessner.[15] The only difference with the standard dimer model is that in this instance there are *three* dimers coming out of every site of the triangular lattice instead of the usual one. Sondhi and Moessner considered an additional "diagonal" term (see below) in the triangular lattice quantum-dimer model, and argued that the model was in a spin-liquid state in portions of the phase diagram. Central to their argument was an exactly soluble point of the model, first exploited by Rokhsar and Kivelson (RK)[3] in the square lattice quantum-dimer model. The additional term is diagonal in S_i^z, and may be written $\mathcal{H}_{\rm nf} = u_4 \Sigma_{r\in\bowtie} \hat{P}_{\rm flip}(r)$, where

$$\hat{P}_{\rm flip}(r) = \sum_{\sigma=\pm 1} \prod_{j\in r=1}^{4} \left(\frac{1}{2} + \sigma(-1)^j S_j^z\right). \tag{8}$$

The operator $\hat{P}_{\rm flip}(r)$ is a projection operator onto the two flippable states of the bow-tie ring r. This term in the Hamiltonian can be combined with $\mathcal{H}_{\rm ring}$ and written in the suggestive form

$$\mathcal{H}_{\rm ring} + \mathcal{H}_{\rm nf} = \sum_r \hat{P}_{\rm flip}(r)\left\{-J_{\rm ring}\prod_{j=1}^{4} 2S_j^x + u_4\right\}. \tag{9}$$

When $u_4 = J_{\rm ring}$ one can write down exact ground state(s) which have the product of $2S_x$ equalling one on all bow-tie rings. One such state is the XY ferromagnet with $S_j^x = 1/2$ on every site. In the hard-core boson description, this corresponds to a superfluid state (albeit an unusual one with no zero-point fluctuations). One must project back into the subspace in which there are three bosons on every hexagon ($S_{\hexagon}^z = 0$), since otherwise this state will not be an eigenstate of $\mathcal{H}_0$. (Actually, several distinct projections are generally possible, onto different sectors disconnected from one another under the action of $\mathcal{H}_{\rm ring}$. These give degenerate ground states.) This projection of a superfluid wave function to obtain a bosonic insulating state is analogous to the Gutzwiller projections of superconducting wave functions to obtain variational states for quantum spin models,[16] but there is an important difference. In the present instance, the constraints (of three bosons on every hexagon) *commute* with the Hamiltonian $\mathcal{H}_{\rm ring}$ which hops the bosons, in contrast to the no-double occupancy constraint which does not commute with the electron kinetic energy term in Hubbard-type models. Thus, in our case the wave function after projection is still an exact eigenstate of the full Hamiltonian.

IV. DUALITY, VISONS, AND SPINONS

Before studying this wave function, it is convenient to expose the vison degrees of freedom via a duality transformation. Specifically, we will employ the standard $2+1$ dimensional Ising duality which connects a global spin model to a Z_2 gauge theory with gauge fields living on the links of the dual lattice.[19] Ising duality transformations for quantum-dimer models have been extensively discussed in Ref. 11. In our case the global spin model is the Kagome model $\mathcal{H}_{\rm ring} + \mathcal{H}_{\rm nf}$ in Eq. (9), so that the dual lattice is the "dice" lattice, which can conveniently be constructed in terms of two interpenetrating honeycomb lattices as depicted in Fig. 2. On the operator level, the duality transformation is implemented by re-expressing S^x and S^z directly in terms of the dual gauge fields, σ_{ij}^μ, a set of Pauli matrices living on the links of the dice lattice

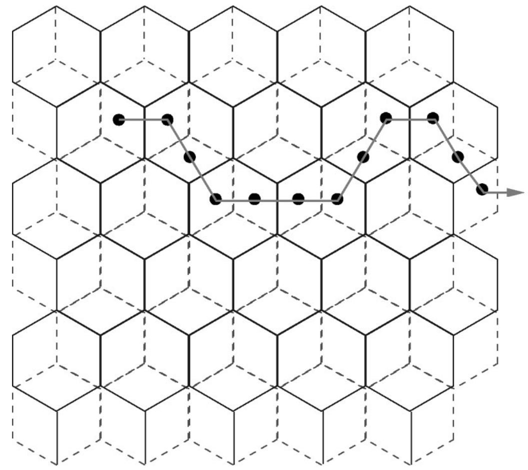

FIG. 2. Dice lattice shown as two interpenetrating honeycombs: blue (solid lines) and red (dashed lines). A blue vison is created geometrically by multiplying $2S_i^z$ over the underlying Kagome sites (centers of parallelograms, shown by solid dots) through which the "string" indicated passes. In the dual variables, this product is given by the product of blue gauge fields σ_{ij}^x cut by the string shown. The "blueness" of the vison shown owes to the fact that only a single spin S_i^z is contained within the originating blue hexagon.

$$S_i^x = \frac{1}{2}\prod_{jl\in\diamond} \sigma_{jl}^z \tag{10}$$

and

$$S_i^z = \frac{1}{2}\prod_{jl=i}^{\infty} \sigma_{jl}^x. \tag{11}$$

Here, the first product is taken around an elementary four-sided plaquette on the dice lattice which surrounds the spin S_i^x. The second product involves an infinite string which connects sites of the Kagome lattice, eminating from the site S_i^z and running off to spatial infinity. For every bond of the dual dice lattice which is bisected by this string, a factor of σ_{ij}^x is present in the product. To assure that this definition is independent of the precise path taken by the string, requires imposing the constraint that the product of σ_{ij}^x on all bonds connected to each site on the dice lattice is set equal to unity

$$\mathcal{G}_i = \prod_{\langle ji\rangle} \sigma_{ij}^x = 1, \tag{12}$$

where here j labels the near-neighbor sites to i. These local Z_2 gauge constraints must be imposed on the Hilbert space of the dual theory. In the resulting dual gauge theory, these constraints are analogous to Coulomb's law ($\boldsymbol{\nabla}\cdot E = 0$) in conventional electromagnetism. The necessity of including the constraints can be simply seen by counting degrees of freedom: there are twice as many (six) bonds per unit cell on the dice lattice as sites (three) per unit cell on the Kagome lattice, hence to maintain the physical Hilbert space of the original spins (site variables) requires restricting the gauge fields (bond variables).

The dual Hamiltonian takes the form

$$\mathcal{H}_{\rm dual}=\sum_r \hat{P}_{\rm flip}(r)\left\{-J_{\rm ring}\prod_{\hexagon_r}\sigma^z\prod_{\hexagon_b}\sigma^z+u_4\right\}, \quad (13)$$

where the products are taken around the two hexagonal plaquettes of the dice lattice which surround the given Kagome site r. These products measure ''magnetic flux'' (in the dual gauge fields) through hexagons belonging to the two honeycomb sublattices. The flip term becomes

$$\hat{P}_{\rm flip}(r)=\prod_{j\in r=1}^{4}(1-\sigma^x_{ij}\sigma^x_{jl}), \quad (14)$$

where the product is taken over pairs of bonds on the elementary dice plaquette which both connect to the same site, j.

One can readily verify that the operators which implement a local gauge transformation, $\mathcal{G}_i$ in Eq. (12), commute with this dual Hamiltonian. Equivalently, since $\mathcal{G}_i\sigma^z_{ij}\mathcal{G}_i=-\sigma^z_{ij}$, the dual Hamiltonian is invariant under the general Z_2 gauge transformation,

$$\sigma^z_{ij}\rightarrow\epsilon_i\sigma^z_{ij}\epsilon_j, \quad (15)$$

with arbitrary $\epsilon_i=\pm 1$. Remarkably, though, it turns out that this gauge theory actually has an additional set of local Z_2 symmetries. In particular, it is possible to transform the σ^z gauge fields living on the blue (or red) links separately, and still leave the Hamiltonian invariant. Eqivalently, one can define local red or blue gauge operators which commute with the Hamiltonian

$$\mathcal{G}^r_i=\prod_{\langle ij\rangle_r}\sigma^x_{ij}, \quad (16)$$

with the product over red links which eminate from site i, and similarly for the blue links. On the dice lattice, for each six-fold coordinate site there corresponds both a blue and red local gauge operator, whereas the three-fold coordinated sites are either red or blue.

The presence of this additional local symmetry can be directly traced to the conservation of the magnetization $S^z_{\hexagon}$ on each hexagon of the original Kagome lattice ring spin-model (note that this is the conservation of dimer number emerging from each site on the equivalent triangular lattice dimer model). Indeed, upon using Eq. (11), one can show that for the six-fold coordinate sites of the dice lattice

$$\mathcal{G}^{\rm red}_i=\mathcal{G}^{\rm blue}_i=-\exp[i\pi S^z_{\hexagon,i}], \quad (17)$$

where the center of the hexagon is at site i on the dice lattice. The right-hand side of this expression can be interpreted as a Z_2 ''charge'' living on the six-fold coordinate sites of the dice lattice, since it equals the (lattice) divergence of the Z_2 ''electric fields.'' For the ''singlet'' sector of the theory with $S^z_{\hexagon}=0$ for all hexagons, the right-hand side is simply minus one. But more generally, this expression indicates that hexagons with a nonzero (odd integer) value of the globally conserved spin, $S^z_{\hexagon}=\pm 1$, also carry both a red and a blue Z_2 gauge charge.

This fact allows us to identify both the spinon and vison excitations in the theory. Specifically, consider starting in the ''singlet'' sector of the theory with zero magnetization on every hexagon of the Kagome lattice, and flipping a single spin. Since each site of the Kagome lattice is shared by two hexagons, this creates two hexagons each with $S^z_{\hexagon}=1$. By adding a small near-neighbor spin exchange it is possible to hop these two magnetized hexagons, and to spatially separate them. As we demonstrated above, such magnetized hexagons also carry both a red and a blue Z_2 charge. Provided the dual gauge theory is in it's deconfined phase, these magnetized hexagons can propagate as independent particles. Since two such magnetized hexagons were created when we added spin-one to the system (by flipping the single spin), each magnetized hexagon must carry spin $S^z=1/2$, and we can thereby identify these excitations as the deconfined spinons.

A 2D spin liquid with deconfined spinons must necessarily support topological vortex like excitations—the visons.[10] The vison acts as a source of Z_2 ''flux'' for the spinon, whose wave function changes sign as it is transported around a vison. In the Z_2 gauge theory formulation of 2D fractionalization, the flux of the vison corresponds generally to a plaquette with $\Pi_{\rm plaq}\sigma^z=-1$. Since the spinons which hop on the six-fold coordinated sites of our dice lattice carry both a red and a blue Z_2 gauge charge, it is clear that this spin liquid phase will support two flavors of visons—a red (blue) vison corresponding to a flux penetrating one red (blue) hexagon of the dice lattice.

Due to the long-ranged statistical interaction between visons and spinons, it is not possible to have both excitations present and freely propagating. In particular, if the visons are gapped excitations they will be expelled from the ground state and the spinons will be deconfined. On the other hand, a proliferation and condensation of visons will lead to spinon confinement. Thus, in order to establish 2D fractionalization it is adequate to show an absence of vison condensation. A useful diagnostic for this is the vison two-point correlation function

$$V(r_i-r_j)=\langle\hat{v}_i\hat{v}_j\rangle, \quad (18)$$

where $\hat{v}$ denotes a vison creation operator. When this correlation function is short ranged, the visons are not condensed, and the system is fractionalized.

In order to evaluate this correlation function for the Kagome spin model, it is necessary to express this vison two-point function in terms of the original spin operators. To this end, we first note that from the definition in Eq. (11), it is apparent that the operator $2S^z_i$ creates both a red and a blue vison, $2S^z_i=\hat{v}^r_i\hat{v}^b_i$, since it introduces a Z_2 flux through the red and blue hexagons of the dice lattice which enclose the spin. Since $\hat{v}_i\hat{v}_i=1$, a single (red) vison (say) can be created by stringing together an infinitely long product of spin operators S^z. This ''string'' starts at the given site of the Kagome lattice and joins neighboring spins making only

$\pm 30^0$ turns eventually running off to spatial infinity, but otherwise is arbitrary. Explicitly, the vison two-point function is then

$$V_{ij} = \left| \langle 0 | \prod_{k=i}^{j} 2S_k^z | 0 \rangle \right|, \tag{19}$$

where $|0\rangle$ denotes the ground state, and the product in Eq. (19) is taken, as described above, along some path on the Kagome lattice starting at site *i* and ending at site *j*, containing an even number of sites, and making only "$\pm 30°$" turns left or right. Due to the constraint $S^z_{\rm hex}=0$, the latter product is path-independent up to an overall sign [hence the absolute value in Eq. (19)]. We also define for convenience the physically interesting spin-spin correlator

$$C_{ij} = \langle 0 | S_i^z S_j^z | 0 \rangle, \tag{20}$$

V. CORRELATORS AT THE SOLUBLE POINT

With *V* and *C* defined appropriately in terms of the spins, we are now in a position to evaluate them using the exact RK wave function. Specifically, we consider exact ground state wave functions (at the RK point) on the torus defined by identifying sites connected by the two winding vectors $\vec{W}_1 = n_1 \vec{a}_1$ and $\vec{W}_2 = n_2 \vec{a}_2$, where n_1, n_2 are positive even integers, and $\vec{a}_1, \vec{a}_2$ are primitive vectors (see Fig. 1). The degeneracies, etc., of such wave functions are nearly identical to that discussed by Moessner and Sondhi, so we do not go into detail here. We focus on the wave function $|0\rangle$, which has been projected onto a single topologically connected sector.

The expectation values of interest can be evaluated stochastically using a *classical* "infinite temperature" Monte Carlo algorithm, which "random walks" through the various components of the wave function. Our numerical results for a torus with $n_1 = n_2 = 20$ are shown in Fig. 3 plotting $\ln V$, $\ln C$ versus distance. Apart from a saturation due primarily to round-off error, both correlators clearly display exponential decay, $\ln C_{ij}$, $\ln V_{ij} \sim -|r_i - r_j|/\xi$ with apparently the same correlation length $\xi \approx 1$.

Short-range exponentially decaying correlations in C_{ij} establish the absence of spin order, but do not preclude broken translational symmetries such as plaquette or bond order. The exponential decay of the vison correlator *V*, however, implies that the phase is necessarily fractionalized with deconfined $S_z = 1/2$ spinon excitations, regardless of the presence or absence of broken translational symmetries. The exponential spatial decay of V_{ij} is suggestive of a vison gap, i.e., the existence of a minimum energy required to excite a vison. A vison gap, however, strictly speaking requires exponential decay of the vison correlator in *imaginary time*, not in space at equal time. Conceivably, the latter condition could occur in the absence of a vison gap, provided that these visons were localized. Given the peculiarities of the present model, we desire a direct argument for a vison gap. Fortunately, such an argument can also be made using the properties of the RK point. Our construction closely follows and only slightly refines an argument used in Ref. 20. We consider a family of models defined by singling out a single "central" triangle of the Kagome lattice, and the associated three bow-tie ring moves centered on the three sites of this triangle. For these three ring terms, we let $u_4' = J_{\rm ring}'$ vary independently of $u_4 = J_{\rm ring}$ for all other bow ties. Independent of the choice of $J_{\rm ring}' \geq 0$, the original, translationally invariant $|0\rangle$ state remains a zero energy ground-state wave function. However, for the special case $J_{\rm ring}' = 0$, we can find an additional exact zero-energy eigenstate,

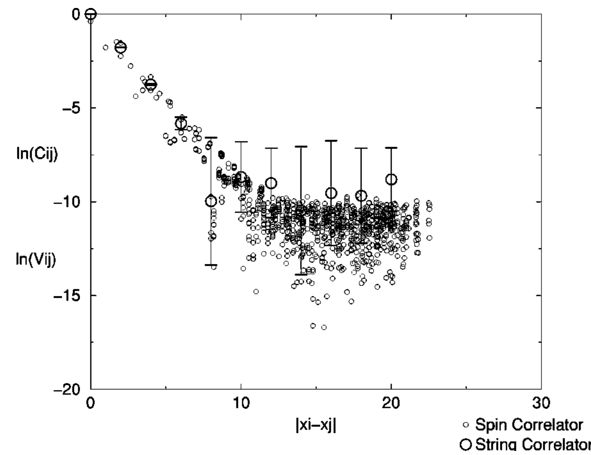

FIG. 3. Correlators for the (20,20) Kagome torus. The horizontal axis $\min|\vec{x}_i - \vec{x}_j|$ is the shortest distance between sites *i* and *j* on the torus, in units of the intersite distance. Large circles with error bars show the logarithm of V_{ij}, while small circles represent $\ln(C_{ij})$. Each small circle represents an average over all pairs of sites with a fixed separation, while the large circles with error bars represent the distribution of string correlators of pairs of sites connected by a *horizontal* string. Both correlators clearly decay exponentially, with apparently the same exponent, albeit with larger numerical errors for the string correlator, due presumably to the fact that this data is less spatially averaged.

$$|v\rangle = \hat{v}_i |0\rangle, \tag{21}$$

where $\hat{v}_i$ is the (string) vison creation operator emerging from any of the sites of the central triangle. Thus for $J_{\rm ring}' = 0$, there is no vison gap. We interpret this result to mean that by reducing the ring couplings on the central triangle, a vison has been *bound* to this triangle, with a binding energy that exactly equals its gap in the bulk. To test this hypothesis, we calculate the first-order energy shift as $J_{\rm ring}'$ is increased from zero to positive values using perturbation theory. To leading order, one finds that

$$E_v = \langle v | \mathcal{H}' | v \rangle = \langle 0 | \hat{v}_i \mathcal{H}' \hat{v}_i | 0 \rangle = 6 J_{\rm ring}' \langle 0 | \hat{P}_{\rm flip} | 0 \rangle, \tag{22}$$

where $\mathcal{H}'$ is the sum of ring terms for the three sites of the triangle. To obtain the final result, we use $\hat{v}_i S_j^x \hat{v}_i = \pm S_j^x$, where the minus (plus) sign obtains if *j* is (is not) on the string. The energy of the no-vison state $E_0 = 0$ for all $J_{\rm ring}'$. Hence the gap in this approximation is proportional to $J_{\rm ring}'$ multiplied by the probability that any given bow tie is flippable. The latter probability is determined directly by the classical Monte Carlo procedure, and we find $\langle 0|\hat{P}_{\rm flip}|0\rangle \approx 0.257$, hence

$$E_v \approx 1.54 J'_{\rm ring}. \tag{23}$$

Note that this is only the first-order approximation to the vison gap. The naive extension $J'_{\rm ring} \to J_{\rm ring}$ gives a reasonable extrapolation (this is used in Ref. 20), but there is no obvious reason to expect it to be exact.

VI. DISCUSSION

We have demonstrated the equivalence of the generalized Kagome antiferromagnet in the easy-axis limit to an XY ring model, and moreover shown that, with the addition of the u_4 interaction, this model is in a topologically ordered phase at the RK point where $u_4 = J_{\rm ring}$. At this point, the model is a true "short-range" spin liquid, insofar as there is evidently no order or broken symmetry, and all excitations are gapped. But more importantly by computing the vison two-point function we explicitly demonstrate that this spin-liquid phase is fractionalized, and supports $S^z = 1/2$ spinon excitations whose gap is $O(J_z)$. Since the visons also are gapped—with a gap of $O(J_{\rm ring}) = O(J_\perp^2/J_z)$—this spin-liquid state is resilient.

In particular, we may consider a variety of perturbations away from the special soluble model. Following similar arguments to those of Moessner and Sondhi,[15] the spin-liquid state remains the ground state for $A < u_4/J_{\rm ring} < 1$, where A is some unknown dimensionless number. If $A < 0$, then the fractionalized phase persists to the pure ring model, but it is possible that an intermediate phase (or phases) intervene(s) such that $A > 0$. Quantum Monte Carlo simulations could be useful in deciding this question.

Perhaps more novel perturbations consist in deviations from the condition $J_1 = J_2 = J_3$ imposed initially. Again, the presence of a complete gap in the spectrum rules out the destruction of the spin liquid by these perturbations (provided they are weak). It is interesting to consider the effect of small changes in J_1, in particular $J_{1z} = J_z + \delta J_{1z}$, $J_{1\perp} = J_\perp + \delta J_{1\perp}$. Viewing these deviations as perturbations, the change in the easy-axis coupling can be rewritten as

$$\delta\mathcal{H}_z = \delta J_{1z} \sum_{\langle ij\rangle} S_i^z S_j^z = \frac{1}{4} \delta J_{1z} \sum_{\langle ij\rangle} \hat{v}_i \hat{v}_j, \tag{24}$$

where the sum in the first line is taken over nearest-neighbor site of the Kagome lattice, and is equivalent in the second line to a sum over hexagons that are nearest neighbors within either the red or blue honeycomb sublattice of the dual dice lattice. Remarkably, the latter form, Eq. (24) corresponds to a vison hopping or kinetic energy term. Because the visons are already gapped, this clearly will not destabilize the ground state provided the kinetic energy gain remains small relative to the vison gap, i.e., $\delta J_{1z} \lesssim J_{\rm ring}$.

The change in the in-plane exchange can be written

$$\delta\mathcal{H}_\perp = \frac{1}{2} \delta J_{1\perp} \sum_{\langle ij\rangle} (S_i^+ S_j^- + S_i^- S_j^+). \tag{25}$$

For any given bond on the Kagome lattice, the associated term in Eq. (25) raises $S^z_{\hexagon}$ of one hexagon by $+1$ and lowers $S^z_{\hexagon}$ of another by -1. Clearly, acting upon the ground state, this takes the system outside the low-energy manifold of $S^z_{\hexagon} = 0$. Hence due to the large spin gap, it generates only weak second-order virtual processes that renormalize couplings of the effective ring model. However, its effects are more interesting on some of the excited states. In particular, for a single spinon excitation, one has for $\delta J_{1\perp} = 0$ a single magnetized hexagon with $S^z_{\hexagon} = \pm 1$. The spinon is completely localized, and there is an associated degeneracy of these excited states reflecting translational invariance due to the arbitrariness of which hexagon is magnetized. For $J_{1\perp} \neq 0$, this degeneracy is lifted, since Eq. (25) allows the magnetized hexagon to move. Thus Eq. (25) has the effect of giving the spinons some kinetic energy, and the associated states broaden into a band. Indeed, it is possible to formally rewrite Eq. (25) explicitly as a spinon hopping term. As above, because of the existing spinon gap, the ground state is expected to be stable to this perturbation for $\delta J_{1\perp} \lesssim J_z$.

A universal aspect of fractionalized quantum spin liquids is ground state degeneracy when the model is defined on nonsimply connected spaces, which is a signature of the underlying topological order.[8] In the phenomenological square lattice Z_2 gauge theory model, this can be understood as arising due to the option of passing or not passing a vison through each "hole" (noncontractible loop). For instance, on a cylinder (strip with periodic boundary conditions), the square lattice Z_2 gauge theory predicts a twofold ground state degeneracy. Following this argument, and the fact that the Kagome model studied here supports two "colors" of visons, one might naively expect a fourfold ground state degeneracy on the cylinder. This conclusion, however, overcounts the physical degenerate states within the gauge theory. To see this, consider the operator which creates a red or blue vison through the hole of the cylinder,

$$\hat{W}_{r/b} = \overleftrightarrow{\prod_{\langle jl\rangle_{r/b}}^{\infty}} \sigma^x_{jl}, \tag{26}$$

where the product is taken over all red or blue bonds that cut a loop making a full circumference of the cylinder. Both $\hat{W}_r$ and $\hat{W}_b$ commute with the Hamiltonian $\mathcal{H}_{\rm dual}$. Next, consider the duality transformation defining σ^x_{ij}, Eq. (10), carried out on a cylinder. For this tranformation to be invertible (and hence not introduce unphysical decoupled states—and false degeneracies—into the dual description), one must require that the full "string" (product of σ^x_{ij}) around the cylinder equal unity. Hence

$$\hat{W}_r \hat{W}_b = 1 \tag{27}$$

for all physical states. Thus if one finds a ground state $|0\rangle$, another generally inequivalent one may be obtained as $\hat{W}_r|0\rangle$. Due to Eq. (27), however, this state is *identical* to

$\hat{W}_b|0\rangle$, since $\hat{W}_{r/b}^2=1$. The correct result is the same as that of the Z_2 gauge theory: we expect a twofold ground state degeneracy on a cylinder (and, e.g., a fourfold degeneracy on a torus).

We conclude with a comparison of our results to some related discussions in the literature. An interesting aspect of the spinons in the generalized Kagome antiferromagnet we have considered is that they are *bosonic*. Despite the close relation of the topologically ordered state described here to a Gutzwiller-projected superfluid, this is in contrast to what is obtained by such projections on SU(2)-invariant superconducting states,[16] as are naturally suggested by work arising from various slave-fermion theories.[5] As in our work, the large N approaches to the spin liquid[6] also find bosonic spinons.

One of the most intriguing aspects of the numerical results on the spin-1/2 nearest-neighbor Kagome antiferromagnet[13] is the proliferation of a large number of very low-energy singlet excitations. Our approach does not shed too much light on this phenomenon, since the spin liquid ground state found here is in fact fully gapped. It is, however, true that in the easy-axis limit considered above the visons [which are "singlets" under U(1) rotations about the S^z axis] have a much smaller gap ($J_{\rm ring}=J_\perp^2/J_z \ll J_z$) than the spinons. Moreover, it is natural to expect in our effective ring model that as the ratio $u_4/J_{\rm ring}$ is decreased, some confinement transition should occur. At such a confinement transition, provided it is second order, the vison gap must vanish. Should the pure ring model lie near to this critical point, one would indeed expect a large number of low-lying singlet excitations, which on the deconfined side of the critical point are understood as weakly gapped visons.

In light of these results it should be interesting to look at other bosonic ring models, well away from the integrable RK point. For instance, the properties of the pure XY ring model, Eq. (5), defined on four-site plaquettes on diverse lattices (Kagome, square, triangular, etc.) are very poorly understood. With the insight that such terms strongly favor vortex pairing, these seem to be excellent candidate models that might exhibit quantum number fractionalization. A variety of numerical[21] and novel analytical techniques[22] might profitably be applied to these systems.

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