Dissipation and quantum fluctuations in granular superconductivity

Matthew P. A. Fisher
IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598
(Received 24 December 1986)

The effects of dissipation and quantum fluctuations on the onset of superconductivity are discussed. A model for a granular superconductor is considered which consists of a d-dimensional array of resistively shunted Josephson junctions with charging energy incorporating the long-ranged Coulomb interaction. In one dimension the model exhibits a $T=0$ *dynamical* transition into a state with vanishing resistivity at a critical value of the shunt resistance, R_S. Most surprisingly the system is always *statically* disordered even in the superconducting state. For $d \geq 2$ both the dynamical response and static ordering depend sensitively on R_S. Specifically, for R_S less than a critical value of order the quantum of resistance, $h/4e^2$, the dissipation suppresses quantum fluctuations enabling the array to order at $T=0$ for arbitrarily weak Josephson coupling. Above this critical resistance and for weak coupling, the order parameter suffers phase slips due to quantum tunneling driving the system normal.

I. INTRODUCTION

In recent years there has been a resurgence of interest in understanding the onset of superconductivity in disordered systems.[1-5] In granular superconductors it has been appreciated for some time that charging effects might play an important role.[6] Indeed, considerable theoretical effort[7-11] has been devoted to analyzing the effects of quantum fluctuations on the ordering of superconducting arrays.

In a recent experiment on thin granular Sn films, Orr, Jaeger, Goldman, and Kuper[12] observed the presence of an apparently universal resistance threshold for the onset of superconductivity. Only those samples with a normal-state sheet resistance smaller than about $h/4e^2$ became superconducting. They suggested that this threshold was due to the effect of dissipative processes on the quantum fluctuations of the order parameter. Following this suggestion several authors[13,14] analyzed the properties of superconducting arrays with dissipation introduced phenomenologically by shunt resistors between grains. It was found that in the presence of quantum fluctuations both the system's thermodynamics and dynamics depended sensitively on the shunt resistance R_S changing qualitatively at a value of order $h/4e^2$. Reasonable agreement with experiment was obtained thus highlighting the importance of incorporating dissipation in understanding granular superconductors.

The purpose of this paper is to study in further detail the effects of dissipation and quantum fluctuations on the onset of superconductivity in granular systems. Attention will focus on both the *static* ordering and the *dynamical* response, specifically the resistivity. The resistivity, in contrast to the static ordering, is only *well defined* (i.e., not infinite) in the presence of dissipative processes. The analysis is based on a model consisting of an array of resistively shunted Josephson junctions. As usual, the phase of the order parameter on each grain is treated as a quantum operator canonically conjugate to the Cooper-pair number.[15] In contrast to previous theoretical work, though, a form of the charging energy is chosen which incorporates the long-ranged Coulomb interaction. Consequently, the linearized version of the theory supports a dispersionless plasma wave. This has a significant effect on the system's behavior, particularly in the one-dimensional (1D) case. In Sec. II the model is introduced and discussed in detail.

Owing to the choice of charging energy the theoretical analysis for the one-dimensional chain is relatively simple. As shown in Sec. III, several unexpected results emerge in this case. At $T=0$ the chain exhibits a *dynamical* superconducting phase transition into a zero resistance state at a critical value of the dissipation. Specifically, the transition occurs when the shunt resistance R_S is equal to the quantum of resistance $R_Q = h/4e^2$. A simple physical argument, based on the noncommutivity of the phase with the Cooper-pair number, is presented which explains the origin of the transition at R_Q. It is found that the chain is always *statically* disordered, even in the regime with vanishing resistivity $R_S < R_Q$. Thus, most surprisingly, the model exhibits, simultaneously, static short-range order and an infinite conductivity. This result provides a concrete counterexample to the conventional wisdom[16] that static long-range phase coherence is tantamount to vanishing resistivity. Moreover, in the resistive state at $T=0$, the *value* of the chain resistance is completely insensitive to the static correlation length, which can be made arbitrarily large by tuning an appropriate parameter. This apparent decoupling of the static and dynamical behavior arises only because the system's constituents are quantum mechanical.

The array resistivity ρ in higher dimensions is analyzed in Sec. IV by deriving a formally exact Kubo-formula expression. The resistivity is evaluated perturbatively in the Josephson coupling between grains which, while inadequate for studying explicitly a superconducting transition,

does provide nontrivial information. As for the $d=1$ chain, the shunt resistance emerges as a critical parameter. Specifically, for R_S larger than a threshold value of order $h/(2e)^2$, the array resistivity is found to vary nonmonotonically with temperature, initially falling upon cooling but rising again at low temperatures and approaching a finite (nonvanishing) limit as $T\to 0$. The low-temperature behavior is due entirely to phase slippage of the order parameter mediated by quantum tunneling. In contrast, for R_S below the threshold the dissipation is sufficiently strong to suppress this quantum tunneling and $\rho(T)$ falls monotonically upon cooling. The temperature variation of $\rho(T)$ in the two regimes compares favorably with the resistivity measurements by Orr *et al.*[12]

The system's static ordering in higher dimensions is studied in Sec. V by analyzing the quantum-mechanical partition function. Upon combining a perturbative renormalization group with heuristic arguments a static phase diagram is obtained. Not surprisingly, R_S enters once again in an important way. For R_S less than the critical threshold the array orders at $T=0$ for arbitrarily weak Josephson coupling, much as it would were the phases treated classically. In contrast, when R_S is above the threshold, the dissipation is insufficient to suppress the important quantum fluctuations and for weak coupling the system remains disordered down to $T=0$.

Finally, the static and dynamical behavior for $d\geq 2$ is compared. It is speculated that in a particular range of parameters the system might exhibit concurrently static long-ranged order yet a nonvanishing resistivity.

II. THE MODEL

The model consists of a d-dimensional hyper cubical array of superconducting islands interconnected to nearest-neighbor islands by Josephson junction links. Associated with each island is a phase ϕ_r. Amplitude fluctuations of the order parameter are ignored. Each junction is modeled by the so-called resistively shunted junction model[17] with associated capacitance C and shunt resistance R_S, as depicted in Fig. 1. For simplicity each junction is assumed identical. A *classical* Langevin equation for the dynamics of the phases ϕ_r can be written by observing that the total current into a given node (see Fig. 1) must vanish:

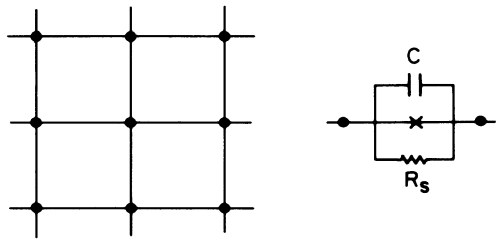


FIG. 1. The model consists of a cubic lattice with the sites representing the superconducting islands and the links the interconnecting Josephson junctions. Each junction is modeled as an ideal Josephson element, with associated critical current $(2e/\hbar)E_J$, shunted by a capacitance C and resistance R_S.

$$\sum_{\pm}\sum_{\nu=1}^{d}\left[C(\dot{V}_r-\dot{V}_{r\pm\nu})+R_S^{-1}(V_r-V_{r\pm\nu})+\frac{2e}{\hbar}E_J\sin(\phi_r-\phi_{r\pm\nu})\pm\xi_{r,\pm\nu}(t)\right]=0\ . \tag{2.1}$$

Here ν denotes a nearest-neighbor unit vector $(\hat{\mathbf{x}}_1,\ldots,\hat{\mathbf{x}}_d)$. The voltage on the island at site r, V_r, is given in terms of ϕ_r by the Josephson relation

$$V_r=(\hbar/2e)\dot{\phi}_r\ . \tag{2.2}$$

The fluctuating noise term $\xi_{r,\pm\nu}$ is associated with current fluctuations through the shunt resistor on the link adjoining site r to $r\pm\nu$ and satisfies the usual white-noise form

$$\langle\xi_{r,\nu}(t)\xi_{r,\nu}(0)\rangle=\frac{2k_BT}{R_S}\delta(t)\ . \tag{2.3}$$

The classical dynamics of the system is fully specified by Eqs. (2.1)–(2.3).

As in the single-junction case,[18,19] in order to incorporate quantum fluctuations of the phases, it is desirable to consider a system-plus-bath Lagrangian which, as far as the motion of ϕ_r is concerned, is equivalent to (2.1),

$$L=\sum_{r,\nu}\left[\frac{C}{2}(V_r-V_{r+\nu})^2+E_J\cos(\phi_r-\phi_{r+\nu})-(V_r-V_{r+\nu})\sum_j\lambda_j^{r,\nu}x_j^{r,\nu}\right]+L_{\rm osc}\ . \tag{2.4}$$

Here the voltage across *each* link, $V_r-V_{r+\nu}$, is coupled to an independent bath of harmonic oscillators, with positions $\{x_j^{r,\nu}\}$. The Lagrangian for the oscillator bath is denoted $L_{\rm osc}$. The motion of these oscillators represent the flow of normal current $I_N^{r,\nu}$ thru the shunt resistors,[19]

$$I_N^{r,\nu}=-\sum_j\lambda_j^{r,\nu}\dot{x}_j^{r,\nu}\ . \tag{2.5}$$

As usual,[18] the coupling constants λ_j are chosen so that the classical equation of motion which follows from (2.4), after elimination of the oscillator degrees of freedom, is precisely (2.1),

$$\sum_j\frac{(\lambda_j^{r,\nu})^2}{m_j^{r,\nu}}\delta(\omega-\omega_j^{r,\nu})=R_S^{-1}\ . \tag{2.6}$$

Here m_j and ω_j refer to the oscillator masses and frequencies, respectively.

The Hamiltonian corresponding to (2.4) is

$$H=\tfrac{1}{2}\sum_{r,r'}(C^{-1})_{rr'}Q_rQ_{r'}-\sum_{r,\nu}E_J\cos(\phi_r-\phi_{r+\nu})+H_{\rm osc}\ , \tag{2.7}$$

$$Q_r=2en_r+\sum_j(\lambda_j^{r,\nu}x_j^{r,\nu}-\lambda_j^{r-\nu,\nu}x_j^{r-\nu,\nu})\ , \tag{2.8}$$

where $\hbar n_r$ is the angular momentum conjugate to the phase ϕ_r. The capacitance matrix $C_{rr'}$ is defined by the condition that $\frac{1}{2}\sum_{r,r'}C_{rr'}V_rV_{r'}$ is equivalent to the first term in (2.4); that is, the Fourier transform of $C_{r-r'}$ satisfies

$$C_k = C \sum_{\nu} 2(1-\cos k_\nu) \ . \tag{2.9}$$

In a fully-quantum-mechanical treatment the commutation relations

$$[\phi_r, n_{r'}] = i\delta_{r,r'} \tag{2.10}$$

are imposed and, to be consistent, the harmonic oscillators are also treated quantum mechanically. The quantized angular momentum n_r can then, up to an additive constant be interpreted as the number of Cooper-pair charges on each island. The total charge Q_r in (2.8) is a sum of the Cooper-pair charge and the normal charge transferred to the island via the neighboring shunt resistors.

Equations (2.6)–(2.10) fully specify the model. The behavior of the system will depend on the temperature T and the parameters which enter the Hamiltonian, E_J, C, and R_S. At times it will be more convenient to express results in terms of the charging energy

$$E_C = (2e)^2/C \tag{2.11}$$

and a dimensionless shunting conductance

$$\alpha = R_Q/R_S, \quad R_Q = h/(2e)^2 \tag{2.12}$$

rather than C and R_S. As usual the quantum behavior described by this model should reduce to the classical limit when $\hbar \to 0$. Slight care is needed in the present case, though, since $\hbar$ already appears in the *classical* equations of motion (2.1). However, since $\hbar$ enters (2.1) only in the combination $\hbar/e$, the classical limit follows by taking $\hbar \to 0$, keeping the ratio $\hbar/e$ fixed. Notice that in this limit E_C vanishes as $\hbar^2$ and α diverges as $\hbar^{-1}$.

The form of the charging energy in the Hamiltonian (2.7) incorporates the long-range nature of the Coulomb interaction. In particular,[20] since $C_k \sim k^2$ for $k \to 0$ from (2.9), the inverse matrix $(C^{-1})_{rr'}$ falls off as $1/r$ in $d=3$. The capacitance matrix itself $C_{rr'}$ is, however, short ranged, corresponding to a nearest-neighbor form in the Lagrangian (2.4). Physically, this near-neighbor form amounts to the assumption that a charge on a given island is completely screened out by the near-neighbor islands alone. This is reasonable for bulk three-dimensional samples. In the past it has been standard practice to study models with diagonal charging energies[7,9,13] $C_{rr'} = C\delta_{r,r'}$. For a bulk Coulombic system this is inappropriate since it corresponds to a complete *absence* of screening by the other islands in the array. For systems with short-range interactions, such as helium, it may be appropriate. The difference between these two forms of the charging energy is most apparent in the system's excitation spectrum which follows by linearizing the sine term in the equation of motion (2.1). For the present model one finds a damped dispersionless plasma wave, which at weak damping has an associated complex frequency[20]

$$\omega(k) = \omega_p + i\gamma/2 \ , \tag{2.13}$$

with $\hbar\omega_p = (E_J E_C)^{1/2}$ and $\gamma = (R_S C)^{-1}$. In contrast, a system with short-range interactions and diagonal capacitance matrix supports a sound wave with linear dispersion at long wavelengths, $\omega(k) \sim k$, and a k-dependent damping, $\gamma(k) \sim k^2$.

It should be emphasized that the nonvanishing of $\omega(k)$ as $k \to 0$ in (2.13) implies that at temperatures smaller than $\hbar\omega_p$ modes with *arbitrarily* long wavelength behave quantum mechanically. The conventional arguments[21] on the irrelevancy of quantum mechanics to a system's finite-temperature critical dynamics, which relies on the fact that $\hbar\omega(k) \ll k_B T$, for $k \to 0$, does *not* apply in this case. A correct analysis of the critical dynamics must necessarily incorporate the effects of quantum fluctuations, even for nonzero temperatures.

When the Hamiltonian (2.7) is treated quantum mechanically, the Heisenberg operators $\hat{\phi}_r(t)$ satisfy an equation of motion, as in (2.1), except the fluctuating current is an operator which depends on the initial state of the oscillator bath. It is given by

$$\hat{\xi}(t) = i \sum_{\alpha} \lambda_\alpha (\hbar\omega_\alpha/2m_\alpha)^{1/2} (\hat{a}_\alpha e^{-i\omega_\alpha t} - \hat{a}^\dagger_\alpha e^{i\omega_\alpha t}) \ , \tag{2.14}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are harmonic-oscillator raising and lowering operators. Averaging over an initial Boltzman distribution $\exp(-\beta H_{\rm osc})$ and using (2.6) one finds

$$\tfrac{1}{2}\langle \{\hat{\xi}(t), \hat{\xi}(0)\} \rangle = (\hbar/R_s) \int_{-\infty}^{\infty} d\omega \, \omega \coth(\beta\hbar\omega/2) e^{-i\omega t} \ , \tag{2.15}$$

which reduces to the white-noise form (2.3) in the $\hbar \to 0$ limit.

In the remaining sections we will be interested in extracting information concerning the system's superconducting behavior. Emphasis will be placed on two physical quantities, the helicity modulus (or superfluid density) (Ref. 22) and the dc resistivity. The helicity modulus, a *static* property of the system, is defined as the coefficient of $1/L^2$ $(L \to \infty)$ in the excess free-energy density due to antiperiodic boundary conditions imposed on a system of length L. Antiperiodic boundary conditions correspond to a *constant* phase difference of π maintained across the system. Adequate information is obtained more easily by studying the static spin-spin correlation function for an infinite system. When this correlation function is short ranged, decaying exponentially with distance, the helicity modulus is zero since the sensitivity to antiperiodic boundary conditions vanish like $\exp(-L/\xi)$. In the statically ordered state the helicity modulus is nonzero.

The dc resistivity, in contrast, is a *dynamical* property of the system. To extract the resistivity it is necessary to perturb the system by, for example, imposing an externally applied current. To be specific, imagine feeding in an external current, $I(t)$, into each of the islands on one plane of the system, say the $x = -L$ plane. Furthermore, consider grounding all of the islands on the $x = +L$ plane, assuring that their phases remain constant in time. In the presence of the external current, the equations of motion (2.1) for those islands on the $x = -L$ plane pick up a forcing term $I(t)$ on the right-hand side, which can be thought of as an external *torque* on the phases. In the Lagrangian (2.4) this corresponds to an additional piece of the form $L_{\rm ext} = -\sum_r V_r(t) \int^t I \, dt$, where the sum runs only over sites on the $x = -L$ plane. Since the phases at $x = L$ are held fixed in time this can alternatively be written

$$L_{\rm ext} = -\sum_r (V_r - V_{r+\hat{x}}) \int^t I(t')dt' \ , \tag{2.16}$$

where the sum now runs over all sites between the two planes and including the $x=-L$ plane. When convenient the $L\to\infty$ limit can then be taken. The corresponding modification to the Hamiltonian (2.7) is that for sites on the $x=-L$ plane the charge operator (2.8) is modified by $Q_r \to Q_r + \int^t I\, dt$.

The resistivity follows from the system's *dynamical* response to this external perturbation, that is, to the external torque on the phases. Of interest is the steady-state voltage response to a time-independent torque. Since the array is homogeneous the voltage drop across any x-direction link is all that is required. The resistivity of the array, ρ, is then defined as $\langle V_r - V_{r+\hat{x}} \rangle / I$ as $I\to 0$ (the lattice spacing has been set to 1).

It should be emphasized that the resistivity, in contrast to helicity modulus, is only well defined in the presence of the dissipative processes, represented by the oscillator baths. The external current feeds energy into the system which, in turn, must be dissipated if the system is to maintain a steady-state response. Classically, it is customary to model dissipative processes by a Langevin description. The oscillator bath approach enables one to extend the Langevin description of dissipation into the quantum regime.

III. ONE-DIMENSIONAL CHAIN: STATICS AND DYNAMICS

It is instructive, initially, to specialize to a one-dimensional chain of junctions. In this case the Lagrangian (2.4) factorizes into a sum of *independent* Lagrangians, one for each junction. The single-junction Lagrangians depend on the phase *difference* between neighboring sites and the associated oscillator baths. This factorization is possible only when the charging energy incorporates the Coulomb interaction, i.e., is of a nearest-neighbor form in the Lagrangian.

Consider first the system's static behavior and, in particular, the static spin-spin correlation function

$$G(r-r') = \langle \cos(\phi_r - \phi_{r'}) \rangle \ . \tag{3.1}$$

In the 1D case this factorizes into a product of single-junction averages,

$$G(r) = (\langle \cos\phi \rangle)^r = e^{-r/\xi} \ , \tag{3.2}$$

where $\phi \equiv \phi_{r+1} - \phi_r$ is the phase difference between any two neighboring islands. In terms of ϕ, the Lagrangian (2.4) describes a quantum pendulum with inertia C, external field E_J, and damping R_S^{-1}. At sufficiently high temperatures the field is unimportant and $\langle \cos\phi \rangle \simeq 0$. Upon cooling the pendulum tends to align with the field, increasing both $\langle \cos\phi \rangle$ and the correlation length ξ in (3.2). If the phase is treated classically $\langle \cos\phi \rangle$ approaches 1, and $\xi\to\infty$, as $T\to 0$; the spins order at $T=0$. However, in the quantum case due to the zero-point fluctuations ξ never diverges, even at $T=0$. Specifically, in the large-E_J limit the zero-point fluctuations of the damped pendulum can be bounded by the fluctuations in a harmonic potential $U(\phi) = -E_J(1-\phi^2/2)$:

$$\langle \cos\phi \rangle \lesssim \exp(-\tfrac{1}{2}\langle \phi^2 \rangle_0) \simeq 1 - \tfrac{1}{2}\langle \phi^2 \rangle_0 \ , \tag{3.3}$$

where $\langle \phi^2 \rangle_0$ is the variance of a *damped* quantum harmonic oscillator,

$$\langle \phi^2 \rangle_0 = (E_C/\hbar) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\omega^2 + \gamma\,|\,\omega\,| + \omega_p^2)^{-1} \ , \tag{3.4}$$

$$\gamma = (R_S C)^{-1}, \quad \hbar\omega_p = (E_J E_C)^{1/2} \ . \tag{3.5}$$

The correlation length ξ, although always finite, will depend sensitively on E_J, C, and R_S. For example, it can be made arbitrarily large by increasing the coupling E_J, or decreasing the charging energy E_C, appropriately. Nevertheless, since $G(r)$ is always short ranged in 1D, the helicity modulus vanishes.

Consider next the system's dynamics and, in particular, the chain's dc resistivity. Even in the presence of an external current (2.16), the total Lagrangian factorizes into a sum of independent, single-junction Lagrangians. The current imposes a relative torque between adjacent phases in the chain which tends to accelerate the phase difference $\phi(t) = \phi_{r+1} - \phi_r$. Since the voltage response is simply a sum of the voltage drops across each of the junctions, the chain resistance is also additive and one need only consider the resistance of a single junction. Previous results[14,23,24] for the dynamics of a single junction can then be applied directly. In terms of the quantum pendulum analog, the external current corresponds to an applied torque. The junction resistance $R(T)$, defined as the ratio of the voltage $V = \hbar\dot{\phi}/2e$ to the current I as $I\to 0$, is essentially the pendulum's angular mobility. Extensive calculations have shown[23-27] that the dimensionless shunting conductor α, defined in (2.12), is a critical parameter.[28] There are two distinct regimes.[23]

(i) For $\alpha > 1$, as the temperature is lowered $R(T)$ falls monotonically and vanishes identically at $T=0$, regardless of the other junction parameters such as E_J and C. The junction is superconducting at $T=0$.

(ii) For $\alpha < 1$, $R(T)$ also falls initially upon cooling, but below some crossover the temperature rises, due to quantum tunneling of the phase, and as $T\to 0$ approaches R_S, the shunt resistance. At $T=0$ all of the conduction through the junction is via the shunting resistor.

At first glance these results are indeed somewhat surprising: the junction's (and chain's) $T=0$ resistance is determined *completely* by the value of R_S, being equal to zero for $R_S < R_Q$ but equal to R_S for $R_S > R_Q$. The coupling energy E_J and capacitance play no role. In Ref. 14 these single-junction results were used, in conjunction with a percolation argument, to explain the observed universal resistance threshold[12] in granular films of Sn and Ga. The observed threshold was, within experimental uncertainties, equal to the theoretical threshold, the quantum of resistance $R_Q = h/4e^2$. The presence of this threshold and the insensitivity to E_J and C can be understood theoretically in terms of the following simple argument.

Consider the charge transported through the shunting resistor when the phase difference ϕ changes by 2π. The motion of ϕ induces a small voltage $V = \hbar\dot{\phi}/2e$, which, in

turn, forces a small current $I_N(t)$, through the shunt resistor. The total charge transported through the resistor is given by

$$\Delta Q_N = \int dt\, I_N(t) = R_S^{-1} \int dt(\hbar\dot{\phi}/2e) \ . \tag{3.6}$$

Under one cycle of motion, the integral $\int \dot{\phi}\, dt$ can be set equal to 2π, giving

$$\Delta Q_N = 2e(R_Q/R_S) = 2e\alpha \ . \tag{3.7}$$

Now imagine a current incident on the junction. The charge can either pass through the Josephson element in units of $2e$ by Cooper-pair tunneling, get carried through the shunting resistor in an average unit of $2e(R_Q/R_S)$, with an accompanying 2π phase slip in ϕ, or remain on the junction plates. Since storing charge at the junction costs (charging) energy, it is energetically favorable to keep this at a minimum. This is achieved by transporting the charge through the junction in the smallest possible units. For $R_S < R_Q$ the Cooper-pair channel is, in this respect, favorable and all of the current should pass through the ideal Josephson element. The junction should carry current with no resistive losses. However, when $R_S > R_Q$ it becomes favorable to transport charge via the shunting resistor, and the Cooper-pair channel will no longer contribute to the conduction. The junction will be in a resistive state, with resistance equal to R_S.

The above argument relies implicitly on the noncommutitivity of the phase with the Cooper-pair number $[\phi, n] = i$, using the fact that the system is 2π periodic in ϕ *and* that the Cooper pairs come in quantized units. Had the phase been treated classically, the angular momentum n (i.e., the Cooper-pair number) would not take on quantized integer values.

The static and dynamical results can best be contrasted by considering the zero-temperature phase diagram, Fig. 2. The dimensionless coupling $J = E_J/E_C$ is plotted versus α. The static correlation function is short ranged over the entire (J,α) plane. The static correlation length

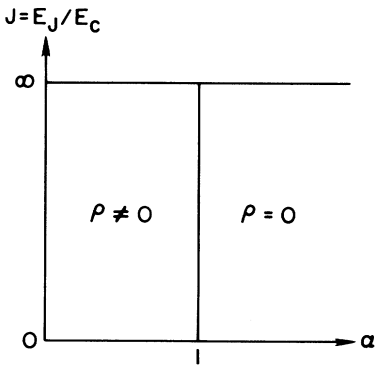


FIG. 2. The $T=0$ phase diagram for the one-dimensional chain. The line $\alpha=1$ is a *dynamical* phase boundary separating the superconducting state, with vanishing resistivity $\rho=0$, from the resistive state $\rho\neq 0$. The chain is *statically* disordered in *both* phases. For $\alpha > 1$ the system exhibits, simultaneously, static short-range order and an infinite conductivity.

does, however, vary appreciably throughout the plane, growing arbitrarily large as J tends to ∞. In marked contrast, the system exhibits a *dynamical* superconducting *phase transition* at a critical value of α, shown as the solid line in Fig. 2. For $\alpha > 1$ the 1D chain has strictly zero resistivity, whereas for $\alpha < 1$ it is resistive. Most surprisingly, for $\alpha > 1$ this model exhibits, simultaneously, static short-range order and an infinite conductivity. Both the helicity modulus and the resistivity vanish. The superconductivity persists for arbitrarily small J (when $\alpha > 1$) despite the fact that in this limit the *static* spin-spin correlation function falls off extremely rapidly with distance.

This coexistence of static short-range order and infinite conductivity is possible only because the phase are quantum degrees of freedom. It can be understood physically by considering the quantum pendulum analog for each junction in the chain. For small external field J, due to the quantum zero-point motion, the *static* probability distribution of ϕ (on the interval $[0,2\pi]$) will be only slightly distorted from uniform, $\langle \cos\phi \rangle$ will be close to zero, and the static correlation function (3.1) short ranged. The junction's resistance, in contrast, is determined not by $\langle \cos\phi \rangle$ but by the pendulum's angular mobility, a *dynamical* feature of the system. In fact, as has recently been shown,[19] this *angular* mobility is formally equivalent to the *linear* mobility of a damped quantum particle moving in an *extended* washboard potential with ϕ defined on the interval $[-\infty, \infty]$. For $\alpha > 1$ and $T=0$ the ground-state wave function in this extended potential is, in fact, localized.[23,24] The discrete translational symmetry of the cosine potential is broken by the coupling to the dissipative degrees of freedom. Consequently, in the presence of an external force the mobility (and the junction's resistance) vanishes identically. The chain of junctions is superconducting.

The behavior of the chain for $\alpha < 1$ and J large is also somewhat surprising. As J increases the static correlation length ξ increases as the phases attempt to order. Although long-range order is only obtained in the unphysical limit $J = \infty$, ξ can be made arbitrarily large. Surprisingly, the dynamics of the system at $T=0$ is unaffected by this (growing) static correlation length. Indeed, for arbitrary J, each junction in the chain is completely resistive (for $\alpha < 1$) with *all* of the incident current passing via the shunt resistors. This regime can also be understood in terms of the quantum pendulum analog. For large J the pendulum aligns with the field with small zero-point fluctuations. The static correlation function falls off slowly with distance. Since J is proportional to the energy barriers through which the pendulum must tunnel in order to rotate, one might have expected the pendulum angular mobility also to depend sensitively on J. This expectation, however, relies on our intuition for *incoherent* quantum tunneling. Crucial in the present case is quantum *coherence*. In the absence of a shunt resistor ($\alpha=0$) the pendulum's states are coherent Bloch waves with bandwidth proportional to $\exp(-J^{1/2})$. Under an external torque the pendulum clearly accelerates indefinitely. The dissipation cuts off this coherent runaway, leading to a finite response, which, at $T=0$, is independent of bandwidth.

The above discussion for the 1D chain has indicated that, upon inclusion of quantum fluctuations, the *static* ordering may be completely unrelated to the *dynamical* response such as the dc resistivity. Indeed the $T=0$ chain for $\alpha>1$, which is statically disordered but nevertheless superconducting, provides a concrete counterexample to the conventional wisdom that static long-range order is equivalent to infinite conductivity. In the remainder of the paper the generalization to higher-dimensional arrays is considered.

IV. ARRAY RESISTIVITY IN HIGHER DIMENSIONS

In this section I consider the dynamical behavior of the array of junctions for $d\geq 2$. Specifically, an exact Kubo formula for the resistivity of the array is obtained. The resistivity is evaluated perturbatively in the coupling strength E_J (to second order) at arbitrary temperature.

The array Lagrangian is given in (2.4). In the presence of an imposed current there is an additional term, Eq. (2.16). It will be convenient to study, rather than this Lagrangian, a closely related model,

$$L_{\rm CL}=\sum_{r,\nu}\left[\frac{C}{2}(V_r-V_{r+\nu})^2+E_J\cos(\phi_r-\phi_{r+\nu})\right] +I\frac{\hbar}{2e}\sum_r(\phi_r-\phi_{r+\hat{x}})+L_d \ , \tag{4.1}$$

where

$$L_d=-\sum_{r,\nu,j}\left[\frac{\hbar\phi_{r,\nu}}{2e}\lambda_j^{r,\nu}\omega_j^{r,\nu}x_j^{r,\nu} +\left[\frac{\hbar\phi_{r,\nu}}{2e}\lambda_j^{r,\nu}\right]^2\Big/2m_j^{r,\nu}\right]+L_{\rm osc} \tag{4.2}$$

and the notation $\phi_{r,\nu}\equiv\phi_r-\phi_{r+\nu}$ has been introduced. Note that in (4.2), as in the original Caldeira-Leggett approach,[18] the phase itself is now coupled to the oscillator bath, rather than the voltage, $\dot{\phi}_r$, as in (2.4). Upon performing a canonical transformation on the oscillators, interchanging position and momenta, the Lagrangian (4.1) differs from the original model only by a total time derivative. Therefore the two models generate equivalent *classical* dynamics for the phase $\{\phi_r\}$. Since $L_{\rm CL}$ is not invariant under $\phi_r\rightarrow\phi_r+2\pi$ a quantum treatment is only possible if the phases are treated as extended coordinates $-\infty<\phi_r<\infty$. It can be shown,[29] however, that the formal expression for the voltage response ($\dot{\phi}_r$) which follows from $L_{\rm CL}$ with extended quantum phases is *identical* to the voltage response of the Lagrangian (2.4) and (2.16) with phases defined on the compact interval $[0,2\pi]$, provided the shunt resistance R_S is not infinite. For purposes of computing the array resistivity it is therefore perfectly valid to use $L_{\rm CL}$ with extended phases.

A Kubo formula for the array resistivity can be obtained by treating perturbatively the external current, which enters the Hamiltonian (associated with $L_{\rm CL}$) in the form

$$H_{\rm ext}=-(\hbar/2e)I\sum_r(\phi_r-\phi_{r+\hat{x}}) \ . \tag{4.3}$$

The resistivity ρ, defined by

$$a^{2-d}\rho\equiv\lim_{I\rightarrow 0}\langle V_r-V_{r+\hat{x}}\rangle/I \ , \tag{4.4}$$

measures the voltage response across an x-direction link. Here, a is the lattice spacing which henceforth is set equal to 1. A straightforward application of linear-response theory then gives[30]

$$\rho=\frac{R_Q}{2\pi}\lim_{\omega\rightarrow 0}\lim_{k\rightarrow 0}i\omega k_x^2D_{21}(\mathbf{k},\omega+i0^+) \tag{4.5}$$

where $D_{21}(\mathbf{k},\omega)$ is the space-time Fourier transform of the retarded phase-phase correlation function

$$iD_{21}(r-r',t-t')=\Theta(t-t')\langle[\hat{\phi}_r(t),\hat{\phi}_{r'}(t')]\rangle \ . \tag{4.6}$$

Here, $\hat{\phi}_r(t)$ is a Heisenberg operator in the absence of the external current, and the angular brackets refer to an equilibrium expectation value with respect to the system-plus-bath Hamiltonian [corresponding to $L_{\rm CL}$ in (4.1)].

The above application of linear-response theory should be contrasted with the more conventional approach which expresses the *conductivity* (ρ^{-1}) as a current-current correlation function. Although this approach must clearly give the same answer as (4.5), as can in fact be verified to order E_J^2 by direct calculation,[31] it is computationally less convenient since the oscillator degrees of freedom enter explicitly via the *normal* current operator (2.5). In contrast, the above expressions for ρ do *not* depend explicitly on correlation functions for the bath degrees of freedom.

In general, the evaluation of the correlation function (4.6) is an exceedingly complicated task due to the nonlinearities associated with the cosine term in (4.1). Since the remaining terms in the model are quadratic the correlation function can be computed in the $E_J=0$ limit. The aim here is to compute the resistivity to leading nonvanishing order in powers of E_J. Such a calculation was recently performed for a single junction using Feynman-Vernon theory[23] and gave nontrivial information. The single-junction results have since been rederived using an elegant and compact Keldysh formalism.[27] I now generalize the Keldysh approach to the array. The calculation will be sketched briefly. The interested reader is referred to Ref. 27 for more details.

In the Keldysh approach one introduces a time contour C with two branches, the first running from $-\infty$ to ∞ and the second from $+\infty$ back to $-\infty$. Heisenberg operators, such as $\hat{\phi}_r(t)$, acquire an additional index $\beta=1,2$ to indicate which segment of the contour they lie on. A time-ordering operator T_C is also introduced which orders along the contour $-\infty\rightarrow+\infty\rightarrow-\infty$; for example,

$$T_C\phi_{r,1}(t)\phi_{r,2}(t')=\phi_{r,2}(t')\phi_{r,1}(t)$$

for all t,t'. It is useful also to perform a rotation in path-index space and define operators

$$\psi_{r,1}=(\phi_{r,1}-\phi_{r,2})/\sqrt{2}$$

and

$$\psi_{r,2}=(\phi_{r,1}+\phi_{r,2})/\sqrt{2} \ .$$

As in Ref. 27, I introduce a generating functional which can be used to generate expectation values as in (4.6), namely

$$\Omega[\mathbf{F}]=\left\langle T_C \exp\left[i\int_{-\infty}^{\infty} dt \sum_{r,\beta} F_\beta(r,t)\psi_{r,\beta}(t) \right] \right\rangle \ . \tag{4.7}$$

For convenience it is assumed in taking this expectation value that in the distant past, $t=-\infty$, all of the phases are aligned $\phi_r(-\infty)=0$ and the oscillator baths are in a Boltzmann distribution at temperature T. Various correlation functions can be obtained from Ω by functional differentiation with respect to $\mathbf{F}$,

$$D_{\beta\beta'}(r-r',t-t')=i\frac{\delta^2\Omega[\mathbf{F}]}{\delta F_\beta(r,t)\delta F_{\beta'}(r',t')}\bigg|_{\mathbf{F}=0} \ . \tag{4.8}$$

From the definition (4.7) it follows that D_{21} corresponds to the retarded correlation function defined in (4.6). Moreover, $D_{12}(r,t-t')=D_{21}(r,t'-t)$, $D_{11}=0$, and

$$iD_{22}(r-r',t-t')=\langle\{\phi_r(t),\phi_{r'}(t')\}\rangle \ . \tag{4.9}$$

The generating functional can, in turn, be related to the *bare* generating functional when $E_J=0$, denoted Ω^0. One finds formally

$$\Omega[\mathbf{F}]=\exp\left[iS_J\left[-i\frac{\delta}{\delta\mathbf{F}} \right] \right]\Omega^0[\mathbf{F}] \ , \tag{4.10}$$

with

$$S_J[\psi]=-(2E_J/\hbar)\sum_{r,\nu}\int dt \sin\left[\frac{1}{\sqrt{2}}(\psi_{r,1}-\psi_{r+\nu,1}) \right] \times\sin\left[\frac{1}{\sqrt{2}}(\psi_{r,2}-\psi_{r+\nu,2}) \right] \ . \tag{4.11}$$

The bare generating function, which can be computed exactly since the Hamiltonian is quadratic when $E_J=0$, takes the form

$$\Omega^0[\mathbf{F}]=\exp\left[-\frac{i}{2}\int dt\,dt' \sum_{r,r'}\sum_{\beta,\beta'} F_\beta(r,t) \times D^0_{\beta\beta'}(r-r',t-t') \times F_{\beta'}(r',t') \right] \ , \tag{4.12}$$

where $D^0_{\beta\beta'}$ are the correlation functions, as defined in (4.6) and (4.9), evaluated for $E_J=0$. These correlation functions are most readily obtained by solving the system of differential equations (2.1) with $E_J=0$. A spatial Fourier transform diagonalizes the set of equations, leaving a first-order, linear inhomogeneous differential equation for the time dependence of each Fourier component. After integration the averages in (4.6) and (4.9) can be performed readily by using the operator representation of the noise, Eq. (2.14). One finds

$$D^0_{21}(\mathbf{k},\omega)=D^0_{12}(\mathbf{k},-\omega) =2\frac{\pi}{\alpha}[A(k)]^{-1}\frac{\gamma}{\omega(\omega+i\gamma)} \ , \tag{4.13}$$

$$D^0_{22}(\mathbf{k},\omega)=2i\coth(\beta\hbar\omega/2)\mathrm{Im}D^0_{21}(\mathbf{k},\omega) \ , \tag{4.14}$$

and $D^0_{11}=0$. Here, $\gamma=(R_SC)^{-1}$ and $A(k)$ is a k-space representation of the lattice Laplacian,

$$A(k)=\sum_{\nu=1}^{d} 2(1-\cos k_\nu) \ . \tag{4.15}$$

Equations (4.10)–(4.15) are a formally exact representation of the generating function from which the resistivity of the array can, in principle, be extracted. When $E_J=0$ the resistivity of the array should be simply $\rho=R_S$, since in that limit the system reduces to an array of shunt resistors alone. This can readily be checked by inserting (4.13) into (4.5).

A perturbation expansion in powers of E_J can be developed by expanding the exponential in (4.10). All details are shown in the Appendix. Specifically, it is shown there that, regardless of the lattice structure, the term linear in E_J vanishes and the leading correction to the resistivity is of order E_J^2. Moreover, for a hypercubic lattice all odd terms in E_J can be shown to vanish. Physically, this results from the fact that on a cubic lattice the Hamiltonian is invariant under $E_J\rightarrow -E_J$ with a corresponding shift $\phi_r\rightarrow\phi_r+\pi$ for r on one sublattice. Since the correlation function (4.6) is invariant under this transformation, the resistivity should be an even function of E_J.

To order E_J^2 the final result for the array resistivity is

$$\frac{\rho}{R_S}=1-\frac{2\pi}{\alpha}(E_J/\hbar)^2\int_0^\infty t\Phi_0^{(z)}(t)dt+O(E_J^4) \ , \tag{4.16}$$

$$\Phi_0^{(z)}(t)=\mathrm{Im}\exp\left[\frac{2}{\alpha z}[iQ_1(t)-Q_2(t)] \right] \ , \tag{4.17}$$

where $Q_2(t)$ is defined by

$$Q_2(t)=\int_0^\infty d\omega\,\omega^{-1}(1-\cos\omega t)\coth(\beta\hbar\omega/2)f(\omega/\gamma) \ , \tag{4.18}$$

$Q_1(t)$ being the $\beta\rightarrow 0$ limit of $\frac{1}{2}\hbar\beta\partial_t Q_2(t)$ and $f(x)=(1+x^2)^{-1}$. For a cubic lattice the parameter z is equal to d, the lattice dimensionality, but more generally can be interpreted as the ratio

$$z=R_S/\tilde{R}_S \ , \tag{4.19}$$

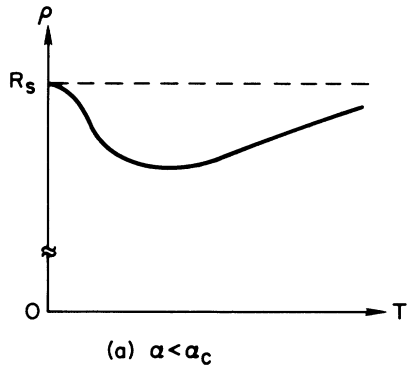
where $\tilde{R}_S$ is the effective resistance measured *between* two adjacent nodes in a network of resistors R_S. For the 1D chain the resistivity reduces to the known single-junction results[23,27] as expected, since in that case the dynamics of each junction is independent as noted in Sec. III.

The parameter z arises in (4.17) as a k-space sum over all of the modes of the system, which for a cubic array takes the form

$$z^{-1}=N^{-d}\sum_k 2(1-\cos k_x)/A(k) \ , \tag{4.20}$$

where N is the number of sites in the array and the lattice Laplacian $A(k)$ is defined in (4.15). From this one sees that each of the modes, including those at arbitrarily long wavelengths, contribute in the same way to the resistivity, at least at order E_J^2. This would not be the case for a model with diagonal charging energy. In that case[31] the damping γ in (4.18) is replaced by a k-dependent form $\gamma(k) \sim k^2$. Then at sufficiently long wavelengths $\hbar\gamma(k) \ll k_B T$, and the coth in (4.18) can be expanded for small argument. Since $\alpha \sim 1/\hbar$ these modes contribute an $\hbar$ independent factor in (4.17); in effect, the long-wavelength modes are behaving classically. For the model (2.4), which incorporates the Coulomb interaction, since γ is independent of wavelength [see (2.13)], *all* of the modes behave quantum mechanically for $k_B T \lesssim \hbar\gamma$.

The low-temperature behavior of $\rho(T,\alpha)$ can be inferred by replacing the soft cutoff $f(x)=(1+x^2)^{-1}$ in (4.18) by $f(x)=\exp(-x)$. Then the integrals can be performed analytically for $k_B T \ll \hbar\gamma$, giving a coefficient of the E_J^2 term varying as $T^{-2(1-1/\alpha z)}$. As in the $d=1$ case the low-temperature resistivity depends critically on the dimensionless shunting conductance α. More generally, the integrals can be performed numerically. The resulting resistivity is plotted schematically as a function of T in Fig. 3. For α smaller than a critical value $\alpha_c = 1/z$, the resistivity varies nonmonotonically with temperature, rising below a crossover temperature $\tilde{T}$ and approaching R_S as $T \to 0$. The crossover temperature is given by $\tilde{T} = g(\alpha/\alpha_c) E_C / 12z$ where $g(z)$ is a monotonic function which approaches 0 as $x \to 1^-$ and 1 as $x \to 0$. In contrast, for $\alpha > \alpha_c$, the resistivity decreases monotonically upon cooling. In this case as $T \to 0$ the coefficient of the E_J^2 term diverges indicating a breakdown of the perturbation expansion.



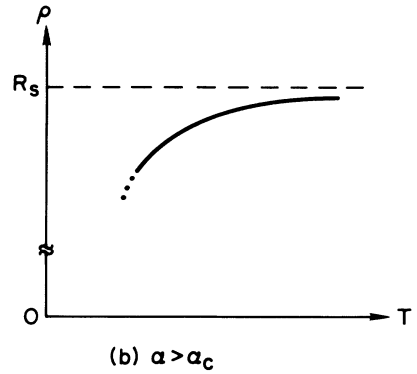


FIG. 3. Plot of the array resistivity $\rho(T)$ to order E_J^2 obtained from Eqs. (4.16)–(4.18) in the text. When $\alpha < \alpha_c = 1/z$ [with z defined in (4.19)] the resistivity is enhanced at low temperatures by a coherent quantum tunneling of the order parameter phase. For $\alpha > \alpha_c$ the quantum coherence is suppressed and the resistivity varies monotonically with temperature. In this case the $T \to 0$ behavior cannot be extracted from the perturbation calculation.

In the classical limit, $\hbar \to 0$ with $\hbar/e$ held fixed, the resistivity reduces to the result

$$\rho_{\rm cl}(T)/R_S = 1 - \frac{z}{2}(E_J/k_B T)^2 e^K K^{-K+1} \gamma(K,K) , \qquad (4.21)$$

where $K = (2e/\hbar)^2 k_B T R_S^2 C/z$ and $\gamma(K,K)$ is the incomplete γ function. This result also follows straightforwardly by solving perturbatively the equations of motions (2.1), treating the phases $\{\phi_r\}$ as classical c numbers.

The classical result (4.21) gives a monotonic temperature dependence much as in the quantum case when $\alpha > \alpha_c$ (it also breaks down as $T \to 0$). It is therefore apparent that the nonmonotonicity of the resistivity for $\alpha < \alpha_c$ is entirely due to the quantum-mechanical character of the phases. Below some crossover temperature the phases $\{\phi_r\}$ can rotate more easily with respect to one another due to quantum-tunneling-induced phase slips, thus enhancing the voltage drop across the array in response to the applied current. Most surprisingly, at $T=0$ (and to leading order in E_J) the phases are as free to rotate as they are in the absence of the Josephson coupling term, $E_J = 0$. The phases are apparently rotating freely and coherently even in the presence of a nonzero coupling.

In the 1D case the behavior of the resistivity for larger E_J can be inferred by studying the system's dynamics in the tight-binding ($E_J \to \infty$) limit.[23–26] As discussed in Sec. III, it is found that for $\alpha > \alpha_c = 1$ the resistivity falls monotonically upon cooling and vanishes identically at $T=0$. For $\alpha < 1$ and in the large $J = E_J/E_C$ limit, ρ drops drastically upon cooling but at exponentially low temperatures,[32] $k_B T \lesssim (E_J E_C)^{1/2} \exp(-J^{1/2})$, rises again due to coherent quantum tunneling of ϕ and as $T \to 0$ approaches R_S. That is, *regardless* of the ratio E_J/E_C, in the 1D case, $\rho(T=0, \alpha < \alpha_c) = R_S$.

For $d \geq 2$ it is tempting to speculate as to the behavior for larger E_J. One is tempted to guess that, as in $d=1$, the result $\rho(T=0, \alpha < \alpha_c) = R_S$ survives beyond the order-E_J^2 term and remains true for arbitrarily strong E_J. Moreover, when $\alpha > \alpha_c$, since the perturbation expansion indicates that quantum tunneling has been suppressed, it is reasonable to suppose that at sufficiently low temperatures the resistivity should drop to zero. In any event, the perturbative results indicate that α enters as a critical parameter in determining the behavior of $\rho(T)$ at low temperatures.

It is instructive to compare the above results for $\rho(T)$ with the resistivity measurements by Orr *et al.* on thin Sn and Ga films. In these measurements the normal-state

sheet resistance R_N emerged as the critical parameter for the film's superconductivity. For $R_N \lesssim R_Q = h/4e^2$ the sheet resistance $R(T)$ dropped monotonically into the superconducting state upon cooling, whereas for $R_N \gtrsim R_Q$, $R(T)$ was generally nonmonotonic, rising, and tending to a finite limit at the lowest temperatures measured.

A comparison with these experiments requires an assumption about the temperature dependence of the shunt resistance R_S. If R_S is associated with a thermally excited quasiparticle resistance, it would depend sensitively on temperature, diverging as $\exp(\Delta/k_B T)$ for $T \to 0$, with Δ the superconducting gap. This would, in turn, imply that the sheet resistance of those films which were not (globally) superconducting would grow exponentially at low temperatures, in marked contrast to that observed experimentally.[12] In practice, however, real Josephson junctions tend to exhibit a sizable shunt conductance even for temperatures well below the gap. In particular, there is indirect evidence based on experiments on single junctions[15(a)] that the effective shunt at low temperatures is roughly equal to the *normal*-state resistance of the junction. While the origin of this shunt resistance is not well understood from a microscopic point of view, based on this evidence it seems plausible to assume that R_S is roughly independent of temperature.

For the model considered here, in $d=2$ the normal-state sheet resistance is simply equal to the shunt resistance R_S. If R_S is indeed temperature independent, the above results then imply that the low-temperature behavior of $\rho(T)$ depends sensitively on the normal-state sheet resistance. Indeed, $\rho(T)$ in the two regimes, Fig. 3, compares favorably with the experiments. Theoretically, the threshold resistance is R_Q/z, with $z=2$ for a square lattice, which is off by a factor of 2 from the experimental value. However, in real granular films the grains do not lie on an ordered lattice and, moreover, the interconnecting junctions will be inequivalent so that close quantitative agreement is not expected.

V. ORDERING IN HIGHER DIMENSIONS

This section is devoted to a study of the *static* ordering of the d-dimensional array. For this purpose it suffices to analyze the partition function. In Sec. V A a path-integral representation for the quantum partition function is obtained. By combining a perturbative renormalization group with heuristic arguments, a static phase diagram will be constructed in Sec. V B. Finally, a brief discussion of some remaining questions is given in Sec. V C.

A. Partition function

Consider then the quantum-mechanical partition function for the Hamiltonian (2.7), which can be written

$$Z = \mathrm{Tr}_{\mathrm{osc}} \int_0^{2\pi} \prod_r d\phi_r \langle \{\phi_r\} | e^{-\beta H} | \{\phi_r\} \rangle' , \tag{5.1}$$

where $\mathrm{Tr}_{\mathrm{osc}}$ denotes a trace over the harmonic-oscillator degrees of freedom and the prime on the propagator $\langle \phi | e^{-\beta H} | \phi \rangle'$ indicates that the phases $\{\phi_r\}$ are restricted to the compact interval $[0,2\pi]$. This propagator can be written in terms of a propagator for noncompact variables,[33] defined on the unrestricted interval $[-\infty,\infty]$, by introducing a set of winding numbers $\{m_r\}$,

$$Z = \mathrm{Tr}_{\mathrm{osc}} \int_0^{2\pi} \prod_r d\phi_r \sum_{\{m_r\}} \langle \{\phi_r + 2\pi m_r\} | e^{-\beta H} | \{\phi_r\} \rangle . \tag{5.2}$$

These winding numbers are integers running from $-\infty$ to ∞. It is convenient to express (5.2) in a path-integral representation over paths $\{\phi_r(\tau)\}$ and $\{x_j^{r,\nu}(\tau)\}$,

$$Z = \int_0^{2\pi} \prod_r d\phi_r \sum_{\{m_r\}} \int_{\phi_r}^{\phi_r + 2\pi m_r} \prod_r D\phi_r(\tau) \int \prod_{r,\nu} \prod_j Dx_j^{r,\nu}(\tau) \exp\left[-(1/\hbar) \int_0^{\beta\hbar} d\tau L_E \right] , \tag{5.3}$$

where L_E is the Euclidian Lagrangian which follows from (2.4) by the replacement $t \to -i\tau$. The harmonic-oscillator path integrals are Gaussian and can be performed exactly. This leaves an effective action just in terms of the paths $\{\phi_r(\tau)\}$.

Upon integration over the harmonic-oscillator degrees of freedom, a term is generated of the form

$$\exp\left[-\sum_{\langle rr' \rangle} (m_r - m_{r'})^2 F_{rr'} \right] , \tag{5.4}$$

with

$$F_{rr'} = \alpha \int_0^{\infty} d\omega\, \omega^{-1} \coth(\beta\hbar\omega/2) . \tag{5.5}$$

In (5.4) the sum is over nearest-neighbor pairs. Notice that the factor $F_{rr'}$ is infinite due to a divergence at low frequencies. Therefore all of the winding numbers $\{m_r\}$ can be set equal, since the configurations in which they differ are suppressed completely. This leaves a remaining sum over a single winding number, m, for the whole array. This sum is a reflection of the fact that the array is ungrounded. To ground the array it suffices to hold the phase of *one* of the islands (say, on the edge of the array) constant in time, so that $V = \hbar\dot{\phi}/2e = 0$. Since the winding number of this grounded island is then identically zero, only the $m=0$ term need be retained. Moreover, since the Lagrangian is invariant under $\phi_r \to \phi_r + 2\pi l_r$ for all integers $\{l_r\}$, the endpoint integration from 0 to 2π in (5.3) can, apart from an overall multiplicative constant, be extended to run from $-\infty$ to ∞. In this way the partition function can finally be expressed in the form

$$Z = \int \prod_r D\phi_r(\tau) \exp(-S) , \tag{5.6}$$

with

$$S = \sum_{\langle rr' \rangle} S_{rr'} , \tag{5.7}$$

$$S_{rr'}=\tfrac{1}{2}(\beta\hbar)^{-1}\sum_m D(\omega_m)\,|\,\phi_{rr'}(\omega_m)\,|^2 - (E_J/\hbar)\int_0^{\beta\hbar} d\tau\cos\phi_{rr'}(\tau)\ , \tag{5.8}$$

and the function D defined by

$$D(\omega_m)=(\alpha/2\pi)\,|\,\omega_m\,|+\hbar\omega_m^2/E_C\ . \tag{5.9}$$

The path integration in (5.6) is over *all* β-periodic paths with no restriction on the endpoints and the sum in (5.7) runs over nearest-neighbor sites only. In (5.8) the shorthand notation $\phi_{rr'}=\phi_r-\phi_{r'}$ has been employed and $\phi_r(\omega_m)$ denotes the coefficients in a Fourier decomposition of the paths $\phi_r(\tau)$,

$$\phi_r(\tau)=(\beta\hbar)^{-1}\sum_{m=-\infty}^{\infty} e^{-i\omega_m\tau}\phi_r(\omega_m),\quad \omega_m=\frac{2\pi m}{\beta\hbar}\ . \tag{5.10}$$

Notice that for the 1D chain the partition function factorizes into a product of single-junction models since, via a simple change of variables, the functional integration over $\{\phi_r(\tau)\}$ can be replaced by an integration over (independent) paths corresponding to the difference between neighboring islands, $\phi_{r,r+1}=\phi_r-\phi_{r+1}$. For $d\geq 2$ the paths $\phi_{rr'}(\tau)$ with r and r' nearest neighbors do *not* form an independent set.

B. Static ordering

Consider the equal-time correlation function $G(r)$, defined in (3.11), which gives a measure of the static stiffness of the phase. When $G(r)$ is long ranged, the helicity modulus and superfluid density are nonzero. To study the ordering of $G(r)$ it is useful to consider a spin-wave approximation. For large E_J one expects that typically $[\phi_r(\tau)-\phi_{r+v}(\tau)]^2\lesssim 1$, so that the cosine term in (5.8) can be expanded for small argument. Retaining the quadratic term only, the action takes the form

$$\tilde{S}=\tfrac{1}{2}(\beta\hbar)^{-1}\sum_m\int_k A(k)B(\omega_m)\,|\,\phi(k,\omega_m)\,|^2\ , \tag{5.11}$$

where $A(k)$ is the lattice Laplacian defined in (4.15) and

$$B(\omega_m)=(E_J/\hbar)+(\alpha/2\pi)\,|\,\omega_m\,|+\hbar\omega_m^2/E_C\ . \tag{5.12}$$

As $k\to 0$, $A(k)$ vanishes and there is no stiffness in the τ direction. Within the spin-wave approximation, $G(r)$ can be evaluated, giving

$$\tilde{G}(r)=\exp\left[-(1/K_1)\int_k A^{-1}(k)(1-\cos k\cdot r)\right]\ , \tag{5.13}$$

where K_1^{-1} is the variance of a damped quantum harmonic oscillator,

$$K_1^{-1}=\langle\phi^2\rangle_0=(\beta\hbar)^{-1}\sum_{m=-\infty}^{\infty} B^{-1}(\omega_m)\ . \tag{5.14}$$

If K_1 is interpreted as an effective coupling constant, the form (5.13) is identical to that for a spin-wave analysis of a *classical* x-y model, also in d dimensions. As in the classical case, for $d>2$ the function $G(r)$ is long ranged, falls to zero algebraically in $d=2$, and is short ranged for $d<2$. Thus the helicity modulus is nonvanishing for $d\geq 2$, but vanishes in $d=1$, even in the spin-wave state. The result (5.13) indicates that once the system has statically ordered into the spin-wave state, decreasing the temperature further reduces $\langle\phi^2\rangle_0$ and thus *enhances* the order.

The spin-wave approximation should be valid provided the nearest-neighbor phase-difference fluctuations are small with respect to 1. These fluctuations can be estimated using the spin-wave action (5.11), giving

$$\langle[\phi_r(\tau)-\phi_{r+v}(\tau)]^2\rangle=(zK_1)^{-1}\ , \tag{5.15}$$

where z is the lattice coordination number defined in (4.19). Thus when $(zK_1)^{-1}\ll 1$ it is reasonable to conclude that for $d\geq 2$ the system will be statically ordered. At $T=0$ this inequality is satisfied when $J=E_J/E_C\gg 1$, regardless of α. It is tempting to speculate that the transition into the disordered case occurs when the spin-wave approximation breaks down, that is, roughly,

$$\langle\phi^2\rangle_0=1\ . \tag{5.16}$$

Indeed, precisely this criterion was proposed by Abeles,[6] and has since been used extensively by others.[34] When applied here (5.16) gives a static boundary which depends only weakly on the dissipation α. In particular, the resulting $T=0$ phase boundary, $J(\alpha)$, is a smooth function of α, approaching zero as $\alpha\to\infty$ and a nonzero constant as $\alpha\to 0$. No special behavior is seen at the critical value α_c, which emerged in the resistivity, Sec. IV. It will be argued below that, in fact, the criterion (5.16) gives qualitatively *incorrect* results for the static boundary, particularly when $\alpha>\alpha_c$.

It is worthwhile noting that the above spin-wave analysis suggests that the static ordering transition will be in the universality class of the d-dimensional *classical* x-y model, even when $T=0$ in the quantum system. The reason the $T=0$ quantum system will not order like a $d+1$ classical system is due to the absence of stiffness in the τ direction: the Gaussian kernel in (5.11) varies as $k^2[1+O(\omega)]$ and vanishes as $k\to 0$ regardless of ω. This lack of stiffness is a reflection of the long-range nature of the Coulomb interaction which has been built into the model (2.4) from the start. In contrast, a model with diagonal charging energy, appropriate for a system with short-range interactions, has a Gaussian kernel of the form $k^2+\omega^2$ and presumably orders[11] like a $d+1$ classical system at $T=0$.

I now describe an approach, based on a perturbative renormalization group, which enables an approximate construction of the static phase diagram. The resulting phase diagram is very similar to that obtained for a closely related model by Chakravarty *et al.*,[13] who used a variational approach. The idea is to attempt to generate an effective *classical* partition function from the quantum partition function (5.6). One can then apply prior knowledge about the ordering of classical x-y models. To this end consider splitting each path $\phi_r(\tau)$ in (5.6) into a zero-frequency component and a fluctuating component,

$$\phi_r(\tau)=\phi_r^0+(\beta\hbar)^{-1}\sum_{m\neq 0} e^{-i\omega_m\tau}\phi_r(\omega_m)\equiv\phi_r^0+\phi_r^f(\tau)\ . \tag{5.17}$$

Similarly, split the functional integration in (5.6) into a product of a simple integration over ϕ_r^0 and a functional integration over the fluctuating pieces of the path,

$$\int D\phi_r(\tau) \rightarrow \int_{-\infty}^{\infty} d\phi_r^0 \int D\phi_r^f(\tau) \ . \tag{5.18}$$

If the fluctuating pieces (i.e., the nonzero Matsubara frequencies) are integrated out, an effective *classical* theory in terms of the zero-frequency c numbers $\{\phi_r^0\}$ remains.

While straightforward in principle, this approach is entirely nontrivial in practice. However, progress can be made if one is willing to work perturbatively in E_J. In particular, if the exponential of the cosine term in the action (5.8) is expanded to linear order, the remaining functional integration is Gaussian and can be performed. Integrating out the fluctuating pieces and reexponentiating gives a classical ferromagnetic x-y model,

$$Z_{\rm cl} = \int_{-\infty}^{\infty} \prod_r d\phi_r^0 \exp\left[\sum_{r,r'} K_0 \cos(\phi_r^0 - \phi_{r'}^0)\right] , \tag{5.19}$$

with an effective (renormalized) dimensionless coupling constant given by

$$K_0 = \beta E_J \langle \cos\phi_{rr'}^f(\tau) \rangle_0 \ . \tag{5.20}$$

Here the angular brackets refer to an average over the fluctuating ($\omega_m \neq 0$) degree of freedom, weighted by $\exp(-S_0)$, where S_0 denotes the quadratic terms in the action (5.7) and (5.8).

This effective classical model could alternatively have been obtained by performing a temporal renormalization group; that is, integrating out a shell of high-frequency modes, rescaling in τ, and obtaining an effective action with renormalized couplings. After flowing for a time $\beta\hbar$ one obtains an effective classical model with appropriately renormalized coupling constants. If the renormalization group is implemented perturbatively in E_J, the final renormalized model coincides precisely with (5.18) and (5.19).

Performing the Gaussian average in (5.20), one finds

$$K_0 = \beta E_J \exp\left[-\frac{1}{2z}(\beta\hbar)^{-1} \sum_{m\neq 0} D^{-1}(\omega_m)\right] , \tag{5.21}$$

with z defined in (4.19). In the $T \rightarrow \infty$ limit the sum can be evaluated, giving

$$K_0 = \beta E_J [1 - (\beta E_C)/24z + \cdots], \quad k_B T \gg \alpha E_C, E_C \tag{5.22}$$

where the relative corrections are of order $E_C/k_B T$. In the low-temperature limit one finds

$$K_0 \sim (E_J/k_B T)(k_B T/\alpha z E_C)^{1/\alpha z}, \quad k_B T \ll \alpha E_C \ . \tag{5.23}$$

Low temperature corresponds to the scaling regime in the temporal renormalization-group approach. Indeed, taking a logarithmic derivative with respect to temperature in (5.23) gives the (linear) flow equation

$$\frac{\partial K_0}{\partial \ln T} = [(\alpha z)^{-1} - 1] K_0 \ . \tag{5.24}$$

At high temperatures the coupling constant behaves classically, $K_0 \sim 1/T$. In the low-temperature regime, where quantum effects dominate, the behavior of $K_0(T)$ depends critically on α. In fact, as in the dynamical calculation of the resistivity, the behavior changes sharply at $\alpha = \alpha_c = 1/z$. For $\alpha < \alpha_c$, $K_0(T)$ is nonmonotonic, vanishing in the zero-temperature limit, whereas for $\alpha > \alpha_c$, $K_0(T)$ grows monotonically upon cooling and diverges as $T \rightarrow 0$. Strictly speaking, this diverging behavior cannot be inferred from (5.21) or (5.23) since these expressions were derived perturbatively in E_J. It is most plausible to assume, however, that, for $\alpha > \alpha_c$, $K_0(T)$ continues to increase under the renormalization-group flows, even for large K_0 outside the perturbative regime (in one dimension this assumption is borne out by calculations in the large-E_J limit). One expects that the qualitative behavior of K_0 in (5.21) should be valid throughout the whole temperature range.

The behavior of K_0 can be understood in terms of the fluctuating paths $\phi_r(\tau)$ that enter into the original partition function (5.6). At high temperatures (small $\beta\hbar$) the paths do not have ample time to fluctuate appreciably and are essentially independent of time; the time integration in (5.8) can then be trivially performed, giving $K_0 = \beta E_J$. Upon cooling the paths have more time to fluctuate and can, in particular, phase-slip (rotate) by 2π as they move in imaginary time. These 2π phase slips tend to decouple the cosine of the zero-frequency components ϕ_r^0 of near-neighbor sites, reducing the effective coupling K_0. At low temperatures there is a competition between the entropy of the phase slips and the logarithmic attraction (repulsion) between phase slips of different (the same) sign, mediated by the $\alpha|\omega|$ term in (5.8). For $\alpha < \alpha_c$ the entropy dominates, there is a predominance of phase slips at low temperatures and $K_0(T)$ is driven to zero as $T \rightarrow 0$. In contrast, when $\alpha > \alpha_c$ the logarithmic interaction dominates and the phase slips tend to occur only in "charge-neutral" pairs. In this case $K_0(T)$ increases monotonically upon cooling.

For $d \geq 2$ the effective nearest-neighbor classical model (5.19) will order when K_0 is equal to some constant of order 1. The associated phase transition is that of a classical d-dimensional x-y model. Strictly speaking, however, the nearest-neighbor form is valid only to leading order in E_J. At higher order, upon integration over the nonzero Matsubara modes, further-neighbor interactions are generated. In the disordered state at high temperatures one expects these interactions to fall off with distance, presumably somewhat more rapidly than the spatial correlations present in $\{\phi_r^0\}$. Inclusion of these additional interactions will, of course, modify the precise location of the transition. Nevertheless, one should be able to infer *qualitatively* the location of the static boundary by examination of the near-neighbor coupling only. Specifically, upon cooling, one expects a transition into the ordered state to occur at a temperature T_c satisfying

$$K_0(T_c) = K_c \ , \tag{5.25}$$

with K_c a constant of order 1 and K_0 defined in (5.21).

For $\alpha \geq \alpha_c$, where $K_0(T)$ increases monotonically upon cooling, criterion (5.25) is unambiguous. In contrast, when $\alpha < \alpha_c$, since $K_0(T)$ is nonmonotonic, (5.25) either has no solutions, in which case the array is disordered

down to $T=0$, or two solutions. In the latter event does the lower solution correspond to a reentrant transition? I do not believe so. For temperatures below the upper solution, the effective interactions generated in the classical model will be of infinite range (since the system is ordered), so that the near-neighbor form (5.19) cannot be employed. Moreover, the spin-wave analysis indicated that once in the ordered (or spin-wave) state, further cooling tended to enhance the order. Therefore I expect the system to remain ordered at all temperatures below the upper solution of (5.25). In constructing the static boundary it should then be understood that when two solutions exist, only the higher-temperature solution corresponds to a transition.

The criterion (5.25) gives a single constraint among the three dimensionless variables $J=E_J/E_C$, α, and T/E_C. The static boundary can thus be represented as a two-dimensional surface in this space. The phase with temperatures above (below) the boundary is statically disordered (ordered). A $T=0$ slice of this surface is shown schematically in Fig. 4. The ordered state corresponds to large J and α. The constant J_0 in Fig. 4 is expected to be of order 1, although its *precise* value cannot be inferred reliably from the criterion (5.25). It should be emphasized that this $T=0$ static boundary, $J(\alpha)$, differs *qualitatively* from that obtained from the criterion $\langle\phi^2\rangle_0=1$, Eq. (5.16). Specifically, (5.16) gives a nonzero $J(\alpha)$, even for $\alpha>\alpha_c$.

The phase diagram obtained from (5.25) is similar to that derived by Chakravarty *et al.*[13] for a closely related model. The variational method employed by these authors can be trivially modified to the present model. The resulting static boundary for $d\geq 2$ is, in fact, very similar to Fig. 4, with the same critical value α_c entering. However, when applied to the $d=1$ chain the variational method gives several incorrect results. Specifically, a transition is predicted at nonzero temperatures, in conflict with the results of Sec. III. Moreover, at $T=0$ a phase boundary as in Fig. 4 is obtained (with $\alpha_c=1$) which for large J does not coincide with the (dynamical) phase diagram obtained in Sec. III, Fig. 3.

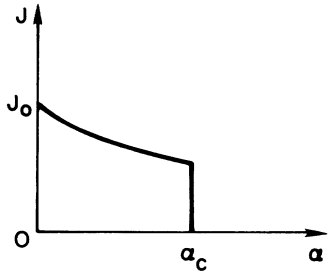


FIG. 4. Schematic plot of the $T=0$ static phase diagram for $d\geq 2$, obtained from the criterion (5.25). The statically ordered phase corresponds to large J and α.

C. Discussion and speculations

The $T=0$ phase diagram, Fig. 4, indicates that for $\alpha>\alpha_c$ the array orders at arbitrarily small coupling E_J. This can be traced to the fact that the effective coupling constant $K_0(T)$ increases monotonically upon cooling, so that regardless of how small E_J is, K_0 exceeds unity as $T\to 0$. The behavior of $K_0(T)$ in this regime is rather similar to the classical result, $K_0^{\rm cl}=\beta E_J$, obtained by taking $\hbar\to 0$ in (5.21). In contrast, for $\alpha<\alpha_c$, $K_0(T)$ crosses over from a high-temperature classical regime to a regime dominated by quantum phase slips (in imaginary time) which drive K_0 to zero as $T\to 0$. For small coupling E_J the quantum effects set in before the system has a chance to order. For sufficiently strong coupling, though, the system orders in the classical regime at temperatures above where quantum phase slips dominate. As evident from the spin-wave analysis, in this case quantum fluctuations, which emerge at low temperatures, are insufficient to disorder the system even as $T\to 0$.

It is instructive to contrast these results for the $d\geq 2$ static ordering with the resistivity, calculated dynamically in Sec. IV. This is particularly important in light of the $d=1$ results showing that the static and dynamical ordering were unrelated. Recall from Sec. IV that the resistivity $\rho(T)$ to second order in E_J was found to depend sensitively on α, varying nonmonotonically and approaching R_S as $T\to 0$ for $\alpha<\alpha_c$ and monotonically decreasing upon cooling for $\alpha>\alpha_c$. The threshold value α_c is precisely the same as that which appears in the $T=0$ static phase diagram, Fig. 4. The statically disordered phase, $\alpha<\alpha_c$ and small E_J/E_C, corresponds to the dynamically resistive state. In the statically ordered phase, $\alpha>\alpha_c$, the dynamical results for $\rho(T)$ break down as $T\to 0$; however, a vanishing resistivity in this limit seems quite plausible (see Fig. 3). In contrast to the $d=1$ case, these results for $d\geq 2$ are *not* inconsistent with the usual assumption that static ordering and vanishing resistivity coincide. There is, however, a remaining question along these lines.

Is it possible that in addition to the static ordering transition, an *additional* transition is manifest in the partition function? The answer is most probably yes, at least at $T=0$. Consider the statically ordered phase (large J) at $T=0$. In this phase *each* (imaginary) time slice in the path-integral representation is ordered, $\langle\cos\phi_r(\tau)\rangle\neq 0$. As a given phase ϕ_r moves in time, however, it will still undergo an occasional 2π slip with respect to its neighbors. In the absence of dissipation, $\alpha=0$, each slip will be essentially independent of those at earlier times, so that at long times the phase difference between neighbors, $\phi(\tau)$, will undergo a random walk $\phi(\tau)\sim\tau^{1/2}$, where $\phi(\tau=0)=0$ has been assumed. For weak dissipation, $\alpha\ll 1$, since successive slips interact logarithmically, one has $\phi(\tau)\sim\ln\tau$. Now for a *single* junction[23] (or the $d=1$ chain) there exists a transition at $\alpha=1$, with $\phi(\tau)\sim\ln\tau$ for $\alpha<1$ and $\phi(\tau)$ approaching a finite constant for $\alpha>1$. This transition is precisely the *dynamical* transition described in Sec. III. For the array one also expects, at sufficiently strong dissipation, a *temporal* phase transition into a state where $\phi(\tau)\to$const as $\tau\to\infty$. Indeed, the flow equations (5.24) would seem to suggest that this tran-

sition occurs at $\alpha=\alpha_c$. In contrast to the static ordering transition, which corresponds to a breaking of the symmetry under global rotations, this transition would represent a broken local symmetry in the ground-state wave function. Specifically, the Hamiltonian (2.5) is invariant under the local transformation $\phi_r \rightarrow \phi_r + 2\pi m$ for *each* site. By analogy with the single junction,[23] I suspect that this discrete translational symmetry is broken and a particular integer value of m selected out.

How would the presence of such an additional transition manifest itself in the behavior of the array? In $d=1$ the transition is evident directly in the *dynamics,* with $\rho(T=0)=0$ for $\alpha>1$ and $\rho(T=0)=R_S$ for $\alpha<1$. One might be tempted to speculate that for $d\geq 2$ this also is the case, with a resistive state for $\alpha<\alpha_c$ and a superconducting state ($\rho=0$) for $\alpha>\alpha_c$. Were this the case, the regime $\alpha<\alpha_c$ and J large would be rather exotic, exhibiting simultaneously static long-ranged order yet a finite conductivity. Clearly, further work is necessary to establish more carefully and clarify further the role played by such an additional transition.

An additional and perhaps related theoretical question concerns the effects of the nonvanishing dispersion, $\omega(k\rightarrow 0)=\omega_p$, on the low-temperature $k_B T \lesssim \hbar\omega_p$ dynamical critical behavior. Since all of the modes, even as $k\rightarrow 0$, behave quantum mechanically, the $T\neq 0$ dynamical critical phenomena might possess interesting nonclassical features.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge stimulating and clarifying conversations with A. T. Dorsey, D. S. Fisher, M. E. Fisher, G. Grinstein, A. J. Leggett, and O. Martin.

APPENDIX

In this Appendix a formal perturbation expansion for the array resistivity (in powers of E_J) is developed. The approach follows closely that contained in Ref. 27 for the single-junction resistance. The starting point is the exact expression for the generating function, (4.10)–(4.15). The idea is to simply expand the exponential in (4.10) as a power series in E_J. It is convenient to decompose the sines in (4.11) into a sum of two exponentials by introducing a set of charges σ_j ($=\pm$) and ϵ_j ($=\pm$), where j runs from one to n, at nth order in the expansion. Upon defining a charge density

$$\rho_\beta(r,t)=\frac{1}{\sqrt{2}}\sum_{j=1}^{n}\binom{\sigma_j}{\epsilon_j}\delta(t-t_j)(\delta_{r,r_j}-\delta_{r,r_j+\nu_j}) , \tag{A1}$$

the exponential in (4.10) can be written

$$\exp\left[iS_J\left[-i\frac{\delta}{\delta \mathbf{F}}\right]\right]=\hat{\sigma}_c \exp\left[\int dt\,\boldsymbol{\rho}(t)\cdot\frac{\delta}{\delta \mathbf{F}(t)}\right] , \tag{A2}$$

where the dot product refers to a sum over r and $\beta=1,2$. Here the operator $\hat{\sigma}_c$, defined by

$$\hat{\sigma}_c=\sum_{n=0}^{\infty}\frac{1}{n!}(iE_J/2\hbar)^n\prod_{j=1}^{n}\left[\sum_{\sigma_j=\pm}\sigma_j\sum_{\epsilon_j=\pm}\epsilon_j\int dt_j\sum_{r_j}\sum_{\nu_j}\right] \tag{A3}$$

performs a sum over different charge configurations. The differential operator on the right-hand side of (A2), when acting on the bare generating functional $\Omega^0[\mathbf{F}]$, simply shifts $\mathbf{F}\rightarrow\mathbf{F}+\boldsymbol{\rho}$,

$$\exp\left[\int dt\,\boldsymbol{\rho}\cdot\frac{\delta}{\delta \mathbf{F}(t)}\right]\Omega^0[\mathbf{F}]=\Omega^0[\mathbf{F}+\boldsymbol{\rho}] . \tag{A4}$$

Therefore the exact generating functional, defined in (4.10), can be written

$$\Omega[\mathbf{F}]=\hat{\sigma}_c\Omega^0[\mathbf{F}+\boldsymbol{\rho}] . \tag{A5}$$

The correlation functions of interest, defined via (4.8), can then be expressed as derivatives with respect to the charge density $\boldsymbol{\rho}$, rather than the source fields,

$$D_{\beta\beta'}(r-r',t-t')=i\hat{\sigma}_c\frac{\delta^2\Omega^0\{\boldsymbol{\rho}\}}{\delta\rho_\beta(r,t)\delta\rho_{\beta'}(r',t')} . \tag{A6}$$

Inserting the Gaussian form for $\Omega^0\{\boldsymbol{\rho}\}$, (4.12), into (A6) and performing the functional differentiation gives

$$D_{21}(r-r',t-t')=D^0_{21}(r-r',t-t')-i\int d\tau\,d\tau'\sum_{R,R'}D^0_{21}(r-R,t-\tau)\langle\langle\rho_1(R,\tau)\rho_2(R',\tau')\rangle\rangle D^0_{21}(R'-r',\tau'-t') , \tag{A7}$$

with the definition

$$\langle\langle A[\boldsymbol{\rho}]\rangle\rangle=\hat{\sigma}_c A(\boldsymbol{\rho})\Omega^0[\boldsymbol{\rho}] . \tag{A8}$$

In arriving at (A7) use has been made of the fact that $D^0_{11}=0$ and that[27] $\langle\langle A[\rho_2]\rangle\rangle=A(0)$.

At this stage it is straightforward to evaluate the retarded correlation function D_{21} (and resistivity) perturbatively by simply keeping only those charge configurations corresponding to the first two terms in (A1). The contribution from the $n=1$ term vanishes so that the leading correction to the resistivity is of order E_J^2. The resulting expression is given in the text, (4.16)–(4.18).

The vanishing of the linear term in E_J can be seen quite generally by noting the following charge-neutrality constraint.[23,27] The bare generating function when expressed in Fourier space takes the form

$$\Omega^0[\boldsymbol{\rho}]=\exp\left[-\frac{i}{2}\int\frac{d\omega}{2\pi}\int\frac{d^dk}{(2\pi)^d}\rho_\beta(k,\omega)\times D^0_{\beta\beta'}(k,\omega)\rho_{\beta'}(-k,-\omega)\right] . \tag{A9}$$

Since $iD^0_{22}(k,\omega)$ diverges as $\omega \to 0$ [see (4.14)], the only charge configurations which contribute to the average in (A7) are those satisfying $\rho_2(k, \omega=0)=0$ for all k, or equivalently $\rho_2(r, \omega=0)=0$ for all sites r. Using (A1) this charge-neutrality condition can be written alternatively as

$$\sum_{j=1}^{n} \epsilon_j(\delta_{r,r_j}-\delta_{r,r_j+\nu_j})=0, \quad \forall r \ . \tag{A10}$$

One sees immediately that for $n=1$, regardless of ϵ_1, r_1, and ν_1, the left-hand side of (A10) *cannot* vanish for all sites r. Therefore the contribution to D_{21} linear in E_J vanishes identically. More generally, it is convenient to visualize each term in the parentheses in (A10) as a dipole with positive charge at site r_j and negative charge at a neighboring site $r_j+\nu_j$. The condition (A10) is then equivalent to the constraint that the dipoles must be arranged on the lattice in such a way that *each* site is charge neutral. When $n=2$ the two dipoles must lie head to tail canceling each other's charge. For a cubic lattice it is apparent that charge neutrality can only be maintained when n is even. Therefore the resistivity will be an even function of E_J. In contrast, on a triangular lattice, for example, odd terms in $n\,(>1)$ can and will contribute to the resistivity.

[1]J. M. Graybeal and M. R. Beasley, Phys. Rev. B **29**, 4167 (1984).

[2]A. F. Hebard and M. A. Paalanen, Phys. Rev. Lett. **54**, 2155 (1985).

[3]A. E. White, R. C. Dynes, and J. P. Garno, Phys. Rev. B **33**, 3549 (1986).

[4]A. Kapitulnik and B. G. Kotliar, Phys. Rev. Lett. **54**, 473 (1985).

[5]M. Ma, B. I. Halperin, and P. A. Lee, Phys. Rev. B **34**, 3136 (1986).

[6]B. Abeles, Phys. Rev. B **15**, 2828 (1977).

[7]S. Maekawa, H. Fukuyama, and S. Kobayashi, Solid State Commun. **37**, 45 (1980).

[8]E. Simánek, Solid State Commun. **31**, 419 (1979).

[9]P. Fazekas, B. Mühlschlegel, and M. Schröter, Z. Phys. B **57**, 193 (1984).

[10]L. Jacobs, J. V. José, and M. A. Novotny, Phys. Rev. Lett. **53**, 2177 (1984).

[11]S. Doniach, Phys. Rev. B **24**, 5063 (1981); R. M. Bradley and S. Doniach, *ibid.* **30**, 1138 (1984).

[12]B. G. Orr, H. M. Jaeger, A. M. Goldman, and C. G. Kuper, Phys. Rev. Lett. **56**, 378 (1986). See also H. M. Jaeger, D. B. Haviland, A. M. Goldman, and B. G. Orr, Phys. Rev. B **34**, 4920 (1986).

[13]S. Chakravarty, G. Ingold, S. Kivelson, and A. Luther, Phys. Rev. Lett. **56**, 2303 (1986).

[14]M. P. A. Fisher, Phys. Rev. Lett. **57**, 885 (1986).

[15]There is convincing experimental evidence that in sufficiently small, single, fabricated Josephson junctions the phase behaves as a quantum-mechanical degree of freedom. See (a) S. Washburn, R. A. Webb, R. F. Voss, and S. M. Faris, Phys. Rev. Lett. **54**, 2712 (1985); (b) J. M. Martinis, M. H. Devoret, and J. Clarke, *ibid.* **55**, 1543 (1985); (c) D. B. Schwartz, B. Sen, C. N. Archie, and J. E. Lukens, *ibid.* **55**, 1547 (1985).

[16]See, for example, P. W. Anderson, *Basic Notions of Condensed Matter Physics* (Benjamin/Cummings, Menlo Park, CA, 1984), pp. 229–248.

[17]See, A Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982), Chap. 6.2.

[18]A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983).

[19]W. Zwerger, A. T. Dorsey, and M. P. A. Fisher, Phys. Rev. B **34**, 6518 (1986).

[20]The capacitance matrix in (2.9) corresponds to an interaction between charges varying as $\ln r$ in $d=2$ and linearly in r in $d=1$. A more appropriate choice for real *low*-dimensional charged systems would correspond to a $1/r$ intraction in both cases, leading to a dispersion relation (2.13) varying as $k^{1/2}$ in $d=2$ and $k(\ln k)^{1/2}$ in $d=1$ for $k \to 0$.

[21]See for example, P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[22]M. E. Fisher, M. N. Barber, and D. Jasnow, Phys. Rev. A **8**, 1111 (1973).

[23]M. P. A. Fisher and W. Zwerger, Phys. Rev. B **32**, 6190 (1985); W. Zwerger, Phys. Rev. B **35**, 4737 (1987).

[24]F. Guinea, V. Hakim, and A. Muramatsu, Phys. Rev. Lett. **54**, 263 (1985).

[25]U. Weiss and H. Grabert, Phys. Lett. **108A**, 63 (1985).

[26]C. Aslangul, N. Pottier, and D. Saint-James, Phys. Lett. **111A**, 175 (1985).

[27]U. Eckern and F. Pelzer, Europhys. Lett. **3**, 131 (1987).

[28]In the closely related problem of a single resistively shunted junction in a SQUID ring, the parameter α also enters as a critical parameter. See S. Chakravarty, Phys. Rev. Lett. **49**, 681 (1982); A. J. Bray and M. A. Moore, *ibid.* **49**, 1545 (1982); S. Chakravarty and A. J. Leggett, *ibid.* **52**, 5 (1984).

[29]M. P. A. Fisher (unpublished). For a *single* junction the equivalence between the voltage response of the two Lagrangians has been shown in some detail in Ref. 19.

[30]See, for example, A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971), Sec. 13.

[31]M. P. A Fisher (unpublished).

[32]M. P. A. Fisher, thesis, University of Illinois at Urbana-Champaign, 1986, p. 53.

[33]L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981), Chap. 23.

[34]B. G. Orr, J. R. Clem, H. M. Jaeger, and A. M. Goldman, Phys. Rev. B **34**, 3491 (1986).