

## Roton Fermi liquid: A metallic phase of two-dimensional electrons

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We introduce and analyze a metallic phase of two-dimensional (2D) electrons, the roton Fermi liquid (RFL), which, in contrast to the Landau–Fermi liquid, supports both gapless fermionic and bosonic quasiparticle excitations. The RFL is accessed using a reformulation of 2D electrons consisting of fermionic quasiparticles and  $hc/2e$  vortices interacting with a mutual long-ranged statistical interaction. In the presence of a strong vortex-antivortex (i.e., roton) hopping term, the RFL phase emerges as an exotic yet eminently tractable quantum ground state. The RFL phase exhibits a “Bose surface” of gapless roton excitations describing transverse current fluctuations, has off-diagonal quasi-long-ranged order at zero temperature ( $T=0$ ), but is not superconducting, having zero superfluid density and no Meissner effect. The electrical resistance *vanishes* as  $T \rightarrow 0$  with a power of temperature (and frequency),  $R(T) \sim T^\gamma$  (with  $\gamma > 1$ ), independent of the impurity concentration. The RFL phase also has a full Fermi surface of quasiparticle excitations just as in a Landau–Fermi liquid. Electrons can, however, scatter anomalously from rotonic “current fluctuations” and “superconducting fluctuations.” Current fluctuations induced by the gapless rotors scatter anomalously only at “hot spots” on the Fermi surface (with tangents parallel to the crystalline axes), while superconducting fluctuations give rise to an anomalous lifetime over the entire Fermi surface *except* near the (incipient) nodal points (“cold spots”). Fermionic quasiparticles dominate the Hall electrical transport. We also find three dominant instabilities of the RFL phase: an instability to a conventional Fermi-liquid phase driven by vortex condensation, a BCS-type instability toward fermion pairing, and a (nonpairing) superconducting instability. Precisely *at* the instability into the Fermi-liquid state, the exponent  $\gamma$  saturates the bound,  $\gamma=1$ , so that  $R(T) \sim T$ . Upon entering the superconducting state the rotors are gapped out, and the anomalous quasiparticle scattering is strongly suppressed. We discuss how the RFL phase might underlie the strange metallic state of the cuprates near optimal doping, and outline a phenomenological picture to accommodate the underdoped pseudogap regime and the overdoped Landau–Fermi-liquid phase.

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### I. INTRODUCTION

Despite the appeal of spin-charge separation as an underpinning to superconductivity in the cuprates, there are seemingly fatal obstacles with this approach. Ever since Anderson’s initial suggestion<sup>1</sup> of a spinon Fermi surface in the normal state at optimal doping, there have been nagging questions about the *charge* sector of the theory. The concept of the “holon,” a charge  $e$  spinless boson, was introduced in the context of the doped spin liquid state,<sup>2</sup> and was presumed to be responsible for the electrical conduction. In the simplest theoretical scenario, spin-charge separation occurs on electronic energy scales, thereby liberating the electron’s charge from its Fermi statistics. This idea has been actively investigated over the past 15 years—see Ref. 3 for a review. A pervasive challenge to this perspective, however, is the difficulty of avoiding holon condensation and superconductivity at *inappropriately* high temperatures. In addition, recent theoretical work, which has elucidated the phenomenology of putative spin-charge separated states, has led to further conflicts with observations. One class of theories have shown how spin-charge separation can emerge from a superconducting phase by pairing and condensing vortices.<sup>4</sup> Following this work, a  $Z_2$  gauge theory formulation greatly clarified the nature of fractionalization of electronic (and other) quantum numbers.<sup>5</sup> It has become clear that a necessary requirement for true spin-charge separation in two di-

mensions is the existence of a “vison” excitation with a gap.<sup>6–9</sup> The vison is perhaps most simplest thought of within the vortex pairing picture, as the remnant of an unbound single vortex. If these ideas were to apply to the cuprates, one would expect this gap to be of order the pseudogap scale  $k_B T^*$ . Unfortunately for spin-charge separation advocates, experiments designed to detect the vison and measure its gap<sup>10,11</sup> have determined an unnaturally low upper bound of approximately 150 K for the vison gap in underdoped YBCO. Is this the death knell for spin-charge separation?

In a very recent paper focusing on the effects of ring exchange in simple models of bosons hopping on a two-dimensional (2D) square lattice,<sup>12</sup> we have identified a zero-temperature *normal fluid* phase—(re)named the “exciton Bose liquid” (EBL). In the EBL phase boson–antiboson pairs (i.e., an exciton) are mobile, being carried by a set of gapless collective excitations, while single bosons cannot propagate. The resulting quantum state is “almost an insulator,” with the dc conductivity vanishing as a power of temperature  $\sigma(T) \sim T^\alpha$  with  $\alpha \geq 1$ . This is in contrast to the “strange metallic phase” in the optimally doped cuprates, which is “almost superconducting”<sup>13</sup> with an extrapolated zero resistance at  $T=0$ , as if it were a superconductor with  $T_c=0$ . This phenomenology suggests the need for a nonsuperconducting quantum phase in which the vortices are strongly immobilized at low temperatures.

Motivated by this, we revisit the  $Z_2$  vortex-spinon field theory of interacting electrons,<sup>4,5</sup> in which the  $hc/2e$  vortices and the spinons have a long-ranged statistical interaction mediated by two  $Z_2$  gauge fields. Rather than gapping out single vortices while condensing pairs (which leads to a spin-charge separated insulator),<sup>5</sup> we would like to find a quantum phase in which the vortices are gapless but nevertheless immobile. To this end, we add an additional “vortex-ring” term to the earlier vortex-field theory. This term is effectively a kinetic energy for vortex-antivortex pairs, that is, for rotons. To access the limit of strong roton hopping requires a further reformulation of the  $Z_2$  vortex-spinon field theory, replacing the  $Z_2$  gauge fields by U(1) gauge fields. Similar U(1) vortex-fermion field theories have been explored in Refs. 14. The resulting U(1) vortex-spinon formulation is tractable in the limit of a very strong roton hopping, and describes the “roton Fermi liquid” (RFL) phase, a metallic ground state qualitatively different than Landau’s Fermi liquid phase.

Both the  $Z_2$  vortex-spinon field theory developed in Refs. 4,5 and our U(1) vortex-fermion field theory, described in Sec. II below, are constructed in terms of operators which create the excitations of a conventional BCS superconductor: the Bogoliubov quasiparticles, the  $hc/2e$  vortices, and the collective plasmon mode. This choice of “basis,” however, does not presume that the system is necessarily superconducting at low temperatures, and indeed we intend to employ such a formulation to describe nonsuperconducting states. When superconductivity is present at low temperatures, the formulation will also be employed to describe the “normal” state above  $T_c$ . In these approaches, the Bogoliubov quasiparticle is electrically neutralized,<sup>4</sup> and the resulting “spinon” excitation transported around an  $hc/2e$  vortex within an ordinary 2D BCS superconductor, acquire a Berry’s phase of  $\pi$ . Within the  $Z_2$  and U(1) vortex-spinon formulations, one introduces spinon creation operators  $f_{\mathbf{r}\sigma}^\dagger$ , where we let  $\mathbf{r}$  denote the sites of a 2D square lattice and with  $\sigma$  the spin. Vortex creation operators are also introduced, conveniently represented in a “rotor” representation as  $e^{i\theta}$ , which live on the plaquettes of the 2D lattice. The vortices are minimally coupled to a gauge field, living on the links of the dual lattice. The “flux” in the gauge field describes charge density fluctuations on the original lattice sites, and, for example, encapsulates the plasmon mode inside the superconducting phase.<sup>15</sup> Finally, the long-ranged statistical interaction between the spinons and vortices is incorporated by introducing two Chern–Simons  $Z_2$  or U(1) gauge fields. The important new element in the present paper is the inclusion of a roton hopping term. As we shall see, the RFL phase is readily accessed within the U(1) formulation when the magnitude of this roton hopping term is taken significantly larger than the single vortex hopping strength.

Here, we briefly summarize the main results established in the following sections. In Sec. II the  $Z_2$  vortex-spinon field theory formulation of Ref. 5 is recast in terms of a lattice Hamiltonian. Via a sequence of exact unitary transformations on the Hamiltonian, we demonstrate that it is possible to exchange the  $Z_2$  Chern–Simons gauge fields for their U(1) counterparts. Within a Lagrangian representation of the resulting U(1) Hamiltonian which we employ throughout the

paper, it is possible to choose a gauge for one of the U(1) Chern–Simons fields so that the spinon is recharged, and has finite overlap with a bare electron. (In Appendix D we show that this gauge choice can effectively be made at the Hamiltonian level, and construct a Hamiltonian theory in terms of vortices and fermionic operators which carry the electron charge and have a finite overlap with a bare electron.)

Initially, in Sec. III, we ignore the fermions entirely, and focus on the bosonic charge (or vortex) sector of the theory. A “spin-wave” expansion valid in the presence of a large roton-hopping term, leads to a simple theory which is quadratic—*except* for a single vortex-hopping term. Dropping this vortex-hopping term then leads to a soluble harmonic theory of the “roton-liquid” (RL) phase. In addition to the gapless 2D plasmon, the RL phase is shown to support a “Bose surface” of gapless roton excitations. We compute the Cooper, pair propagator in the RL phase, and show that it exhibits off-diagonal quasi-long-range order (ODQLRO) at zero temperature, but *not* a Meissner effect. The RL phase exhibits a high degree of “emergent” symmetry—the number of vortices on every row and column of the 2D dual lattice is asymptotically conserved at low energies. This symmetry implies that the harmonic “fixed point” theory of the RL phase has an infinite conductivity at any temperature.

In Sec. IV we study the legitimacy of the approximations used to arrive at the harmonic RL theory, focusing first on the neglected vortex hopping term. We show that for a range of parameters the vortex hopping term is “irrelevant,” scaling to zero at low energies whenever its associated scaling dimension satisfies  $\Delta_v \geq 2$ . Nevertheless, at finite temperatures vortex hopping leads to dissipation, giving a resistance which vanishes as a power law in temperature  $R(T) \sim T^\gamma$  with  $\gamma = 2\Delta_v - 3 \geq 1$ . A “plaquette duality” transformation<sup>12</sup> allows us to next address the legitimacy of the initial spin-wave expansion, used to obtain the harmonic RL theory. Of paramount importance is the presence of a term in the dual theory which hops a “charged” quasiparticle excitation, a term *not* present in the harmonic fixed-point theory. We find that the “charge” hopping process is irrelevant over a range of parameters—approximately the complement of the range where vortex hopping was irrelevant—implying stability of the RL phase. When relevant, on the other hand, the “charge” quasiparticle condenses, leading to a superconducting ground state.

The fermions are reintroduced back into the theory in Sec. V, where we argue for the stability of the Bose surface of rotons and the 2D plasmon in the presence of a *gapless* Fermi sea of fermionic quasiparticles. We denote the corresponding phase by the roton Fermi liquid (RFL). The gaplessness of the fermions is somewhat surprising, and deserves some comment. Indeed, it is in sharp contrast to the gapped nature of the quasiparticles in both the superconducting phase and  $Z_2$  fractionalized insulator, in which the spinons experience a BCS-like “pair field.” The cause of this difference is the existence of gapless single vortex excitations (and fluctuations) in the RFL, which according to our analysis leads to the “irrelevance” of the fermion pairing term. Crudely, because the bosonic pair field exhibits only OD quasi-LRO rather than ODLRO, there is no average pair field felt by the quasiparticles, and hence no gap. In this

sense, the RFL is in fact closer to a Fermi-liquid state than it is a superconductor.

Section VI is devoted to an analysis of the properties of the RFL phase. We study both the longitudinal and Hall conductivities, and find that the dissipative electrical resistance vanishes with a power of temperature  $R(T) \sim T^\gamma$  with  $\gamma \geq 1$ , similar to the behavior without fermions present. But the fermions are found to dominate the Hall response, leading within a naive Drude treatment to an inverse Hall angle varying as  $\Theta_H^{-1} \sim 1/(\tau_f^2 T^\gamma)$ , with  $\tau_f$  the fermionic quasiparticle transport lifetime. Due to the presence of the Bose surface of gapless rotons, electrons at finite energy  $\omega$  experience anomalous scattering, *not* present in a Landau–Fermi liquid. Specifically, quasiparticles scatter due to rotonic “current fluctuations” and “superconducting fluctuations,” which contribute additively to the electron decay rate. The former gives rise to especially strong electron scattering at “hot spots”—points on the Fermi surface with tangents parallel to the axes of the square lattice. At such hot spots the associated electron decay rate varies with an anomalous power of energy  $\omega^{(\gamma+2)/2}$  for  $1 < \gamma < 2$ . The decay rate from superconducting fluctuations is present everywhere along the Fermi surface *except* near “cold spots” at the incipient  $d$ -wave nodal points. This contribution grows strongly with decreasing energy/temperature, although it has a smaller overall amplitude than the current fluctuation contribution. Upon entering a superconducting state, the rotons—gapless within the RFL phase—become gapped out, and all anomalous scattering is strongly suppressed.

Finally, in Sec. VII we briefly discuss possible connections of the present work to the cuprates. We suggest that the RFL phase might underlie the unusual behavior observed near optimal doping in the cuprates, in particular the “strange metal” normal state above  $T_c$ . A scenario is outlined which also incorporates the pseudogap regime and the conventional Fermi liquid behavior in the strongly overdoped limit.

## II. THE MODEL

We are interested in electrons hopping on a 2D square lattice, with electron creation operators  $c_{\mathbf{r}\sigma}^\dagger$ . Here, the sites are denoted (in bold roman characters) as,  $\mathbf{r} = x_1 \hat{\mathbf{x}} + x_2 \hat{\mathbf{y}}$ , where  $x_1, x_2$  are integers,  $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}$ ,  $\hat{\mathbf{x}}_2 = \hat{\mathbf{y}}$  are unit vectors along the  $x$  and  $y$  axes, and  $\sigma = \uparrow, \downarrow$  denotes the two spin polarizations of the electron (such a spin index will be distinguished from Pauli matrices  $\sigma^\mu$  by the lack of any superscript).

### A. $Z_2$ chargon-spinon formulation

We begin by formulating the electron problem in spin-charge separated variables using the  $Z_2$  gauge theory Hamiltonian. We emphasize that this formulation does not imply that spin-charge separated excitations are deconfined, and indeed this formulation correctly describes the low-energy physics of conventional confined phases as well.

The  $Z_2$  gauge theory is most readily formulated in terms of a charge  $e$  singlet bosonic chargon  $b_{\mathbf{r}}, b_{\mathbf{r}}^\dagger$ , a neutral spin-1/2 fermionic spinon  $f_{\mathbf{r}\sigma}, f_{\mathbf{r}\sigma}^\dagger$ , and an Ising gauge field (Pauli matrix)  $\sigma_j^\mu(\mathbf{r})$  residing on the link between sites  $\mathbf{r}$  and  $\mathbf{r} + \hat{\mathbf{x}}_j$ .

It is convenient to use a rotor representation for the chargons  $b_{\mathbf{r}} = e^{-i\phi_{\mathbf{r}}}$ ,  $b_{\mathbf{r}}^\dagger b_{\mathbf{r}'} = n_{\mathbf{r}'}$ , with  $[\phi_{\mathbf{r}}, n_{\mathbf{r}'}] = i\delta_{\mathbf{r},\mathbf{r}'}$ .

The  $Z_2$  Hamiltonian is conveniently expressed in terms of the Hamiltonian density  $H_{Z_2} = \sum_{\mathbf{r}} \mathcal{H}_{Z_2}$ , which in turn is decomposed into a bosonic charge sector, a fermionic sector, and a gauge field contribution  $\mathcal{H}_{Z_2} = \mathcal{H}_c + \mathcal{H}_f^Z + \mathcal{H}_g$ :

$$\mathcal{H}_c = -t_c \sum_j \sigma_j^z(\mathbf{r}) \cos(\partial_j \phi_{\mathbf{r}} - A_j(\mathbf{r})) + u_c (n_{\mathbf{r}} - \rho_0)^2, \quad (1)$$

$$\mathcal{H}_g = -t_v \sum_j \sigma_j^x(\mathbf{r}) - K \prod_{\square(\mathbf{r}+\mathbf{w})} \sigma^z - \kappa_r \prod_j \sigma_j^x(\mathbf{r}) \sigma_j^x(\mathbf{r} + \hat{\mathbf{x}}_j), \quad (2)$$

$$\begin{aligned} \mathcal{H}_f^Z = & - \sum_j \sigma_j^z(\mathbf{r}) [t_s f_{\mathbf{r}+\hat{\mathbf{x}}_j \sigma}^\dagger f_{\mathbf{r}\sigma} + \Delta_j f_{\mathbf{r}+\hat{\mathbf{x}}_j \sigma} \epsilon_{\sigma\sigma'} f_{\mathbf{r}\sigma'} + \text{H.c.}] \\ & - t_e \sum_j f_{\mathbf{r}+\hat{\mathbf{x}}_j \sigma}^\dagger \sigma_j^x e^{i(\partial_j \phi_{\mathbf{r}} - A_j(\mathbf{r}))} f_{\mathbf{r}\sigma} + \text{H.c.} \end{aligned} \quad (3)$$

Here,  $\rho_0 \equiv 1 - x$  is the electron (charge) density with  $x$  measuring deviations from half filling. Throughout the paper,  $\partial_j$  with  $j=1,2$  denotes a *discrete* (forward) spatial lattice derivatives in the  $x_1$  and  $x_2$  directions, for example,  $\partial_1 \phi_{\mathbf{r}} = \partial_x \phi_{\mathbf{r}} = \phi_{\mathbf{r}+\hat{\mathbf{x}}} - \phi_{\mathbf{r}}$ . We have included an external (physical) vector potential  $A_j(\mathbf{r})$  in order to calculate electromagnetic response and to include applied fields. The Hamiltonian  $\mathcal{H}_c$  describes the dynamics of the chargons hopping with strength  $t_c$ , which are minimally coupled to the  $Z_2$  gauge field. The dynamics of the gauge fields is primarily determined from  $\mathcal{H}_g$ , the first two terms of which constitute the standard pure  $Z_2$  gauge theory Hamiltonian. The “magnetic” contribution involves the plaquette product

$$\prod_{\square(\mathbf{r}+\mathbf{w})} \sigma^z \equiv \sigma_1^z(\mathbf{r}) \sigma_1^z(\mathbf{r} + \hat{\mathbf{y}}) \sigma_2^z(\mathbf{r}) \sigma_2^z(\mathbf{r} + \hat{\mathbf{x}}), \quad (4)$$

which is the  $Z_2$  analog of the lattice curl. Here we have defined  $\mathbf{w} = (1/2)(\hat{x} + \hat{y})$ , and  $\mathbf{r} + \mathbf{w}$  denotes the center of the plaquette. We have also included an additional contribution bilinear in  $\sigma^x$ , which in the dual vortex representation below will become a “roton” hopping term. In the fermion Hamiltonian  $\mathcal{H}_f^Z$ , we have defined the antisymmetric matrix  $\epsilon_{\sigma\sigma'} = i\sigma_{\sigma\sigma'}^y$ , and take  $\Delta_j = (-1)^j \Delta$ , which describes a nearest-neighbor pair field with  $d$ -wave symmetry. Apart from the first two terms familiar to aficionados of the  $Z_2$  gauge theory,<sup>5</sup> we have included a less exotic bare electron hopping amplitude  $t_e$ . We will primarily be interested in the limit that the spinon hopping strength is significantly larger than the electron hopping strength  $t_s \gg t_e$ .

In most of the analysis of this paper, we will consider the limit of small spinon pairing  $\Delta_j \rightarrow 0$ . This can be justified either by the assumption  $\Delta_j \ll t_s$ , or by the irrelevance in the renormalization group (RG) sense, which will occur in some regimes. If strictly  $\Delta_j = 0$ , both fermion (spinon) number and boson charge are conserved, and in principle may be separately fixed. However, for  $\Delta_j \rightarrow 0$ , even infinitesimal, this is not the case. Instead the spinons will *equilibrate* in some time that diverges as  $\Delta_j \rightarrow 0$  but is otherwise finite, and the system will choose a unique fermion density to minimize its

(free) energy. We will return to this point in the U(1) formulation in Sec. II C 1.

The full Hamiltonian above has a set of local  $Z_2$  gauge symmetries, commuting with each of the local operators,

$$\mathcal{C}_{\mathbf{r}}^1 = (-1)^{n_{\mathbf{r}} + n_{\mathbf{r}}^f} \prod_{+\langle \mathbf{r} \rangle} \sigma^x, \quad (5)$$

where  $n_{\mathbf{r}}^f = f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma}$  is the fermion density and the  $Z_2$  lattice divergence is defined as

$$\prod_{+\langle \mathbf{r} \rangle} \sigma^x \equiv \prod_j \sigma_j^x(\mathbf{r}) \sigma_j^x(\mathbf{r} - \hat{\mathbf{x}}_j). \quad (6)$$

Physical states are required to be gauge invariant, which is specified by the set of local constraints:  $\mathcal{C}_{\mathbf{r}}^1 \equiv 1$ . This is the  $Z_2$  analog of Gauss law for conventional electromagnetism.

The connection between the  $Z_2$  gauge theory and a theory of interacting electrons, is most apparent in the limit that  $t_v$  is taken to be much larger than all other couplings. In this limit the electron creation operator is equivalent to the product of the chargon and spinon creation operators  $c_{\mathbf{r}\sigma}^\dagger = b_{\mathbf{r}}^\dagger f_{\mathbf{r}\sigma}^\dagger$ . Indeed, when  $t_v \rightarrow \infty$  the  $Z_2$  ‘‘electric’’ field becomes frozen,  $\sigma_j^x \approx 1$ , and the gauge constraints then imply that on each site of the lattice the sum of the chargon and spinon numbers  $n_{\mathbf{r}} + n_{\mathbf{r}}^f$  is even. Moreover, for large  $t_v$ , the chargon and spinon hopping terms are strongly suppressed, and can be considered perturbatively. Upon integrating out the gauge fields, one will thereby generate an additional electron kinetic energy term with amplitude of order  $t_c t_s / t_v$ . A brief discussion is given in Appendix A.

In what follows, we will study the  $Z_2$  gauge theory more generally, away from the large  $t_v$  limit. Of interest is the electron Green’s function

$$\mathcal{G}_e(\mathbf{r}_1, \tau_1; \mathbf{r}_2, \tau_2) = -\langle T_\tau \hat{c}_{\mathbf{r}_1\sigma}(\tau_1) \hat{c}_{\mathbf{r}_2\sigma}^\dagger(\tau_2) \rangle. \quad (7)$$

We will express the electron operators as

$$c_{\mathbf{r}\sigma} \equiv b_{\mathbf{r}} f_{\mathbf{r}\sigma}, \quad (8)$$

which is exact as  $t_v \rightarrow \infty$ , but more generally should be sufficient to extract the universal low-energy and long-length-scale behavior of the electron Green’s function. We will also be interested in correlation functions involving the Cooper pair creation and destruction operators  $B_{\mathbf{r}}^\dagger, B_{\mathbf{r}}$ , with  $B_{\mathbf{r}} = (b_{\mathbf{r}})^2 = e^{-2i\phi_{\mathbf{r}}}$ .

### B. $Z_2$ Vortex-spinon formulation

In what follows, it will prove particularly convenient to work with vortex degrees of freedom, rather than the chargon fields. To arrive at such a description, we use the U(1) duality transformation,<sup>15</sup> in which the dual variables sit naturally on the 2D dual lattice. We denote the sites of the 2D dual lattice by sans serif characters as  $\mathbf{r} = x_1 \hat{\mathbf{x}} + x_2 \hat{\mathbf{y}} + \mathbf{w}$ , with  $\mathbf{w} = (1/2)(\hat{\mathbf{x}} + \hat{\mathbf{y}})$  and integer  $x_1, x_2$ . The duality transformation itself defines two conjugate gauge fields  $[a_i(\mathbf{r}), e_j(\mathbf{r}')] = i \delta_{ij} \delta_{\mathbf{r}, \mathbf{r}'}$ , where

$$n_{\mathbf{r}} = \frac{1}{\pi} \epsilon_{ij} \partial_i a_j(\mathbf{r} - \mathbf{w}), \quad (9)$$

$$\partial_i \phi_{\mathbf{r}} = \pi \epsilon_{ij} e_j(\mathbf{r} + \mathbf{w} - \hat{\mathbf{x}}_j). \quad (10)$$

Here, as defined, due to the discreteness of the  $n_{\mathbf{r}}$  variables,  $a_j(\mathbf{r})$  takes on values that are integer multiples of  $\pi$ , while  $e_j(\mathbf{r})$  is a periodic variable with period 2. This transformation is faithful provided the constraint

$$(\vec{\nabla} \cdot \vec{e})(\mathbf{r}) \equiv \sum_j \partial_j e_j(\mathbf{r} - \mathbf{x}_j) = 0 \pmod{2} \quad (11)$$

or, equivalently,

$$\mathcal{C}_{\mathbf{r}}^2 = e^{i\pi(\vec{\nabla} \cdot \vec{e})(\mathbf{r})} = 1 \quad (12)$$

is imposed at every site  $\mathbf{r}$  of the dual lattice. Rewriting the charge Hamiltonian, one has

$$\begin{aligned} \mathcal{H}_c = & -t_c \sum_i \sigma_i^z(\mathbf{r}) \cos[\pi \epsilon_{ij} e_j(\mathbf{r} + \mathbf{w} - \hat{\mathbf{x}}_j) - A_i(\mathbf{r})] \\ & + \frac{u_c}{\pi^2} (\epsilon_{ij} \partial_i a_j - \pi \rho_0)^2. \end{aligned} \quad (13)$$

Conventional  $hc/2e$  superconducting vortices are composites of a vison (topological excitation in  $\sigma_j^z$ ) and a half vortex in  $\phi$ . To describe them, we perform a unitary transformation to a new Hamiltonian  $\tilde{H}_{Z_2}$  with new constraints  $\tilde{\mathcal{C}}^a$ ,

$$\tilde{H}_{Z_2} = U^\dagger H_{Z_2} U, \quad \tilde{\mathcal{C}}^a = U^\dagger \mathcal{C}^a U \quad (14)$$

with the unitary operator

$$\begin{aligned} U = & \exp \left[ \frac{i}{2} \sum_{\mathbf{r}, i, j} \epsilon_{ij} a_i(\mathbf{r} + \mathbf{w} - \hat{\mathbf{x}}_i) [\sigma_j^z(\mathbf{r}) - 1] \right] \\ = & \prod_{\mathbf{r}} [\sigma_1^z(\mathbf{r})]^{a_2(\mathbf{r} + \mathbf{w})/\pi} [\sigma_2^z(\mathbf{r})]^{a_1(\mathbf{r} - \mathbf{w})/\pi}, \end{aligned} \quad (15)$$

with  $\bar{\mathbf{w}} = (1/2)(\hat{\mathbf{x}} - \hat{\mathbf{y}}) = \mathbf{w} - \hat{\mathbf{y}}$ . The transformed constraints are

$$\tilde{\mathcal{C}}_{\mathbf{r}}^1 = (-1)^{n_{\mathbf{r}}^f} \prod_{+\langle \mathbf{r} \rangle} \sigma^x = 1, \quad (16)$$

$$\tilde{\mathcal{C}}_{\mathbf{r}}^2 = (-1)^{(\vec{\nabla} \cdot \vec{e})(\mathbf{r})} \prod_{\square(\mathbf{r})} \sigma^z = 1. \quad (17)$$

Under this unitary transformation

$$\tilde{\mathcal{H}}_c = -t_c \sum_j \cos[\pi e_j(\mathbf{r}) + \epsilon_{jk} A_k(\mathbf{r} + \bar{\mathbf{w}})] + \frac{u_c}{\pi^2} (\epsilon_{ij} \partial_i a_j - \pi \rho_0)^2, \quad (18)$$

$$\tilde{\mathcal{H}}_g = -t_v \sum_j \bar{\sigma}_j^x(\mathbf{r}) \cos[a_j(\mathbf{r})] - K(-1)^{(\vec{\nabla} \cdot \vec{e})(\mathbf{r})} + \mathcal{H}_r^{Z_2}. \quad (19)$$

Here we have defined  $\bar{\sigma}_1^x(\mathbf{r}) = \sigma_2^x(\mathbf{r} + \bar{\mathbf{w}})$ ,  $\bar{\sigma}_2^x(\mathbf{r}) = \sigma_1^x(\mathbf{r} - \bar{\mathbf{w}})$ , and have used Eq. (17). The transformed roton hopping term becomes

$$\mathcal{H}_r^{Z_2} = -\kappa_r \bar{\sigma}_1^x(\mathbf{r}) \bar{\sigma}_1^x(\mathbf{r} + \hat{\mathbf{x}}_2) \cos[\partial_y a_x(\mathbf{r})] + (x \leftrightarrow y). \quad (20)$$

The fermion Hamiltonian is almost unchanged in the dual vortex-spinon representation



$$\begin{aligned} \tilde{\mathcal{H}}_f^{Z_2} = & - \sum_j \sigma_j^z(\mathbf{r}) [t_s f_{\mathbf{r}+\hat{\mathbf{x}}_j}^\dagger f_{\mathbf{r}\sigma} + \Delta_j f_{\mathbf{r}+\hat{\mathbf{x}}_j} \epsilon_{\sigma\sigma'} f_{\mathbf{r}\sigma'} + \text{H.c.}] \\ & - t_e \sum_j \sigma_j^z(\mathbf{r}) e^{i[\pi \bar{e}_j(\mathbf{r}) - A_j(\mathbf{r})]} f_{\mathbf{r}+\hat{\mathbf{x}}_j}^\dagger f_{\mathbf{r}\sigma} + \text{H.c.}, \end{aligned} \quad (21)$$

the changes appearing only in the electron-hopping (the last term). Here we have defined

$$\bar{e}_i(\mathbf{r}) = \epsilon_{ij} e_j(\mathbf{r} + \mathbf{w} - \hat{x}_j). \quad (22)$$

Notice that  $\pi \bar{e}_j$  couples ‘‘similar to a gauge field’’ to the spinons in the final electron term.

To arrive at the final  $Z_2$  vortex-spinon theory, we split the electric and magnetic fields into longitudinal and transverse parts  $a_j = a_j^l + a_j^t$ ,  $e_j = e_j^l + e_j^t$ , with  $\vec{\nabla} \cdot \vec{a}^t = \vec{\nabla} \cdot \vec{e}^t = 0$ ,  $\epsilon_{ij} \partial_i a_j^l = \epsilon_{ij} \partial_i e_j^l = 0$ . For future purposes, we note that the longitudinal part of  $e_j$  is related to the transverse part of  $\bar{e}_j$  and vice versa, i.e.,  $\bar{e}_i^l(\mathbf{r}) = \epsilon_{ij} e_j^t(\mathbf{r} + \mathbf{w} - \hat{x}_j)$ . It is then convenient to solve the constraint for the longitudinal fields

$$a_j^l(\mathbf{r}) = -\partial_j \theta_r, \quad (23)$$

$$(\vec{\nabla} \cdot \vec{e}^t)(\mathbf{r}) = N_r. \quad (24)$$

The fields  $e^{i\theta_r}$  and  $N_r$  have the interpretation of vortex creation and number operators, as can be seen by tracking the circulation defined from the original chargin phase variable  $\phi$ . One finds canonical commutation relations

$$[\hat{\theta}_r, \hat{N}_{r'}] = i \delta_{rr'}. \quad (25)$$

At this stage,  $N_r$  is a periodic variable with period 2, and  $-\partial_j \theta_r + a_j^l(\mathbf{r})$  is constrained to be an integer multiple of  $\pi$ . It is convenient to soften the latter constraint, and in order to respect the uncertainty relation implied by Eq. (25), at the same time relax the periodicity of  $N_r$ . Formally, this is accomplished by replacing

$$-t_c \sum_j \cos(\pi e_j + \epsilon_{jk} A_k) \rightarrow \text{const.} + \frac{u_v}{2} \sum_j (e_j + \epsilon_{jk} A_k / \pi)^2, \quad (26)$$

with  $u_v \approx t_c \pi^2$ , and adding a term to the Hamiltonian  $\tilde{H}_{Z_2} \rightarrow \tilde{H}_{Z_2} + \sum_r \mathcal{H}_{2v}^{Z_2}$ , with

$$\mathcal{H}_{2v}^{Z_2} = -t_{2v} \sum_j \cos(2\partial_j \theta - 2a_j^t). \quad (27)$$

The constraint is recovered for large  $t_{2v}$ , but we will consider a renormalized theory in which  $t_{2v}$  may be considered a small perturbation.

It is convenient to regroup the longitudinal contribution to  $\tilde{H}_c$  along with the terms in  $\tilde{H}_g$  and  $\mathcal{H}_{2v}^{Z_2}$  into vortex ‘‘potential’’ and ‘‘kinetic’’ terms  $H_N$  and  $H_{\text{kin}}^{Z_2}$ . We thereby arrive at the final form for the  $Z_2$  vortex-spinon Hamiltonian

$$\tilde{H}_{Z_2} = H_{\text{pl}} + H_N + H_{\text{kin}}^{Z_2} + \tilde{H}_f^{Z_2}, \quad (28)$$

with the fermion Hamiltonian  $\tilde{\mathcal{H}}_f^{Z_2}$  given in Eq. (21) and with

$$\mathcal{H}_{\text{pl}} = \frac{u_v}{2} [e_j^t(\mathbf{r}) + \epsilon_{jk} A_k^l(\mathbf{r} + \bar{\mathbf{w}})]^2 + \frac{v_0^2}{2u_v} [\epsilon_{ij} \partial_i a_j^t(\mathbf{r}) - \pi \rho_0]^2, \quad (29)$$

$$H_N = \frac{u_v}{2} \sum_{r,r'} \left( \hat{N}_r - \frac{B_r}{\pi} \right) \left( \hat{N}_{r'} - \frac{B_{r'}}{\pi} \right) V(\mathbf{r} - \mathbf{r}'), \quad (30)$$

$$\mathcal{H}_{\text{kin}}^{Z_2} = \mathcal{H}_v^{Z_2} + \mathcal{H}_{2v}^{Z_2} + \mathcal{H}_r^{Z_2}. \quad (31)$$

Here  $B_r = \epsilon_{ij} \partial_i A_j(\mathbf{r} - \mathbf{w})$  is the physical flux through the plaquette of the original lattice located at the site  $\mathbf{r}$  of the dual lattice. In the ‘‘plasmon’’ Hamiltonian,  $\mathcal{H}_{\text{pl}}$ , an implicit sum over  $j=1,2$  is understood and we have defined a (bare) plasmon velocity  $v_0$  as  $v_0^2 = 2u_c t_c = 2u_c t_v / \pi^2$ . Inside the superconducting phase, this Hamiltonian describes the Goldstone mode—or sound mode—and can be readily diagonalized to give the dispersion  $\omega_{\text{pl}}^2(\mathbf{k}) = v_0^2 \sum_j [2 \sin(k_j/2)]^2$  with  $|k_j| < \pi$  in the first Brillouin zone. Since the electron charge density is given by  $(1/\pi) \epsilon_{ij} \partial_i a_j$ , one can readily include long-ranged Coulomb interactions which will modify the plasmon at small  $\mathbf{k}$ .

In  $H_N$  above we have set  $K=0$ , dropping henceforth the term  $\delta H_N(K) = -K \sum_r (-1)^{N_r}$ , since it will not play an important role in the phases of interest. The vortex–vortex interaction energy  $V(\mathbf{r})$  is the Fourier transformation of the discrete inverse Laplacian operator  $V^{-1}(\mathbf{k}) \equiv \mathcal{K}^2(\mathbf{k})$ , with  $\mathcal{K}^2(\mathbf{k}) = \sum_j 2(1 - \cos k_j)$ , and has the expected logarithmic behavior at large distances  $V(\mathbf{r}) \sim (1/2\pi) \ln(|r|)$ . The vortex kinetic energy  $\mathcal{H}_{\text{kin}}^{Z_2}$  is a sum of three contributions—a single vortex-hopping term

$$\mathcal{H}_v^{Z_2} = -t_v \sum_j \bar{\sigma}_j^x(\mathbf{r}) \cos(\partial_j \theta_r - a_j^t), \quad (32)$$

a pair-vortex-hopping term  $\mathcal{H}_{2v}^{Z_2}$  given in Eq. (27), and a roton-hopping term

$$\mathcal{H}_r^{Z_2} = -\kappa_r \bar{\sigma}_2^x(\mathbf{r}) \bar{\sigma}_2^x(\mathbf{r} + \hat{\mathbf{x}}_1) \cos[\Delta_{xy} \theta_r - \partial_x a_y^t(\mathbf{r})] + (x \leftrightarrow y), \quad (33)$$

where we have defined

$$\Delta_{xy} \theta_r = \Delta_{yx} \theta_r \equiv \sum_{e_1, e_2=0,1} (-1)^{e_1+e_2} \theta_{\mathbf{r}+e_1 \hat{\mathbf{x}}+e_2 \hat{\mathbf{y}}}. \quad (34)$$

Notice that  $\mathcal{H}_r^{Z_2}$  hops two vortices, originally at sites of the dual lattice on opposite corners of an elementary square, to the other two sites. Equivalently, this term can be interpreted as hopping a vortex–antivortex pair on neighboring sites (i.e., a vortex ‘‘dipole’’ or more simply a *roton*) one lattice spacing in a direction perpendicular to the dipole. Such a roton is a 2D analog of a 3D vortex ring, and in a Galilean invariant superfluid (such as 4–He) vortex rings propagate in precisely this manner. Henceforth we shall refer to this process as a ‘‘roton hopping’’ process.

The above  $Z_2$  vortex-spinon Hamiltonian must be supplemented by the two gauge constraints, which from Eqs. (16), (17), and (24) can be cast into an appealingly simple and symmetrical form

$$\tilde{C}_r^1 = (-1)^{n_r} \prod_{\square(r)} \bar{\sigma}^x = 1, \quad (35)$$

$$\tilde{C}_r^2 = (-1)^{N_r} \prod_{\square(r)} \sigma^z = 1. \quad (36)$$

These constraints correspond to an attachment of a  $Z_2$  flux in  $\sigma^z$  and  $\bar{\sigma}^x$  to the vortex number *parity* and the spinon number parity, respectively. Since the spinons are minimally coupled to  $\sigma^z$  and the vortices to  $\bar{\sigma}^x$ , this implies a sign change upon hopping a spinon around a vortex—or vice versa. Indeed, if the partition function for the vortex-spinon Hamiltonian (without  $\mathcal{H}_r^{Z_2}$  and with  $t_e=0$ ) is expressed as an imaginary time path integral with the  $Z_2$  constraints in Eqs. (35) and (36) imposed, the resulting Euclidean action becomes identical to Eqs. (109)–(113) in Ref. 5.

It is instructive to obtain explicit expressions for the electron and Cooper pair creation operators in terms of the dualized vortex degrees of freedom. From Eq. (9) we can directly obtain an expression for the Cooper pair destruction operator  $B_r = e^{-2i\phi_r}$  as

$$B_r = \prod_{\mathbf{r}} e^{2\pi i \bar{e}_j(\mathbf{r}') d\mathbf{r}'_j}, \quad (37)$$

where the  $\prod$  symbol here denotes a product along a semi-infinite “directed” string running on the links of the original lattice, originating at  $\mathbf{r}$  and terminating at spatial infinity, with  $d\mathbf{r}'$  the unit vector from the point  $\mathbf{r}'$  along the string to the next point. In terms of  $e_j$  (rather than  $\bar{e}_j$ ), the product contains a factor of  $\exp[\pm 2\pi i e_j(\mathbf{r})]$  for each link of the dual lattice that crosses the string, taking the positive/negative sign for directed links crossing the string from right/left to left/right proceeding from  $\mathbf{r}$  to  $\infty$ . We will use the above notation when possible to present precise analytic expressions for such strings. The path independence of the string is assured by the second gauge constraint  $C_r^2=1$ . Since the unitary transformation  $U$  in Eq. (15) commutes with  $e^{2\pi i e_j}$ , this is the correct expression for the Cooper pair operator within the  $Z_2$  vortex-spinon theory.

An expression for the electron operator  $c_{r\sigma} = b_r f_{r\sigma}$  in the dual vortex-spinon theory can be extracted by moreover re-expressing the “charge-on” operator  $b_r = e^{-i\phi_r}$  as a string,

$$b_r = \prod_{\mathbf{r}} e^{i\pi \bar{e}_j d\mathbf{r}'_j} = S_{\text{vort}}(\mathbf{r}) S_{\phi}(\mathbf{r}). \quad (38)$$

For later convenience, we have here decomposed this expression into a piece depending on the vortex configurations through the longitudinal electric field  $e_j^l$  and a contribution depending on the smooth part of the phase  $\phi$  through the transverse field  $e_j^t$ :

$$S_{\text{vort}}(\mathbf{r}) = \prod_{\mathbf{r}} e^{i\pi \bar{e}_j^l d\mathbf{r}'_j}, \quad S_{\phi}(\mathbf{r}) = \prod_{\mathbf{r}} e^{i\pi \bar{e}_j^t d\mathbf{r}'_j}. \quad (39)$$

But unlike the Cooper pair operator, the charge-on operator transforms nontrivially under the unitary transformation in Eq. (15):

$$\tilde{b}_r = U^\dagger b_r U = \prod_{\mathbf{r}} [\sigma_j^z e^{i\pi \bar{e}_j d\mathbf{r}'_j}], \quad (40)$$

now including a factor of  $\sigma_j^z(\mathbf{r})$  for each link of the string. Again, the path independence is guaranteed by the gauge constraint  $\tilde{C}_r^2=1$ . The final expression for the electron operator within the dualized  $Z_2$  vortex-spinon field theory follows simply as

$$\tilde{c}_{r\sigma} = U^\dagger c_{r\sigma} U = \tilde{b}_r f_{r\sigma}. \quad (41)$$

As discussed at length in Ref. 5, the  $Z_2$  vortex-spinon formulation is particularly well suited for accessing spin-charge separated insulating states. Specifically, when pairs of vortices hop around they “see” an average gauge flux of  $2\epsilon_{ij}\partial_i a_j = 2\pi\rho_0$ . Thus at half filling with  $\rho_0=1$ , vortex pairs effectively moving in zero flux and at large pair-hopping strength  $t_{2v} \rightarrow \infty$  will readily condense—driving the system into an insulating state with a charge gap. In the simplifying limit with vanishing *single* vortex hopping strength  $t_v=0$ , all of the vortices are paired,  $(-1)^N=1$ , and the  $Z_2$  gauge constraint in Eq. (36) reduces to  $\prod_{\square} \sigma^z=1$ . The vison excitations (plaquettes with  $\prod_{\square} \sigma^z=-1$ ) are gapped out of the ground state, and the spinons, being minimally coupled to  $\sigma^z$ , can propagate as deconfined excitations. The charge sector supports gapped but deconfined charge-on excitations, which can be viewed as topological defects in the pair-vortex condensate.

But as we shall see below, to access the roton Fermi liquid phase requires taking the strength of the roton hopping strength large, and the  $Z_2$  formulation proves inadequate. To remedy this, we introduce in Sec. II C below, a new U(1) formulation of the vortex-spinon field theory. As we shall demonstrate, the U(1) and  $Z_2$  vortex-spinon formulations are formally equivalent, and by a sequence of unitary transformations it is possible to pass from one representation to the other. Care should be taken when considering operators which transform nontrivially under the unitary operations relating different representations, however. In particular, some operators local in one representation become nonlocal in the other. A third dual vortex formulation involving electron (rather than spinon) operators is briefly discussed below in Appendix D. The Hamiltonian in this “vortex-electron” formulation is equivalent under a sequence of unitary transformations to both the  $Z_2$  and U(1) vortex-spinon Hamiltonians.

To establish these equivalences, it is convenient to “choose a gauge” in the  $Z_2$  theory. As detailed in Appendix B, it is possible to unitarily transform to a basis in which the  $Z_2$  gauge fields are completely “slaved” to the vortex and spinon operators, and can be eliminated completely from the theory. Specifically, in the chosen gauge the  $x$  components of both  $\sigma^z$  and  $\bar{\sigma}^x$  are set to unity on every link of the lattice  $\bar{\sigma}_1^x(\mathbf{r}) = \sigma_1^z(\mathbf{r}) = 1$ . As we shall see in Sec. II C below, the U(1) gauge fields in the U(1) vortex-spinon formulation can be similarly enslaved. Remarkably one arrives at the *identical* “enslaved” Hamiltonian in both cases, thereby establishing the formal equivalence between the two formulations.

### C. U(1) vortex-spinon formulation

In the U(1) formulation of the vortex-spinon field theory, the Pauli matrices  $\sigma^z$ ,  $\bar{\sigma}^x$  which live as  $Z_2$  gauge fields on the links of the original and dual lattice, respectively, are effectively replaced by exponentials of two U(1) gauge fields  $\bar{\sigma}_j^x(\mathbf{r}) \rightarrow \exp[i\alpha_j(\mathbf{r})]$  and  $\sigma_j^z(\mathbf{r}) \rightarrow \exp[i\beta_j(\mathbf{r})]$ . These two U(1) gauge fields are canonically conjugate variables taken to satisfy

$$[\alpha_i(\mathbf{r} - \hat{\mathbf{x}}_i), \beta_j(\mathbf{r}')] = i\pi\epsilon_{ij}\delta^2(\mathbf{r} - \mathbf{r}'), \quad (42)$$

with  $\mathbf{r} = \mathbf{r} + \mathbf{w}$ . For two ‘‘crossing’’ links these commutation relations imply that the exponentials  $e^{i\alpha}$ ,  $e^{i\beta}$  anticommute with one another,  $[e^{i\alpha}, e^{i\beta}]_- = 0$ .

#### 1. U(1) vortex-spinon Hamiltonian

The full Hamiltonian for the U(1) vortex-spinon field theory takes the same form as the  $Z_2$  vortex-spinon Hamiltonian in Eq. (28),

$$H = H_{\text{pl}} + H_N + H_{\text{kin}} + H_f, \quad (43)$$

with  $H_{\text{pl}}$  and  $H_N$  given as before in Eq. (29). Only the vortex kinetic terms and the fermion Hamiltonian are modified. Once again the vortex kinetic energy terms are decomposed into single vortex, pair-vortex, and roton hopping processes:

$$\mathcal{H}_{\text{kin}} = \mathcal{H}_v + \mathcal{H}_{2v} + \mathcal{H}_r. \quad (44)$$

Since the vortices in the U(1) formulation are minimally coupled to the U(1) gauge field  $\alpha_j(\mathbf{r})$ , each of these three terms will be modified from their  $Z_2$  forms. Specifically, in terms of the associated Hamiltonian densities we have

$$\mathcal{H}_v = -t_v \sum_{j=1,2} \cos(\partial_j \theta - a_j^t + \alpha_j), \quad (45)$$

$$\mathcal{H}_{2v} = -t_{2v} \sum_{j=1,2} \cos(2\partial_j \theta - 2a_j^t + 2\alpha_j), \quad (46)$$

$$\mathcal{H}_r = -\frac{\kappa_r}{2} \cos[\Delta_{xy}\theta - \partial_x(a_y^t - \alpha_y)] + (x \leftrightarrow y), \quad (47)$$

with  $\Delta_{xy}\theta_r$  defined in Eq. (34).

The Hamiltonian density for the fermions in the U(1) formulation is given by

$$\begin{aligned} \mathcal{H}_f = & - \sum_j e^{i\beta_j(\mathbf{r})} [(t_s + t_e e^{i[\pi\bar{\sigma}_j^x(\mathbf{r}) - A_j(\mathbf{r})]}) f_{\mathbf{r}+\hat{\mathbf{x}}_j\sigma}^\dagger f_{\mathbf{r}\sigma} \\ & + \Delta_j [\mathcal{S}_{\mathbf{r}}]^2 f_{\mathbf{r}+\hat{\mathbf{x}}_j\sigma} \epsilon_{\sigma\sigma'} f_{\mathbf{r}\sigma'} + \text{H.c.}] \end{aligned} \quad (48)$$

In the U(1) formulation, the (average) density of spinons  $\langle f_{\sigma}^\dagger f_{\sigma} \rangle = \langle (1/\pi) \epsilon_{ij} \partial_i \alpha_j \rangle$  [as follows from Eq. (54)] is taken to be *equal* to the (average) charge density  $\langle (1/\pi) \epsilon_{ij} \partial_i a_j \rangle$ . A new element, not present in the  $Z_2$  fermion Hamiltonian,  $\tilde{\mathcal{H}}_s^{Z_2}$  in Eq. (21), is the ‘‘string operator’’  $\mathcal{S}_{\mathbf{r}}$ . The string operator is given as a product of  $e^{i\beta}$  running along directed links of the original lattice from the site  $\mathbf{r}$  to spatial infinity:

$$\mathcal{S}_{\mathbf{r}} = \prod_{\mathbf{r}}^{\infty} e^{i\beta_j d\mathbf{r}'_j}. \quad (49)$$

As we shall discuss below, once we restrict the Hilbert space to gauge invariant states other choices for the ‘‘path’’ of the string are formally equivalent. Physically, as we will see below explicitly,  $\mathcal{S}_{\mathbf{r}}^2$  represents the ‘‘transverse part’’ of the phase factor of the Cooper pair field due to vortices/antivortices in the sample.

In addition to global spin and charge conservation, the full U(1) vortex-spinon Hamiltonian,  $\mathcal{H}$  above, has a number of *local* gauge symmetries. To fully define the model we need to specify the set of gauge invariant states that are allowed. Associated with each of the Chern–Simons fields  $\alpha$  and  $\beta$ , is a U(1) gauge symmetry. Specifically, the full Hamiltonian is invariant under *independent* gauge transformations

$$\begin{aligned} e^{-i\theta_{\mathbf{r}}} & \rightarrow e^{-i\theta_{\mathbf{r}}} e^{i\chi_{\mathbf{r}}}, \\ \alpha_j(\mathbf{r}) & \rightarrow \alpha_j(\mathbf{r}) + \delta_j \chi_{\mathbf{r}} \end{aligned} \quad (50)$$

and

$$\begin{aligned} f_{\mathbf{r}\sigma} & \rightarrow f_{\mathbf{r}\sigma} e^{i\Lambda_{\mathbf{r}}}, \\ \beta_j(\mathbf{r}) & \rightarrow \beta_j(\mathbf{r}) + \partial_j \Lambda_{\mathbf{r}} \end{aligned} \quad (51)$$

for arbitrary real functions  $\Lambda_{\mathbf{r}}$  and  $\chi_{\mathbf{r}}$ , living on the original and dual lattices, respectively. The corresponding operators which transform the fields in this way are

$$\mathcal{G}_v(\chi_{\mathbf{r}}) = e^{-i\sum_{\mathbf{r}} \chi_{\mathbf{r}} [N_{\mathbf{r}} - (1/\pi) \epsilon_{ij} \partial_i \beta_j(\mathbf{r} - \mathbf{w})]} \quad (52)$$

and

$$\mathcal{G}_f(\Lambda_{\mathbf{r}}) = e^{i\sum_{\mathbf{r}} \Lambda_{\mathbf{r}} [f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma} - (1/\pi) \epsilon_{ij} \partial_i \alpha_j(\mathbf{r} - \mathbf{w})]}. \quad (53)$$

Both of these operators commute with the full Hamiltonian  $H$ . The U(1) sector is specified by simply setting  $\mathcal{G}_v = \mathcal{G}_f = 1$  for arbitrary  $\chi_{\mathbf{r}}$  and  $\Lambda_{\mathbf{r}}$ . From Eqs. (52) and (53) this is equivalent to attaching  $\pi$  flux in the statistical gauge fields  $\alpha$  and  $\beta$  to the spinons and vortices

$$\epsilon_{ij} \partial_i \alpha_j(\mathbf{r} - \mathbf{w}) = \pi f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma}, \quad \epsilon_{ij} \partial_i \beta_j(\mathbf{r} - \mathbf{w}) = \pi N_{\mathbf{r}}. \quad (54)$$

Notice that this is simply the U(1) analog of the  $Z_2$  flux attachment in Eqs. (35) and (36) and implies the same sign change when a spinon is transported around a vortex or vice versa. The only difference is that in the U(1) formulation the phase of  $\pi$  is accumulated gradually when the spinon is taken around the vortex, whereas the sign change in the  $Z_2$  theory can occur when the spinon hops across a single link. The formal equivalence of the  $Z_2$  and U(1) formulations will be established below.

As we shall see, the ‘‘smearing’’ of the accumulated  $\pi$  phase change, makes the theory in the U(1) formulation eminently more tractable. The one notable complication is the square of the string operator, which in the U(1) sector is a nontrivial function of  $e^{2i\beta}$  along the string, rather than equaling unity as in the  $Z_2$  sector. However, it is worth emphasizing that within the U(1) sector of the theory, the value of the operator  $\mathcal{O}_{\mathbf{r}} \equiv \mathcal{S}_{\mathbf{r}}^2$  is *independent* of the chosen path. Specifi-

cally, consider two (unitary) string operators, denoted  $O_1, O_2$  with different paths running from the same site  $\mathbf{r}$  off to spatial infinity. The “difference” between the two string operators  $O_1^{-1}O_2$  is a product of  $e^{2i\beta}$  around *closed loops*. But due to the U(1) gauge constraint in Eq. (54),  $\epsilon_{ij}\partial_i\beta_j = \pi N$ , this product is an exponential of the total vorticity  $N_{\text{tot}}$  inside the closed loops  $O_1^{-1}O_2 = \exp(2\pi i N_{\text{tot}})$ . Since the vorticity is integer, one deduces that the string operator is indeed path independent,  $O_1 = O_2$ .

It will prove useful to obtain expressions for the electron- and Cooper pair creation operators within the U(1) vortex-spinon formulation. The Cooper pair operator has the same form as in the  $Z_2$  vortex-spinon formulation, given explicitly in Eq. (37), but the electron operator is modified in a non-trivial way. As we shall check explicitly below, the electron operator within the U(1) formulation involves a string depending on both the dual “electric field”  $e_j$ , as well as the statistical gauge field  $\beta_j$ :

$$c_{\mathbf{r}\sigma} = f_{\mathbf{r}\sigma} \prod_{\mathbf{r}}^{\infty} e^{i[\pi\bar{e}_i + \beta_i]d\mathbf{r}'_i}, \quad (55)$$

with  $\bar{e}_i(\mathbf{r})$  defined in Eq. (22). The path independence follows from the condition in Eq. (24), together with the second U(1) gauge constraint in Eq. (54) above:

$$\pi\partial_j e_j(\mathbf{r} - \mathbf{x}_j) = \epsilon_{ij}\partial_i\beta_j(\mathbf{r} - \mathbf{w}). \quad (56)$$

This implies an equality between the longitudinal electric field and the transverse statistical field

$$\beta_1^t(\mathbf{r}) = \pi e_2^l(\mathbf{r} + \bar{\mathbf{w}}), \quad \beta_2^t(\mathbf{r}) = -e_1^l(\mathbf{r} - \bar{\mathbf{w}}) \quad (57)$$

or

$$\beta_i^t = -\pi\bar{e}_i^t, \quad (58)$$

and enables the electron destruction operator to be reexpressed as

$$c_{\mathbf{r}\sigma} = f_{\mathbf{r}\sigma} S_{\phi}(\mathbf{r}) \prod_{\mathbf{r}}^{\infty} e^{i\beta_j^l d\mathbf{r}'_j} \quad (59)$$

with  $S_{\phi}(\mathbf{r})$  defined in Eq. (39). Similarly, the string operator that enters into the U(1) fermion Hamiltonian  $H_f$  in Eq. (48), can be written as

$$S_{\mathbf{r}} = S_{\text{vort}}(\mathbf{r}) \prod_{\mathbf{r}}^{\infty} e^{i\beta_j^l d\mathbf{r}'_j}. \quad (60)$$

Upon combining the above two equations, and recalling the identity for the charge-on operator in the original  $Z_2$  theory,  $b_{\mathbf{r}} = S_{\text{vort}}(\mathbf{r}) S_{\phi}(\mathbf{r})$  in Eq. (38) implies that  $S_{\mathbf{r}}^{\dagger} f_{\mathbf{r}\sigma} = b_{\mathbf{r}}^{\dagger} c_{\mathbf{r}\sigma}$ . Consequently, spinon pairing terms in the  $Z_2$  gauge theory are seen to be equivalent to the usual Bogoliubov–deGennes form

$$[S_{\mathbf{r}}]^2 \epsilon_{\sigma\sigma'} f_{\mathbf{r}\sigma} f_{\mathbf{r}\sigma'} = B_{\mathbf{r}}^{\dagger} \epsilon_{\sigma\sigma'} c_{\mathbf{r}\sigma} c_{\mathbf{r}\sigma'}, \quad (61)$$

with  $B_{\mathbf{r}}^{\dagger}$  the Cooper pair creation operator. For  $d$ -wave pairing, the pair field lives on links, and a similar identity obtains

$$e^{i\beta_j(\mathbf{r})} [S_{\mathbf{r}}]^2 f_{\mathbf{r}+\hat{\mathbf{x}}_j\sigma} \epsilon_{\sigma\sigma'} f_{\mathbf{r}\sigma'} = B_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}_j}^{\dagger} c_{\mathbf{r}+\hat{\mathbf{x}}_j\sigma} \epsilon_{\sigma\sigma'} c_{\mathbf{r}\sigma'}. \quad (62)$$

Here we have introduced a bond-Cooper-pair operator

$$B_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}_j}^{\dagger} = \tilde{b}_{\mathbf{r}}^{\dagger} \tilde{b}_{\mathbf{r}+\hat{\mathbf{x}}_j}^{\dagger}, \quad (63)$$

$$\tilde{b}_{\mathbf{r}}^{\dagger} = \prod_{\mathbf{r}}^{\infty} e^{-i\pi\bar{e}_i d\mathbf{r}'_i}. \quad (64)$$

Note that in the untransformed charge-on variables of the  $Z_2$  gauge theory formulation  $B_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}_j}^{\dagger} = \sigma_j^z(\mathbf{r}) b_{\mathbf{r}}^{\dagger} b_{\mathbf{r}+\hat{\mathbf{x}}_j}^{\dagger}$ .

Recalling the discussion in Sec. II A, it is still the case that for  $\Delta_j=0$ , when this pairing term is absent, the fermion number is conserved, and naively may be chosen arbitrarily. However, in the limit  $\Delta_j \rightarrow 0$ , which we consider here, this conservation is weakly violated, and only the total charge is conserved. The physics at work is clear from Eq. (61): for nonvanishing  $\Delta_j$ , quasiparticle pairs and boson pairs are interchanged, and the two charged fluids come to equilibrium. Thus in what follows, we should choose to divide the charge density amongst the fermions and bosons in such a way as to minimize the total (free) energy. This division will therefore shift as parameters of the Hamiltonian are changed. How it does so is crucial to the ultimate low-energy physical properties of the system, as is clear from Eqs. (45)–(47), which show that the vortices experience an effective flux proportional to the difference of the total charge density ( $\epsilon_{ij}\partial_i a_j / \pi$ ) and the fermionic density  $n_f = \epsilon_{ij}\partial_i \alpha_j / \pi$ . As the fermion density is varied, the effective flux seen by the vortices changes. Significantly, in the limit  $t_v \rightarrow \infty$ , vortex-hopping dominates the energetics, and is minimized when the fermion density equals the total charge density  $n_f = \epsilon_{ij}\partial_i a_j / \pi$ . This naturally recovers the Fermi liquid phase (Appendix A) by binding charge  $e$  firmly to each fermion, fully accommodating all the electrical charge.

In the rest of the paper we work exclusively within the U(1) vortex-spinon formulation, which is particularly suitable for extracting the properties of the roton Fermi liquid. Before embarking on that, we first establish the formal equivalence between the two formulations by enslaving the U(1) gauge fields. As detailed in Appendix C, it is possible to unitarily transform to a gauge with  $\vec{\nabla} \cdot \vec{\alpha} = \vec{\nabla} \cdot \vec{\beta}^{\text{slave}} = 0$ . In this gauge, both  $\alpha_j$  and  $\beta_j$  are enslaved, being fully expressible in terms of the spinon and vortex densities  $n_{\mathbf{r}}^f$  and  $N_{\mathbf{r}}$ , respectively. Moreover, the enslaved U(1) Hamiltonian is found to be *identical* to the enslaved  $Z_2$  Hamiltonian obtained in Appendix B, and the enslaved expressions for the electron operators also coincide.

Having thereby established the equivalence between the  $Z_2$  and U(1) vortex-spinon field theories, in the remainder of the paper we choose to work exclusively within the U(1) formulation, employing the Hamiltonian  $H$  defined in Eq. (43), together with the gauge constraints in Eq. (54). In practice, it is far simpler to work within a Lagrangian formulation, where the gauge constraints can be imposed explicitly within a path integral, as detailed in the next subsection.



## 2. Lagrangian for U(1) vortex-spinon theory

In order to impose the U(1) gauge constraints in Eq. (54) on the Hilbert space of the full vortex-spinon Hamiltonian  $H$ , we will pass to a Euclidean path integral representation of the partition function. The associated Euclidean Lagrangian is readily obtained as a sum of three contributions

$$S = \int d\tau [H + L_B + L_{\text{con}}], \quad (65)$$

with  $L_B$  involving the generalized coordinates and conjugate momenta

$$L_B = \sum_{\mathbf{r}=\mathbf{r}+\mathbf{w}} \left[ iN_{\mathbf{r}}\partial_0\theta_{\mathbf{r}} - \frac{i}{\pi}\beta_i(\mathbf{r})\epsilon_{ij}\partial_0\alpha_j(\mathbf{r}+\hat{x}_i) \right] + \sum_{\mathbf{r}=\mathbf{r}+\mathbf{w}} [ie_j^t(\mathbf{r})\partial_0a_j^t(\mathbf{r}) + f_{\mathbf{r}\sigma}^{\dagger}\partial_0f_{\mathbf{r}\sigma}], \quad (66)$$

with  $\partial_0 \equiv \partial/\partial\tau$  denoting an imaginary time derivative and  $L_{\text{con}}$  is a Lagrange multiplier term imposing the two independent U(1) gauge constraints

$$L_{\text{con}} = \frac{i}{\pi} \sum_{\mathbf{r}=\mathbf{r}-\mathbf{w}} \alpha_0(\mathbf{r})[\epsilon_{ij}\partial_i\beta_j(\mathbf{r}) - \pi N_{\mathbf{r}}] + \frac{i}{\pi} \sum_{\mathbf{r}=\mathbf{r}+\mathbf{w}} \beta_0(\mathbf{r})[\epsilon_{ij}\partial_i\alpha_j(\mathbf{r}) - \pi J_{\mathbf{r}\sigma}^{\dagger}f_{\mathbf{r}\sigma}]. \quad (67)$$

Here we have introduced two Chern–Simons scalar potentials as Lagrange multipliers, denoted  $\alpha_0(\mathbf{r})$  and  $\beta_0(\mathbf{r})$ , which live on the dual and original lattice sites respectively.

Upon introducing another scalar potential  $a_0(\mathbf{r})$  living on the sites of the dual lattice, and collecting together the longitudinal and transverse parts of  $a_j$  and  $e_j$ , the full Euclidean action can be compactly cast into a simple form. In order to make the vortex physics more explicit we choose to reintroduce the vortex phase field  $\theta$  within the Lagrangian formulation. Specifically, we shift  $a_{\mu} \rightarrow a_{\mu} + \partial_{\mu}\theta$  with  $\mu=0, x, y$ , and then integrate over *both*  $a_{\mu}$  and  $\theta$ . In this way we arrive at the final form for the full Euclidean Lagrangian:

$$S = S_c + S_f + S_{\text{CS}} + S_A. \quad (68)$$

The charge sector action  $S_c = \int_{\tau} \sum_{\mathbf{r}} \mathcal{L}_c$  can be expressed in terms of the Lagrangian density

$$\mathcal{L}_c = \mathcal{L}_a + \frac{1}{2u_0}(\partial_0\theta - a_0 + \alpha_0)^2 + L_{\text{kin}}, \quad (69)$$

with the  $u_0 \rightarrow 0$  limit enslaving  $a_0 = \partial_0\theta + \alpha_0$  and

$$\mathcal{L}_a = \frac{u_v}{2} \left( \bar{e}_j - \frac{A_j}{\pi} \right)^2 - ie_j(\partial_0a_j - \partial_ja_0) + \frac{v_0^2}{2u_v}(\epsilon_{ij}\partial_ia_j - \pi\rho_0)^2. \quad (70)$$

The vortex kinetic energy terms  $\mathcal{L}_{\text{kin}}$  are given explicitly by  $\mathcal{H}_{\text{kin}}$  in Eqs. (45)–(47) except with  $a_j^t \rightarrow a_j$ .

The fermion action  $S_f = \int_{\tau} \sum_{\mathbf{r}} \mathcal{L}_f$  is given by

$$\mathcal{L}_f = f_{\mathbf{r}\sigma}^{\dagger}(\partial_0 - i\beta_0)f_{\mathbf{r}\sigma} + \mathcal{H}_f, \quad (71)$$

with  $\mathcal{H}_f$  given in Eq. (48). The two sectors are coupled together by the electric field in the electron-hopping term, and by the Chern–Simons action  $S_{\text{CS}} = \int_{\tau} \sum_{\mathbf{r}=\mathbf{r}+\mathbf{w}} \mathcal{L}_{\text{CS}}$  with

$$\mathcal{L}_{\text{CS}} = \frac{i}{\pi}\beta_0(\mathbf{r})\epsilon_{ij}\partial_i\alpha_j(\mathbf{r}) + \frac{i}{\pi}\beta_i(\mathbf{r})\epsilon_{ij}[\partial_j\alpha_0(\mathbf{r}+\hat{x}_i) - \partial_0\alpha_j(\mathbf{r}+\hat{x}_i)]. \quad (72)$$

Notice that in the absence of the electron hopping term ( $t_e = 0$ ), the electric field  $e_j$  enters quadratically in the action, and can then be integrated out to give  $\mathcal{L}_a \rightarrow \tilde{\mathcal{L}}_a$ , with

$$\tilde{\mathcal{L}}_a = \frac{1}{2u_v}(\partial_0a_j - \partial_ja_0)^2 + v_0^2(\epsilon_{ij}\partial_ia_j - \pi\rho_0)^2. \quad (73)$$

Finally, it is useful in some circumstances to treat the external gauge field by making the shift  $\bar{e}_i \rightarrow \bar{e}_i + A_i/\pi$ , which removes all coupling of  $A_i$  to the fermions, and furthermore leaves  $A_i$  linearly coupled to a charge “three-current” of the usual form  $S_A = \int_{\tau} \sum_{\mathbf{r}} \mathcal{L}_A$ , with

$$\mathcal{L}_A = iA_{\mu}(\mathbf{r})J_{\mu}(\mathbf{r}). \quad (74)$$

Here  $J_{\mu}$  is the charge three-current given explicitly by

$$J_0(\mathbf{r}) = \frac{1}{\pi}\epsilon_{ij}\partial_ia_j(\mathbf{r}), \quad (75)$$

$$J_i(\mathbf{r}) = \frac{1}{\pi}\epsilon_{ij}[\partial_ja_0(\mathbf{r}+\hat{x}_i) - \partial_0a_j(\mathbf{r}+\hat{x}_i)], \quad (76)$$

with  $\mathbf{r}=\mathbf{r}+\mathbf{w}$ . Notice that the three-current is conserved as required:  $\partial_0J_0(\mathbf{r}) + \partial_jJ_i(\mathbf{r}-\hat{x}_i) = 0$ . This form is useful for a variety of calculations, particularly within the purely bosonic RL model discussed in Secs. III and IV, but less so in some RFL calculations best done in the electron gauge (see below), which is incompatible with the above shift of  $\bar{e}_i$ .

As can be seen from the equations of motion obtained from  $\delta\mathcal{L}/\delta\alpha=0$  and  $\delta\mathcal{L}/\delta\beta=0$ , the effect of the Chern–Simons term is to attach  $\pi$  flux in  $\alpha$  ( $\beta$ ) to the spinon (vortex) world lines:

$$\epsilon_{\mu\nu\lambda}\partial_{\nu}\alpha_{\lambda} = \pi J_{\mu}^s, \quad \epsilon_{\mu\nu\lambda}\partial_{\nu}\beta_{\lambda} = \pi J_{\mu}^v, \quad (77)$$

where  $J_{\mu}^s$  and  $J_{\mu}^v$  are the spinon and vortex three-currents. Here  $\mu, \nu, \lambda = 0, x, y$  run over the three space-time coordinates.

Finally we comment on the nature of the gauge symmetries of the full action  $S$  in the Lagrangian representation. In particular, associated with the three gauge fields  $a_{\mu}$ ,  $\alpha_{\mu}$ , and  $\beta_{\mu}$  are three *independent* space-time gauge symmetries. Specifically, these are

$$\theta_{\mathbf{r}} \rightarrow \theta_{\mathbf{r}} + \Theta_{\mathbf{r}},$$

$$a_{\mu}(\mathbf{r}) \rightarrow a_{\mu}(\mathbf{r}) + \partial_{\mu}\Theta_{\mathbf{r}}, \quad (78)$$

$$\theta_{\mathbf{r}} \rightarrow \theta_{\mathbf{r}} + \chi_{\mathbf{r}},$$

$$\alpha_{\mu}(\mathbf{r}) \rightarrow \alpha_{\mu}(\mathbf{r}) - \partial_{\mu}\chi_{\mathbf{r}}, \quad (79)$$

$$f_{\mathbf{r}\sigma} \rightarrow f_{\mathbf{r}\sigma} e^{i\Lambda_{\mathbf{r}}},$$

$$\beta_{\mu}(\mathbf{r}) \rightarrow \beta_{\mu}(\mathbf{r}) + \partial_{\mu}\Lambda_{\mathbf{r}}, \quad (80)$$

with  $\Theta_{\mathbf{r}}$ ,  $\chi_{\mathbf{r}}$ , and  $\Lambda_{\mathbf{r}}$  arbitrary functions of space and imaginary time. Because of the gauge invariance of  $S$  under these three distinct transformations, we are free to fix gauges *independently* for the three gauge fields.

In addition to these three gauge symmetries, the full Lagrangian has two *global* symmetries. By construction, the spinon Lagrangian  $\mathcal{L}_s$  conserves the total spin, and since the electrical three-current in Eqs. (75) and (76) satisfies a continuity equation  $\partial_{\mu}J_{\mu}=0$ , the total electrical charge  $Q = \sum_{\mathbf{r}}J_0(\mathbf{r})$  is also conserved.

A particularly convenient gauge choice for the gauge field  $\beta_{\mu}$  is

$$\beta_i^l(\mathbf{r}) = -\pi\bar{e}_i^l(\mathbf{r}), \quad (81)$$

with  $\bar{e}_i$  defined in terms of the electric field  $e_j(\mathbf{r})$  in Eq. (22). Remarkably, in this gauge the electron creation operator equals the spinon creation operator. To see this, it is convenient to shift  $\alpha_0 \rightarrow \alpha_0 + a_0$  and then integrate out  $a_0$ , which constrains  $\vec{\nabla} \times \vec{\beta} = \pi\vec{\nabla} \cdot \vec{e}$ , or equivalently  $\beta_i^l = -\pi\bar{e}_i^l$ . Together with the above gauge choice this implies that  $\beta_i \equiv -\pi\bar{e}_i$ , so that from Eq. (55) one has  $c_{\mathbf{r}\sigma} = f_{\mathbf{r}\sigma}$ . We refer to this as the electron gauge. The possibility to choose a gauge within the Lagrangian formulation of the U(1) vortex-spinon theory which makes  $f_{\mathbf{r}\sigma}$  an electron operator, suggests that it should be possible to reformulate the vortex-spinon *Hamiltonian* entirely in terms of vortices and electrons. This is indeed the case, as we demonstrate briefly in Appendix D.

### III. THE ROTON LIQUID

We first focus on the bosonic charge sector of the theory, entirely ignoring the fermions. Specifically, in the full Euclidean action in Eq. (68) we retain *only* the charge action  $S_c$  and the coupling to the external electromagnetic field  $S_A$ . We take the fermionic density as a nonfluctuating constant. As discussed in Sec. II C 1, this constant should be determined by energetics. We assume here that the largest energy scales in the problem are those of the vortices, i.e.,  $\kappa_r$ ,  $u_v$ , etc. In this case, one expects that vortex kinetic energy (rotonic or otherwise) is minimized when the vortices experience zero average magnetic flux. We therefore choose the fermionic density *equal* to the total charge density, setting  $(1/\pi)\epsilon_{ij}\partial_i\alpha_j = \rho_0$  and also putting  $\alpha_0 = 0$ . Note that this choice is essential to recovering an ordinary Fermi liquid state (see Sec. IV and Appendix A) and hence is also natural in this sense. We remark that, while we will continue to assume the average fermion density is equal to  $\rho_0$  in the bulk of this paper, we will return to another possibility—and its physical regime of relevance—in the discussion section.

It is also convenient to isolate the fluctuations in the charge density by defining

$$a_j = a_j^b + \tilde{a}_j, \quad (82)$$

with background density  $(1/\pi)\epsilon_{ij}\partial_i a_j^b = \rho_0$ . We can then take  $a_j = a_j^b$ , so that  $a_j - \alpha_j = \tilde{a}_j$ . Of the three vortex kinetic energy

processes which enter in  $\mathcal{L}_{\text{kin}}$ , we hereafter and in the rest of the paper drop entirely the vortex pair hopping process in Eq. (46) putting  $t_{2v} = 0$ . For now we also set the single vortex hopping processes to zero, putting  $t_v = 0$  in Eq. (45), but will return to their effects in Sec. IV. Of interest here is the new roton hopping process  $\mathcal{L}_r \equiv \mathcal{H}_r$  in Eq. (47), which can be conveniently recast in the form

$$\mathcal{L}_r = -\kappa_r \mathcal{C}[\tilde{a}] \cos \left[ \Delta_{xy}\theta - \frac{1}{2}(\partial_x \tilde{a}_y + \partial_y \tilde{a}_x) \right], \quad (83)$$

with

$$\mathcal{C}[\tilde{a}] = \cos(\epsilon_{ij}\partial_i \tilde{a}_j / 2). \quad (84)$$

More generally, with spinon fluctuations included one has  $\mathcal{C} = \cos[\epsilon_{ij}\partial_i(a_j - \alpha_j)/2]$ . Clearly,  $\mathcal{C}$  is maximal—and hence the rotonic kinetic energy most negative—for  $\epsilon_{ij}\partial_i \tilde{a}_j = 0$ , which is true on average for this choice of fermion density.

We next choose the gauge  $\vec{\nabla} \cdot \vec{a} = 0$ , and integrate over  $a_0$  (with  $u_0 \rightarrow 0$ ). Having dropped the vortex hopping processes, the remaining charge Lagrangian is then given by

$$\mathcal{L}_c = \mathcal{L}_{\text{pl}} + \mathcal{L}_{\theta}, \quad (85)$$

with

$$\mathcal{L}_{\text{pl}} = \frac{1}{2u_v} [(\partial_0 \tilde{a}_j)^2 + v_0^2 (\epsilon_{ij}\partial_i \tilde{a}_j)^2], \quad (86)$$

$$\mathcal{L}_{\theta} = \frac{1}{2u_v} (\partial_j \partial_0 \theta)^2 - \kappa_r \mathcal{C}[\tilde{a}] \cos \left[ \Delta_{xy}\theta - \frac{1}{2}(\partial_1 \tilde{a}_2 + \partial_2 \tilde{a}_1) \right]. \quad (87)$$

To analyze the phases of this model it is instructive to represent the Lagrangian  $\mathcal{L}_{\theta}$  in Hamiltonian form by re-introducing the vortex number operator  $\hat{N}_r$ :

$$\hat{H}_{\theta} = \frac{u_v}{2} \sum_{\mathbf{r}, \mathbf{r}'} \hat{N}_{\mathbf{r}} \hat{N}_{\mathbf{r}'} V(\mathbf{r} - \mathbf{r}') - \kappa_r \sum_{\mathbf{r}} \mathcal{C}[\tilde{a}] \cos \left( \Delta_{xy} \hat{\theta}_{\mathbf{r}} - \frac{1}{2}(\partial_x \tilde{a}_1 + \partial_y \tilde{a}_x) \right). \quad (88)$$

Again  $V(\mathbf{r})$  is the Fourier transform of the discrete inverse Laplacian operator  $V(\mathbf{k}) \equiv 1/\mathcal{K}^2(\mathbf{k})$ , with  $\mathcal{K}^2(\mathbf{k}) = \sum_j 2(1 - \cos k_j)$ .

The first term in Eq. (88) describes a logarithmically interacting gas of (integer strength) vortices moving on the dual 2D square lattice. When  $\kappa_r = 0$ , this model will undergo a finite-temperature Kosterlitz–Thouless transition<sup>16</sup> from a high-temperature vortex plasma into the low-temperature vortex dielectric. This corresponds, of course, to a transition into a superconducting phase. With  $\kappa_r = 0$  the Kosterlitz–Thouless transition temperature will be set by the vortex interaction strength  $u_v$ . But upon increasing the strength of the roton-hopping, one expects the transition temperature to be suppressed, and for  $\kappa_r \gg u_v$  to be driven all the way to zero. Thus, at zero temperature, upon increasing the single dimen-

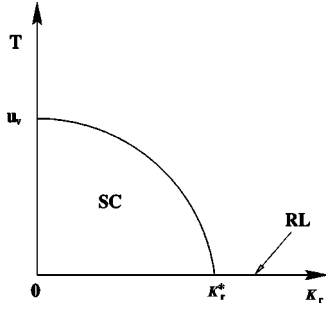


FIG. 1. Schematic phase diagram of the Hamiltonian in the charge sector  $H_c = H_\theta + H_{pl}$ , with zero vortex-hopping strength  $t_v = 0$ . When the roton-hopping vanishes,  $\kappa_r = 0$ ,  $H_c$  describes a classical logarithmically interacting vortex gas, and has a superconductor-to-normal transition at a Kosterlitz-Thouless temperature  $T_{KT} \approx u_v$ . With increasing roton-hopping,  $T_{KT}$  decreases being driven to zero at  $\kappa_r^* \approx u_v$ , where there is a quantum phase transition from the superconductor into the Roton liquid phase.

tionless ratio  $\kappa_r/u_v$ , one expects a quantum phase transition out of the superconducting phase and into a new phase (see Fig. 1)—the roton liquid.

### A. Harmonic theory and excitations

To access the properties of the roton liquid (RL), we consider  $\kappa_r \gg u_v$ , where it is presumably valid to expand the cosine terms in Eq. (88) for small argument, giving  $\hat{H}_\theta = \hat{H}_{rot} + \dots$ , with

$$\hat{H}_{rot} = \frac{u_v}{2} \sum_{r,r'} \hat{N}_r \hat{N}_{r'} V(r-r') + \frac{\kappa_r}{8} \sum_r (\epsilon_{ij} \partial_i \tilde{a}_j)^2 + \frac{\kappa_r}{2} \sum_r \left[ \Delta_{xy} \hat{\theta}_r - \frac{1}{2} (\partial_x \tilde{a}_y + \partial_y \tilde{a}_x) \right]^2. \quad (89)$$

With this expansion, it is no longer legitimate to restrict  $\theta$  to the interval  $[0, 2\pi]$ . Consistency then dictates that the eigenvalues of the vortex number operator no longer be restricted to integers, but allowed to take on any real value from  $[-\infty, \infty]$ .

The full roton-liquid Hamiltonian  $\hat{H}_{RL} = \hat{H}_{pl} + \hat{H}_{rot}$  is quadratic and can be readily diagonalized. This is most conveniently done by returning to the Lagrangian framework, described now by

$$\mathcal{L}_{RL} = \frac{1}{2u_v} [(\partial_0 \tilde{a}_j)^2 + \tilde{v}_0^2 (\epsilon_{ij} \partial_i \tilde{a}_j)^2] + \frac{1}{2u_v} (\partial_j \partial_0 \theta)^2 + \frac{\kappa_r}{2} \left[ \Delta_{xy} \theta - \frac{1}{2} (\partial_x \tilde{a}_y + \partial_y \tilde{a}_x) \right]^2, \quad (90)$$

with  $\tilde{v}_0 = \sqrt{v_0^2 + \kappa_r u_v}/4$ . To proceed to describe the normal modes of this quadratic Lagrangian, we define Fourier transforms

$$O(\mathbf{r}, \tau) = \int_{\mathbf{k}, \omega_n} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_n \tau} O(\mathbf{k}, \omega_n), \quad (91)$$

$$O(\mathbf{r}, \tau) = \int_{\mathbf{k}, \omega_n} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_n \tau} O(\mathbf{k}, \omega_n), \quad (92)$$

for fields  $O, \bar{O}$  on the original and dual lattices, respectively. Here integration  $\int_{\mathbf{k}} \equiv \int d^2\mathbf{k}/(2\pi)^2$  is taken over the Brillouin zone  $|k_1|, |k_2| < \pi$  and  $\int_{\omega_n} \equiv \int_{-\infty}^{\infty} d\omega_n/(2\pi)$  defines the integration measure (at zero temperature) for the Matsubara frequency  $\omega_n$ . At nonzero temperature, one simply replaces  $\int_{\omega_n} \rightarrow \beta^{-1} \sum_{\omega_n}$ , with  $\omega_n = 2\pi n/\beta$ . It is moreover convenient to define

$$\mathcal{K}_j(\mathbf{k}) = -i(e^{ik_j} - 1), \quad (93)$$

so that upon Fourier transformation, the discrete derivatives behave intuitively,

$$\partial_j \rightarrow_{\text{FT}} i\mathcal{K}_j(\mathbf{k}), \quad (94)$$

and of course  $\partial_0 \rightarrow -i\omega_n$  as usual. We also introduce the transverse gauge field

$$\tilde{a}_i(\mathbf{k}) = \epsilon_{ij} \frac{i\mathcal{K}_j^*(\mathbf{k})}{\mathcal{K}(\mathbf{k})} a(\mathbf{k}), \quad (95)$$

with  $\mathcal{K}^2(\mathbf{k}) = \sum_j |\mathcal{K}_j(\mathbf{k})|^2$  and  $|\mathcal{K}_j(\mathbf{k})| = 2|\sin(k_j/2)|$ .

To now diagonalize  $\mathcal{L}_{RL}$ , it is convenient to define a real two-component field  $Y_a$  via

$$a(\mathbf{k}, \omega_n) = \sqrt{u_v} e^{i\mathbf{k}\cdot\mathbf{w}} Y_1(\mathbf{k}, \omega_n), \quad (96)$$

$$\theta(\mathbf{k}, \omega_n) = \frac{\sqrt{u_v}}{\mathcal{K}(\mathbf{k})} Y_2(\mathbf{k}, \omega_n). \quad (97)$$

Then the action  $S_{RL} = \sum_r \int_\tau \mathcal{L}_{RL}$  is

$$S_{RL} = \frac{1}{2} \int_{\mathbf{k}, \omega_n} Y_\alpha(\mathbf{k}, \omega_n) G_{\alpha\beta}^{-1}(\mathbf{k}, \omega_n) Y_\beta(-\mathbf{k}, -\omega_n), \quad (98)$$

with

$$G_{\alpha\beta} = \frac{G^0 \delta_{\alpha\beta} + G^z \sigma_{\alpha\beta}^z + G^x \sigma_{\alpha\beta}^x}{(\omega_n^2 + \omega_{pl}^2)(\omega_n^2 + \omega_{rot}^2)}, \quad (99)$$

where  $\vec{\sigma}$  is the usual vector of Pauli matrices. Here we have defined

$$G^0 = \omega_n^2 + \frac{1}{2} v_+^2 \mathcal{K}^2, \quad (100)$$

$$G^z = -\frac{1}{2} v_+^2 \mathcal{K}^2 + v_1^2 \frac{|\mathcal{K}_x \mathcal{K}_y|^2}{\mathcal{K}^2}, \quad (101)$$

$$G^x = \frac{v_1^2}{2} (|\mathcal{K}_x|^2 - |\mathcal{K}_y|^2) \frac{\tilde{\mathcal{K}}_x \tilde{\mathcal{K}}_y}{\mathcal{K}^2}, \quad (102)$$

with  $v_1 = \sqrt{\kappa_r u_v}$  and  $\tilde{\mathcal{K}}_j = 2 \sin(k_j/2)$ . The poles in  $G_{\alpha\beta}$  at  $\omega = i\omega_n = \pm \omega_{pl}, \pm \omega_{rot}$  describe two types of collective modes.

The first excitation is a plasmon with a renormalized dispersion

$$\omega_{\text{pl}}^2(\mathbf{k}) = \frac{1}{2} [v_+^2 \mathcal{K}^2 + \sqrt{v_-^4 \mathcal{K}^4 + \tilde{v}_0^2 v_1^2 (|\mathcal{K}_x|^2 - |\mathcal{K}_y|^2)^2}] \quad (103)$$

with velocities

$$v_{\pm} = \sqrt{\tilde{v}_0^2 \pm v_1^2/4}. \quad (104)$$

The plasmon frequency vanishes at the center of the Brillouin zone  $\mathbf{k}=0$ , and in the absence of long-ranged Coulomb interactions disperses linearly  $\omega_{\text{pl}}=v_{\text{pl}}|\mathbf{k}|$  at small wave-vectors  $k_j \rightarrow 0$ . But the associated plasmon velocity  $v_{\text{pl}}(\phi)$  depends upon the ratio  $k_y/k_x = \tan(\phi)$ . In particular, along the zone diagonals with  $k_y = \pm k_x$  the velocity is minimal and unaffected by the vortices with  $v_{\text{pl}} = \tilde{v}_0$ , whereas it takes its maximum value  $v_+$  along the  $k_x$  or  $k_y$  axes.

This upward shift in the plasmon frequency is due to a ‘‘level repulsion’’ with the second collective mode—the gapless roton, which disperses as

$$\omega_{\text{rot}}^2(\mathbf{k}) = \frac{1}{2} [v_+^2 \mathcal{K}^2 - \sqrt{v_-^4 \mathcal{K}^4 + \tilde{v}_0^2 v_1^2 (|\mathcal{K}_x|^2 - |\mathcal{K}_y|^2)^2}]. \quad (105)$$

For  $|k_x| \ll 1$  and fixed  $k_y$ , the roton dispersion vanishes,  $\omega_{\text{rot}} \sim v_{\text{rot}}|k_x|$ , with

$$v_{\text{rot}} = \frac{\tilde{v}_0}{v_+} v_1. \quad (106)$$

Remarkably, the roton-liquid phase supports a gapless Bose surface of roton excitations, along the  $k_x=0$  and  $k_y=0$  axes. These roton excitations describe *gapless and transverse* current fluctuations, which are obviously not present in a conventional bosonic superfluid.

With long-range Coulomb interactions present one would have simply  $v_0^2 \rightarrow v_0^2(\mathbf{k}) \sim 1/|\mathbf{k}|$ , giving the familiar 2D plasmon dispersion  $\omega_{\text{pl}} \sim \sqrt{|\mathbf{k}|}$ . In addition, the roton velocity  $v_{\text{rot}}$  becomes dependent upon  $k_y$ . We note in passing that the roton velocity is in either case determined not only from the dynamics of  $\theta$  but also from that of  $\tilde{a}$ , as is evident from its dependence upon  $\tilde{v}_0/v_+$ . It will be sometimes instructive in the following to consider the simple limit  $v_0 \sim \tilde{v}_0 \rightarrow \infty$ , in which the spatial fluctuations of  $\tilde{a}$  vanish and the roton mode is entirely decoupled from  $\tilde{a}$ .

### B. No Meissner effect in roton liquid

We now employ the Gaussian theory to examine some of the electrical properties of the roton liquid phase. Consider first the response of the RL phase to an applied magnetic field. In the presence of a magnetic field  $B = \epsilon_{ij} \partial_i A_j$ , there is an additional term that one must add to the Lagrangian, which from Eq. (74) takes the form

$$\mathcal{L}_A = \frac{i}{\pi} a_0 B. \quad (107)$$

If  $a_0$  is integrated out from  $\mathcal{L}_c$  in Eq. (69) with this additional term present, the Hamiltonian  $H_\theta(\hat{N}_r, \hat{\theta}_r)$  in Eq. (88) becomes simply,  $H_\theta[\hat{N}_r - (1/\pi)B, \hat{\theta}_r]$ . As expected, the vortex density

will become nonzero in the presence of the magnetic field. Since the vortex number operator in  $H_\theta$  has integer eigenvalues, it is not generally possible to shift away the applied  $B$  field. But in the roton liquid phase (at zero temperature) where the cosine term can be expanded to quadratic order (as in  $H_{\text{rot}}$ ), the vortex number operator has a continuous spectra, and one can formally eliminate the  $B$  field by shifting  $\hat{N}_r \rightarrow \hat{N}_r + (1/\pi)B$ , for all  $r$ . Since the ground state energy of the RL phase is thus independent of the applied  $B$  field, both the magnetization and the magnetic susceptibility  $\chi = \partial M / \partial B$  vanish. Unlike in a superconductor, where  $\chi = -1/4\pi$ , there is *no* Meissner effect in the roton liquid (strictly speaking, there is never a Meissner effect in a single two-dimensional layer, but one can consider an infinite stack of electrically decoupled but magnetically coupled layers, which would exhibit a Meissner effect when the layers are true 2D superconductors, but not when they are RLs). Physically, since the RL phase supports gapless roton excitations, the state cannot screen out an applied magnetic field.

### C. Off-diagonal quasi-long-range order

We next consider the Cooper pair propagator in the roton liquid

$$G^{\text{CP}}(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2) = \langle B_{\mathbf{r}_1}(\tau_1) B_{\mathbf{r}_2}^\dagger(\tau_2) \rangle, \quad (108)$$

with the Cooper pair destruction operator  $B_{\mathbf{r}}$  given in Eq. (37) as an infinite product of exponentials  $e^{2\pi i e_j}$  running along the string. The propagator for the  $d$ -wave pair field  $B_{\mathbf{r}, \mathbf{r} + \hat{x}_j}$

$$G_{ij}^{\text{CP}}(\mathbf{r}, \tau) = \langle B_{0, \hat{x}_i}(0) B_{\mathbf{r}, \mathbf{r} + \hat{x}_j}^\dagger(\tau) \rangle, \quad (109)$$

behaves similarly, and will be discussed at the end of this subsection.

#### 1. Equal time correlator

We consider at first the equal time correlator, with  $\tau_1 = \tau_2$ . The path independence of the string rests on the condition  $(\vec{\nabla} \cdot \vec{e})(\mathbf{r}) = N_r$ , with *integer* vortex number  $N_r$ . Unfortunately, within the tractable harmonic approximation valid for most quantities in the roton liquid phase (with cosine terms in the roton-hopping expanded to quadratic order), the condition of integer vortex number is *not* satisfied, and the results for  $G^{\text{CP}}(\mathbf{r}, 0)$  depend upon the choice of string. We believe that the correct behavior can be extracted by taking the string running along the straightest and ‘‘shortest’’ (using the ‘‘city block’’ metric  $|x_1 - x_2| + |y_1 - y_2|$ ) path between the two points  $\mathbf{r}_1$  and  $\mathbf{r}_2$ . As we shall see, the Cooper pair propagator calculated in this way has an anisotropic spatial power-law decay. Preliminary calculations suggest that, once perturbative corrections to the harmonic approximation (using the formalism established in Sec. IV B 2) are taken into account (even if they are irrelevant in the renormalization group sense), simple variations in the string do not modify the power-law decay of  $G^{\text{CP}}(\mathbf{r}, 0)$ , but only change the (nonuniversal) prefactor.<sup>17</sup>

We take  $\mathbf{r}_1 - \mathbf{r}_2 = X\hat{x} + Y\hat{y}$  and  $\mathbf{r}_2 = \mathbf{w}$ , with integer  $X, Y \geq 0$ . Then, upon expressing the correlator  $G^{\text{CP}}(X, Y, 0)$  as an



imaginary time path integral, one obtains an extra term in the Euclidean action  $S = \int_{\tau} \Sigma_{\tau} \mathcal{L}$  in Eq. (68), with

$$\mathcal{L} \rightarrow \mathcal{L} + ie_j(r, \tau) \mathcal{J}_j(r, \tau). \quad (110)$$

The  $c$ -number ‘‘source’’ field is given as

$$\mathcal{J}_j(r, \tau) = 2\pi \delta(\tau) \left[ \delta_{j2} \sum_{x'=0}^{X-1} \delta_{x,x'+1} \delta_{y,0} - \delta_{j1} \sum_{y'=0}^{Y-1} \delta_{x,X} \delta_{y,y'} \right]. \quad (111)$$

The Fourier transform is simply

$$\mathcal{J}_j(\mathbf{k}, \omega_n) = 2\pi \left[ \delta_{j2} \frac{1 - e^{-ik_x X}}{i\mathcal{K}_x} - \delta_{j1} \frac{e^{-ik_x X} (1 - e^{-ik_y Y})}{i\mathcal{K}_y} \right]. \quad (112)$$

Integrating out the electric field  $e_j$  in the gauge  $\vec{\nabla} \cdot \vec{a} = 0$  and decomposing the source fields into transverse and longitudinal parts  $\mathcal{J}_i = i\epsilon_{ij} \mathcal{K}_i \mathcal{J}_j / \mathcal{K}$ ,  $\mathcal{J}_i = i\mathcal{K}_i^* \mathcal{J}_i / \mathcal{K}$ , one obtains the result

$$G^{\text{CP}}(X, Y, 0) = \left\langle \exp \left\{ - \int_{\mathbf{k}, \omega_n} \left[ \frac{i\omega_n}{\sqrt{u_v}} (\mathcal{J}_t Y_1 - \mathcal{J}_l Y_2) + \frac{1}{2u_v} (\mathcal{J}_t^2 + \mathcal{J}_l^2) \right] \right\} \right\rangle_Y, \quad (113)$$

where the Gaussian average over  $Y$  is to be taken with respect to  $S_{\text{RL}}$  in Eq. (98). Performing this Gaussian integral, one obtains

$$G^{\text{CP}}(X, Y, 0) = \exp[-\Gamma_l - \Gamma_t - \Gamma_{lt}], \quad (114)$$

where

$$\Gamma_l = \int_{\mathbf{k}, \omega_n} \frac{|\mathcal{J}_l|^2 (1 - \omega_n^2 G_{22})}{2u_v}, \quad (115)$$

$$\Gamma_t = \int_{\mathbf{k}, \omega_n} \frac{|\mathcal{J}_t|^2 (1 - \omega_n^2 G_{11})}{2u_v}, \quad (116)$$

$$\Gamma_{lt} = - \int_{\mathbf{k}, \omega_n} \frac{\mathcal{J}_l \mathcal{J}_t^* \omega_n^2 G_{12}}{u_v}, \quad (117)$$

with  $G_{ij}$  given in Eq. (99).

Investigation of  $\Gamma_t$  and  $\Gamma_{lt}$  shows that the corresponding integrands are nonsingular at small  $k_x$  or  $k_y$ , and hence go to finite limits for large  $|X|$  and/or large  $|Y|$ . They will thus affect only the amplitude of the Cooper pair propagator at large distances, and we henceforth neglect them. Singular behavior at long distances *does* arise from  $\Gamma_l$ , in line with the intuition that it is vortex fluctuations which disrupt the superconducting phase, since  $\mathcal{J}_l$  couples to the longitudinal electric field, which through  $\vec{\nabla} \cdot \vec{e} = N$  describes the vorticity. To evaluate  $\Gamma_l$ , we first perform the frequency integration to obtain

$$\Gamma_l = \int_{\mathbf{k}} |\mathcal{J}_l|^2 \frac{\tilde{v}_0 v_1 |\mathcal{K}_x \mathcal{K}_y| + v_1^2 |\mathcal{K}_x \mathcal{K}_y|^2 / \mathcal{K}^2}{4u_v (\omega_{\text{pl}} + \omega_{\text{rot}})}. \quad (118)$$

Next, we explicitly express the square of the longitudinal string

$$|\mathcal{J}_l|^2 = \frac{(2\pi)^2}{\mathcal{K}^2} \left[ \left| \frac{\mathcal{K}_x}{\mathcal{K}_y} \right|^2 |1 - e^{-ik_y Y}|^2 + \left| \frac{\mathcal{K}_y}{\mathcal{K}_x} \right|^2 |1 - e^{-ik_x X}|^2 + 2 \operatorname{Re}\{(1 - e^{-ik_x X})(1 - e^{-ik_y Y})\} \right]. \quad (119)$$

The first two terms in Eq. (119) are singular for small  $k_y$ ,  $k_x$  respectively for very large  $Y$ ,  $X$ , leading to a logarithmic dependence when inserted in Eq. (118). The final term in Eq. (119), by contrast, is singular only for *both*  $k_x$ ,  $k_y$  small, and this singularity, inserted into Eq. (118), is weak and integrable. Extracting the logarithmic parts, one finds

$$\Gamma_l \sim \Delta_c (\ln|X| + \ln|Y|), \quad (120)$$

for  $|X|, |Y| \gg 1$ , with

$$\Delta_c = 2\pi \frac{\tilde{v}_0}{v_+} \sqrt{\frac{\kappa_r}{u_v}}. \quad (121)$$

Hence we have

$$G^{\text{CP}}(X, Y, 0) \sim \frac{\text{const.}}{|X|^{\Delta_c} |Y|^{\Delta_c}}. \quad (122)$$

This establishes that the roton-liquid phase has off-diagonal quasilong-ranged order (ODQLRO) at zero temperature.

## 2. Unequal time correlator

We now consider the Cooper-pair propagator at unequal times. Unfortunately, it is difficult to produce a simple and general calculation for arbitrary spatial and time separations. In particular, clearly, by square symmetry, we expect  $G^{\text{CP}}(X, Y, \tau) = G^{\text{CP}}(Y, X, \tau)$ . Any choice of strings, however, necessarily creates an asymmetry between the two spatial directions. As we have been unable to resolve this dilemma, we instead focus on the simple case in which the pair is created and annihilated on a single row of the lattice, i.e.,  $G^{\text{CP}}(X, 0, \tau)$ . We will see that this correlator decays as a power law both in space and time.

To proceed, we take  $\mathbf{r}_1 - \mathbf{r}_2 = X\hat{x}$ ,  $\mathbf{r}_2 = \mathbf{w}$ ,  $\tau_1 = \tau$ ,  $\tau_2 = 0$ , with  $X, Y \geq 0$ . With this choice, the string in Fourier space becomes

$$\mathcal{J}_j(\mathbf{k}, \omega_n) = 2\pi \delta_{j2} \frac{1 - e^{-ik_x X + i\omega_n \tau}}{i\mathcal{K}_x}. \quad (123)$$

Repeating the same manipulations as above, one again obtains (with negligible contributions from the transverse part of the string)

$$G^{\text{CP}}(X, 0, \tau) \sim \exp[-\tilde{\Gamma}] \quad (124)$$

with

$$\tilde{\Gamma}_l \sim \frac{\Delta_c}{2} \ln(X^2 + v_{\text{rot}}^2 \tau^2), \quad (125)$$

where for simplicity we have taken  $v_{\text{rot}}$  independent of  $k_y$  [for a nontrivial  $v(k_y)$ , the logarithm is simply averaged uniformly over the  $k_y$  axis]. This gives

$$G_c(X, 0, \tau) \sim \frac{\text{const.}}{(X^2 + v_{\text{rot}}^2 \tau^2)^{\Delta_c/2}}. \quad (126)$$

Note that this power-law form implies a power-law local tunneling density of states for Cooper pairs  $\rho_{\text{CP}}(\epsilon) \sim \epsilon^{\Delta_c-1}$ .

We conjecture that the full correlator satisfies a simple scaling form with “ $z=1$ ” scaling:

$$G^{\text{CP}}(X, Y, \tau) \sim \frac{\text{const.}}{|X|^{\Delta_c} |Y|^{\Delta_c}} \mathcal{G}\left(\frac{X}{v_{\text{rot}} \tau}, \frac{Y}{v_{\text{rot}} \tau}\right). \quad (127)$$

Combining our two calculations above implies  $\mathcal{G}(\chi, 0) = [\chi^2/(\chi^2+1)]^{\Delta_c/2}$ .

### 3. ODQLRO of the $d$ -wave pair field

We now briefly discuss the analogous ODQLRO of the  $d$ -wave pair field described by  $G_{ij}^{\text{CP}}(\mathbf{r}, \tau)$  in Eq. (109). This quantity is plagued by the same string ambiguities as the local Cooper-pair propagator, but to a larger degree, since it involves *two* separate strings emanating from the two sites shared by the initial bond and ending at the two sites shared by the final bond. Although we are confident  $G_{ij}^{\text{CP}}(\mathbf{r}, \tau)$  has a power-law form consistent with ODQLRO, we are unable to determine the precise nature of these correlations with reliability. For instance, consider the equal time pair field correlator for two bonds along the  $x$  axis  $G_{11}^{\text{CP}}(\mathbf{r}, 0)$ . For two bonds in the same row,  $\mathbf{r} = X\hat{\mathbf{x}}_1$ , the strings can be chosen to line all in the same row, and since the logarithmic divergence controlling the ODQLRO arises from small  $k_x$  in this case, the power-law exponent is unchanged, i.e.,  $G_{11}^{\text{CP}}(X, 0, 0) \sim \text{const.}/|X|^{\Delta_c}$ , with  $\Delta_c$  given above. We believe that, since this choice of string is by far the most natural, this is probably the correct result. If, instead, we choose to separate the two pair fields along a single column  $\mathbf{r} = Y\hat{\mathbf{x}}_2$ , then the two strings involved cannot be taken entirely atop one another. Different choices for the strings then give different results. For instance, making the symmetric choice of two parallel strings (each of strength  $\pi$  rather than  $2\pi$ ) gives a decay exponent reduced from  $\Delta_c$  to  $\Delta_c/2$  in the  $Y$  direction, while choosing the strings to overlap everywhere except the two ends reproduces the previous exponent  $\Delta_c$  without any reduction. Since we are unable to reliably resolve this ambiguity, we are unable to determine the exact form of the  $d$ -wave pair field correlator. Instead, we will take the pragmatic approach of approximating the correlations by those of the local pair field  $G_{ij}^{\text{CP}}(\mathbf{r}, \tau) \approx G^{\text{CP}}(\mathbf{r}, \tau)$ . It should be understood that the decay exponent  $\Delta_c$  may need to be renormalized and/or the correlator corrected slightly to obtain detailed results for a specific model.

### D. Conductivity in the harmonic theory

Given the above result of ODQLRO, it is natural to expect a very large and perhaps infinite conductivity in the

roton liquid. Indeed, neglecting the effects of vortex-hopping  $\mathcal{L}_v$  in Eq. (45), one can readily see [e.g., from Eq. (88)] that the total vortex number on each row and each column of the 2D lattice is separately conserved. Thus it is impossible to set up a vortex flow, and hence by the Josephson relation to generate an electric field. Using the quadratic roton liquid Lagrangian  $\mathcal{L}_{\text{RL}}$ , this expectation can be directly confirmed. In particular, we can integrate out *all* dynamical fields ( $\theta, a$ ) from the Lagrangian leaving only the external gauge field  $A^\mu$ , and thereby extract the polarization tensor and conductivity. This is most conveniently carried out in the gauge  $A_0=0$ , and assuming no magnetic-field  $\partial_x A_y - \partial_y A_x = 0$ . In this case, one may write

$$S_A = \int_{\mathbf{k}, \omega_n} \frac{-i\omega_n \mathcal{K}_j(\mathbf{k})}{\pi \mathcal{K}(\mathbf{k})} e^{-i\mathbf{k}\cdot\mathbf{w}} a(\mathbf{k}, \omega_n) A_j(-\mathbf{k}, -\omega_n). \quad (128)$$

Integrating out  $a$  and  $\theta$  using the Green's function  $G_{11}$  in Eq. (99), one obtains, in the limit  $|\mathbf{k}| \rightarrow 0$ , the effective action

$$S_A^0 = \frac{1}{2} \int_{\mathbf{k}, \omega_n} \Pi_{ij}^0(\mathbf{k}, \omega_n) A_i(\mathbf{k}, \omega_n) A_j(-\mathbf{k}, -\omega_n), \quad (129)$$

with

$$\Pi_{ij}^0(\mathbf{k}, \omega_n) \sim \frac{u_v}{\pi^2} \frac{k_i k_j}{k^2}, \quad (130)$$

as  $|\mathbf{k}| \rightarrow 0$ . The dependence of  $\Pi_{ij}^0$  on the orientation of  $\mathbf{k}$  is due to the fact that we have assumed  $B=0$ , which according to Faraday's law requires  $\nabla \times \mathbf{E} = -\partial_t \mathbf{B} = 0$ . Hence we must choose  $\mathbf{k}$  parallel to the electric field  $\mathbf{A} = \mathbf{E}/(-i\omega)$ . Thus we extract an *isotropic* conductivity tensor  $\sigma_{ij}^0 = \delta_{ij} \sigma^0$ , with

$$\sigma^0(\omega) = \frac{u_v}{\pi^2} \frac{1}{-i\omega}, \quad (131)$$

characteristic of a system with no dissipation.

The above conclusion for the quadratic RL Lagrangian is, however, modified by the vortex hopping terms. As we detail in the next section, the effects of a small vortex hopping term depend sensitively on the parameters that enter in the harmonic theory of the roton liquid—in particular the dimensionless ratio  $u_v/\kappa_r$ . There are two regimes. When this ratio is larger than a critical value, vortex-hopping is “relevant” and grows at low energies destabilizing the roton liquid phase. On the other hand, for small enough  $u_v/\kappa_r$ , the vortex-hopping strength scales to zero and the roton liquid phase is stable. In this latter case, the effects of vortex hopping on physical quantities can be treated perturbatively. In particular, we find that the conductivity in the roton-liquid diverges as a power law in the low-frequency and low-temperature limit.

## IV. INSTABILITIES OF THE ROTON LIQUID

We first consider the instabilities of the roton-liquid due to the presence of a vortex hopping term and examine the effects of such processes on the electrical transport. In the

subsequent subsection we consider the legitimacy of the harmonic expansion required to obtain the quadratic RL Lagrangian. This is achieved by performing a plaquette duality transformation,<sup>12</sup> where it is possible to address this issue perturbatively. Again we find two regimes depending on the parameters in the quadratic roton liquid Lagrangian. A stable regime for small  $u_v/\kappa_r$  wherein the harmonic roton liquid description is valid, and an instability toward a superconducting phase when this ratio is large.

### A. Vortex hopping

The analysis of the stability of the RL to vortex-hopping is mathematically nearly identical to the stability analysis of the exciton Bose liquid (EBL) of Ref. 12 with respect to boson hopping. Taking over those methods, we note that the vortex hopping operator exhibits *one-dimensional power-law correlations*

$$\langle e^{i\partial_y\theta_r(0)} e^{-i\partial_y\theta_{r+\tau}(\tau)} \rangle_{\text{RL}} = \delta_{y,0} \mathcal{R}(x, \tau), \quad (132)$$

with a power law form at large space-time separations

$$\mathcal{R}(x, \tau) = \frac{c}{(x^2 + v_{\text{rot}}^2 \tau^2)^{\Delta_v}}, \quad (133)$$

where  $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$  and  $c$  is a dimensionless constant. Here,  $\mathcal{R}(x, \tau)$  is essentially the single roton Green's function, describing the space-time propagation of a roton with a dipole oriented along the  $\hat{\mathbf{y}}$  axis. When calculated using the RL Lagrangian, one finds

$$\Delta_v = \frac{1}{4\pi} \sqrt{\frac{u_v v_{\pm}}{\kappa_r \tilde{v}_0}}. \quad (134)$$

Simple calculations show that this power-law behavior is not modified by including the fluctuating  $a_y$  field in the vortex hopping operator  $\hat{T}_y(\mathbf{r}, \tau) = e^{i[\partial_y\theta_{r+a_y(\tau)}]}$ , i.e.,

$$\langle \hat{T}_y(\mathbf{r}, 0) \hat{T}_y^*(\mathbf{r} + \mathbf{r}, \tau) \rangle_{\text{RL}} \sim \frac{\delta_{y,0}}{(x^2 + v_{\text{rot}}^2 \tau^2)^{\Delta_v}}, \quad (135)$$

with only a change in the prefactor. Notice that the exponent  $\Delta_v$  characterizing the power-law decay of the roton propagator is inversely proportional to the analogous exponent  $\Delta_c$  in Eq. (121) which gives the power law decay of the Cooper pair propagator. Indeed, for the roton liquid Lagrangian studied here we find the simple identity  $\Delta_v \Delta_c = 1/2$ . But with inclusion of other terms in the original Hamiltonian such as the spinons or further neighbor roton hopping terms, this equality will be modified.

The arguments of Ref. 12 imply that the vortex-hopping term is then relevant for  $\Delta_v < 2$ . In this regime, the vortex-hopping strength grows large when scaling to low energies, and one expects the vortices to condense at zero temperature. In this case it is legitimate to expand the cosine term in Eq. (45) and one generates a ‘‘dual Meissner effect,’’ where the gauge fields that are minimally coupled to the vortices become massive. In the presence of spinons this leads to a mass term of the form  $\mathcal{L}_v \sim (t_v/2)(a_j - \alpha_j)^2$ , which confines one unit of electrical charge to each spinon presumably driving

one into a Fermi-liquid phase. If we ignore fluctuations in the spinon density, or drop the spinons entirely retaining a theory of Cooper pairs, the resulting phase is a charge ordered bosonic insulator. The nature of the charge ordering will in this case depend sensitively on the commensurability of the Cooper-pair density ( $\rho_0/2$ ) with the underlying square lattice. In the simplest commensurate case with one Cooper-pair per site ( $\rho_0=2$ ), a featureless Mott insulating state obtains.

### 1. Electrical resistance in the roton-liquid

When  $\Delta_v > 2$ , on the other hand, vortex-hopping is ‘‘irrelevant,’’ and the effects of the hopping on physical quantities can be treated perturbatively. Physically, this occurs because a vortex sees any background charge as a magnetic field, and accumulates an Aharonov–Bohm flux upon encircling such charge. There are enough charge fluctuations in the roton-liquid (it is nearly superconducting) for  $\Delta_v > 2$  to randomize this phase and render vortex-hopping incoherent over long times/distances. Despite its irrelevance, we expect the vortex-hopping to strongly modify the Gaussian result for the conductivity by introducing dissipation. To understand how this occurs, it is instructive to first consider the simple limit alluded to earlier in which  $v_0 \rightarrow \infty$ . In this limit, the longitudinal density fluctuations described by  $\tilde{a}$  at nonzero wave vector are suppressed, and the roton mode is purely captured by the  $\theta$  field. The zero wave vector (but nonzero frequency) piece of  $a$ , however, remains nonzero in this limit and can be used to calculate the conductivity in an RPA fashion. In particular, we take into account the roton fluctuations and their associated dissipation induced by vortex-hopping by calculating the effective action for  $\tilde{a}, A$  upon integrating out  $\theta$  to second order (the lowest nontrivial contribution) in  $t_v$ . Starting then with the Lagrangian  $\mathcal{L}_{\text{RL}} + \mathcal{L}_A + \mathcal{L}_v$ , expanding to second order in the vortex hopping action  $S_v$ , and integrating over the  $\theta$  field and the gauge field  $a(\mathbf{k} \neq 0)$  with  $v_0 \rightarrow \infty$  gives

$$S_{a,A}^{\text{eff}} = \sum_{\mathbf{r}=\mathbf{r}+\mathbf{w}} \int_{\tau} \left[ \frac{1}{2u_v} (\partial_0 \tilde{a}_j)^2 - \frac{i}{\pi} \epsilon_{ij} \tilde{a}_i(\mathbf{r}) \partial_0 A_j(\mathbf{r} - \hat{\mathbf{x}}_j) \right] + S_{a,A}^{(2)}, \quad (136)$$

where

$$S_{a,A}^{(2)} = - \frac{t_v^2}{2} \sum_{\mathbf{r}, \mathbf{r}'} \int_{\tau, \tau'} \langle \cos(\partial_t \theta - \tilde{a}_i)_{\mathbf{r}\tau} \cos(\partial_j \theta - \tilde{a}_j)_{\mathbf{r}'\tau'} \rangle_{\theta}, \quad (137)$$

where  $\langle \cdots \rangle_{\theta}$  indicates the average with respect to the Gaussian action for  $\theta$ . From Eq. (132), one can carry out this average to obtain

$$S_{a,A}^{(2)} \sim - \frac{t_v^2}{4} \left\{ \sum_{\mathbf{r}, \mathbf{x}} \int_{\tau, \tau'} \mathcal{R}(x, \tau - \tau') \times \cos[\tilde{a}_y(\mathbf{r}, \tau) - \tilde{a}_y(\mathbf{r} + x\hat{\mathbf{x}}, \tau')] + (x \leftrightarrow y) \right\}, \quad (138)$$

with the roton propagator  $\mathcal{R}(x, \tau)$  given as in Eq. (133), ex-

cept with  $\Delta_v \rightarrow \sqrt{u_v/\kappa_r}/(4\pi)$  in this limit. Following the usual RPA strategy, we expand  $S^{(2)}$  to quadratic order in  $\tilde{a}$  to obtain

$$S_{a,A}^{(2)} = \frac{t_v^2}{4} \int_{\mathbf{k}, \omega_n} \tilde{\mathcal{R}}(\omega_n) |\tilde{a}_j(\omega_n)|^2, \quad (139)$$

with the definition  $\tilde{\mathcal{R}}(\omega_n) = \mathcal{R}(0) - \mathcal{R}(\omega_n)$  and

$$\mathcal{R}(\omega_n) \equiv \mathcal{R}(k_x = 0, \omega_n). \quad (140)$$

The  $\mathbf{k}=0$  limit is valid when  $v_0 \rightarrow \infty$ . At low frequencies one has

$$\mathcal{R}(\omega_n) = -C_\gamma \frac{|\omega_n/v_{\text{rot}}|^{1+\gamma}}{v_{\text{rot}} \sin \frac{\pi}{2}(1+\gamma)} + \dots, \quad (141)$$

with  $C_\gamma > 0$  a dimensionless constant. Here we have retained explicitly the leading singular frequency dependence and dropped analytic terms consisting of even powers of  $\omega_n$ . The exponent  $\gamma$  is defined as

$$\gamma = 2\Delta_v - 3 \quad (142)$$

and in the stable regime of the RL phase,  $\gamma > 1$ .

Finally, upon integrating out  $\tilde{a}_j$ , one obtains a renormalized electromagnetic response tensor

$$\Pi_{ij} = \frac{\omega_n^2}{u_v^{-1}(\pi\omega_n)^2 + \frac{\pi^2 t_v^2}{2} \tilde{\mathcal{R}}(\omega_n)} \frac{k_i k_j}{k^2}. \quad (143)$$

It is now straightforward to extract the conductivity by analytic continuation

$$\sigma(\omega) = \frac{\Pi_{xx}(k_x \rightarrow 0, k_y = 0, \omega_n)}{\omega_n} \Big|_{i\omega_n \rightarrow \omega + i\delta}. \quad (144)$$

One obtains an appealing Drude form

$$\sigma(\omega) = \frac{1}{-i\omega(\pi^2/u_v) + i\frac{\pi^2 t_v^2}{2} \tilde{\mathcal{R}}_{\text{ret}}(\omega)/\omega} \quad (145)$$

with the retarded propagator obtained from analytic continuation:

$$\mathcal{R}_{\text{ret}}(\omega) = \mathcal{R}(\omega_n) \Big|_{i\omega_n \rightarrow \omega + i\delta}. \quad (146)$$

The nonanalytic frequency dependence of  $\mathcal{R}_{\text{ret}}(\omega)$  contributes to the dissipative (real) part of the resistance (per square)  $R(\omega) \equiv \text{Re}\sigma^{-1}(\omega)$ , which is quadratic in the vortex-hopping amplitude  $t_v^2$ :

$$R(\omega) = \frac{\pi^2 t_v^2}{2\omega} \text{Im} \mathcal{R}_{\text{ret}}(\omega) = C_\gamma \frac{\pi^2 t_v^2}{2v_{\text{rot}}^2} \left[ \frac{|\omega|}{v_{\text{rot}}} \right]^\gamma. \quad (147)$$

Note that at the point for which vortex-hopping is just marginal,  $\gamma=1$ , the resistance becomes linear in frequency  $\text{Re}\sigma(\omega) \sim 1/\omega$ .

We can readily extend this result to finite temperatures, by using the finite temperature roton propagator

$$\mathcal{R}(\tau) = c_\gamma \left[ \frac{\pi/v_{\text{rot}}\beta}{\sin(\pi\tau/\beta)} \right]^{\gamma+2}. \quad (148)$$

The retarded roton propagator follows upon analytic continuation, and can be most readily extracted by using the identity

$$\text{Im} \mathcal{R}_{\text{ret}}(\omega) = \sinh(\beta\omega/2) \int_{-\infty}^{\infty} dt e^{i\omega t} \mathcal{R}\left(\tau \rightarrow \frac{\beta}{2} + it\right). \quad (149)$$

Upon combining Eqs. (147)–(149) we thereby obtain a general expression for the finite temperature and frequency (dissipative) resistance in the roton liquid phase:

$$R(\omega, T) = c_\gamma \frac{(\pi t_v)^2}{v_{\text{rot}}^2} \left[ \frac{\pi T}{v_{\text{rot}}} \right]^\gamma \tilde{R}_\gamma(\omega/\pi T), \quad (150)$$

with a universal crossover scaling function

$$\tilde{R}_\gamma(X) = \frac{2^\gamma}{\Gamma(2+\gamma)} \left| \Gamma\left(1 + \frac{\gamma + iX}{2}\right) \right|^2 \frac{\sinh(\pi X/2)}{X}, \quad (151)$$

interpolating between the dc resistance at finite temperature and the  $T=0$  ac behavior. Since  $\tilde{R}_\gamma(X \rightarrow 0)$  is finite, the dc resistance varies as a power law in temperature  $R(T) \sim T^\gamma$ . At the boundary of the RL phase with  $\gamma=1$ , a linear temperature dependence is predicted. At large argument,

$$\tilde{R}_\gamma(X \rightarrow \infty) = \frac{\pi}{\Gamma(2+\gamma)} X^\gamma, \quad (152)$$

so that the resistance crosses over smoothly to the zero temperature form  $R(\omega, T=0) \sim |\omega|^\gamma$ .

This RPA treatment has the appeal that it produces the natural physical result that the effect of the weak (irrelevant) vortex-hopping is to generate a small *resistivity*  $\sim t_v^2$ . Formally, it is correct for  $v_0 = \infty$  because the RPA reproduces the exact perturbative result for the electromagnetic response tensor to  $O(t_v^2)$  in this case. Unfortunately, when the spatial fluctuations of  $\tilde{a}_j$  are not negligible, i.e., for  $v_0 < \infty$ , even the  $O(t_v^2)$  term is not obtained correctly. More generally, the fluctuations of  $\tilde{a}_j$  and  $\theta$  must be treated on the same footing. Therefore in the general case we instead integrate out *both* fields and obtain more directly the correction to  $\Pi_{ij}$  to  $O(t_v^2)$ . The calculations are described in Appendix E. This does not yield the appealing ‘‘Drude’’ form in Eq. (145) but instead the Taylor expansion of Eq. (143) to  $O(t_v^2)$ ,

$$\Pi_{ij}^{(2)} \sim -\frac{t_v^2 u_v^2}{2v_{\text{rot}}^{2\Delta_v-1}} |\omega_n|^{2\Delta_v-4} \frac{k_i k_j}{k^2}, \quad (153)$$

except with the scaling dimension  $\Delta_v$ , given explicitly in Eq. (134), now fully renormalized by the plasmon fluctuations. Provided the vortex-hopping is irrelevant, this is sufficient to recover properly the low-frequency behavior of the resistivity (147) to  $O(t_v^2)$ . In particular, formally inverting the perturbative result for  $\sigma(\omega, T)$  to  $O(t_v^2)$ , we infer the appropriate dc dissipative resistance  $R(T) \sim t_v^2 T^\gamma$ , with  $\gamma = 2\Delta_v - 3$ .



### B. “Charge hopping”

In this subsection we examine the legitimacy of the spin wave expansion employed in Sec. III A to obtain the Gaussian roton liquid Lagrangian. This is most readily achieved by passing to a dual representation, which exchanges vortex operators for new charge operators. This procedure is a quantum analog of the mapping from the classical 2D XY model to a sine-Gordon representation,<sup>16</sup> the latter suited to examine corrections to the low temperature spin wave expansion. As we shall find, there are parameter regimes where the charge hopping terms are irrelevant and the RL phase is stable. However, outside of these regimes, the charge quasiparticles become mobile at low energies and condense—driving an instability into a conventional superconducting phase. Throughout this subsection we will drop the vortex hopping term  $H_v$  focusing on the parameter regimes of the roton liquid phase where it is irrelevant (i.e.,  $\Delta_v > 2$ ).

#### 1. Plaquette duality

To this end we now employ the plaquette duality transformation, originally introduced in Ref. 12 in the context of the exciton-Bose-liquid phase. Consider the charge sector of the theory, with Hamiltonian  $H_c = H_{\text{pl}} + H_\theta + H_v$ . This Hamiltonian is a function of the vortex phase field and number operators  $\theta_r, N_r$  (living on the sites of the dual lattice), as well as the (transverse) gauge field  $a_j^i$  and its conjugate transverse electric-field  $e_j^i$ . The plaquette duality exchanges  $\theta_r$  and  $N_r$  for a new set of canonically conjugate fields which live on the sites of the original 2D square lattice. The two new fields, denoted  $\tilde{\phi}_r$  and  $\tilde{n}_r$ , are defined via the relations

$$\pi N_{\mathbf{r}+\mathbf{w}} \equiv \Delta_{xy} \tilde{\phi}_r, \quad (154)$$

$$\pi \tilde{n}_r \equiv \Delta_{xy} \theta_{\mathbf{r}-\mathbf{w}}. \quad (155)$$

Although  $\tilde{\phi}_r$  and  $\tilde{n}_r$  are conjugate fields satisfying

$$[\tilde{n}_r, \tilde{\phi}_{r'}] = i \delta_{\mathbf{r},\mathbf{r}'}, \quad (156)$$

they cannot strictly be interpreted as phase and number operators since the eigenvalues of  $\tilde{\phi}_r = \pi m$  for arbitrary integer  $m$ , whereas  $\pi \tilde{n}_r$  is  $2\pi$  periodic. It is important that  $\tilde{\phi}_r$  and  $\tilde{n}_r$  not be confused with the charge-on phase and number operators introduced in Sec. II which were denoted  $\phi_r, n_r$ —without the tildes. As we discuss below,  $e^{i\tilde{\phi}_r}$  does in fact create a chargelike excitation, but it is *not* the charge-on.

Under the change of variables  $H_\theta(\theta, N) \rightarrow H_\phi(\tilde{\phi}, \tilde{n})$ ,

$$H_\phi = H_u - \kappa_r \sum_{\mathbf{r}} \cos \left[ \pi \tilde{n}_{\mathbf{r}+\mathbf{w}} - \frac{1}{2} (\partial_x \tilde{a}_y + \partial_y \tilde{a}_x) \right], \quad (157)$$

with the definition

$$H_u = \frac{u_v}{2\pi^2} \sum_{\mathbf{r}, \mathbf{r}'} \Delta_{xy} \tilde{\phi}_r \Delta_{xy} \tilde{\phi}_{r'} V(\mathbf{r} - \mathbf{r}'). \quad (158)$$

In the roton liquid phase the cosine term in  $H_\phi$  is expanded to quadratic order, and  $H_\phi + H_{\text{pl}} \rightarrow H_{\text{RL}}$ . To be consistent, both  $\tilde{n}$  and  $\tilde{\phi}$  must then be allowed to take on any real

value, and it is convenient to pass to a Lagrangian written just in terms of  $\tilde{\phi}$ :

$$L_{\text{RL}} = L_{\text{pl}} + H_u + \frac{1}{2} \sum_{\mathbf{r}} \frac{(\partial_0 \tilde{\phi}_r)^2}{\pi^2 \kappa_r} + \frac{i}{2\pi} \sum_{\mathbf{r}} \partial_0 \tilde{\phi}_{\mathbf{r}+\mathbf{w}} (\partial_x \tilde{a}_y + \partial_y \tilde{a}_x). \quad (159)$$

In this dual form the roton liquid Lagrangian depends (quadratically) on the field  $\tilde{\phi}_r$ , which lives on the sites of the direct lattice, and the (transverse) gauge-field  $\tilde{a}_j(\mathbf{r})$  defined on the links of the dual lattice.

#### 2. Superconducting instabilities

This dual formulation is ideal for studying the legitimacy of the spin-wave expansion needed to obtain the quadratic RL Lagrangian. The crucial effect of the spin-wave expansion was in softening the integer constraint on the eigenvalues of  $\tilde{\phi}/\pi$ , allowing  $\tilde{\phi}$  to take on all real values in  $L_{\text{RL}}$ . It is, however, possible to mimic the effects of this constraint by adding a potential term to  $L_{\text{RL}}$  of the form

$$L_\lambda = -\lambda \sum_{\mathbf{r}} \cos(2\tilde{\phi}_r). \quad (160)$$

When  $\lambda \rightarrow \infty$  the integer constraint is enforced, whereas the RL phase corresponds to  $\lambda = 0$ . Stability of the RL phase can be studied by treating  $\lambda$  as a *small* perturbation to the quadratic RL Lagrangian. But as discussed in Ref. 12, one should also consider other perturbations to  $L_{\text{RL}}$  which might be even more relevant. Generally, one can add any local operator involving  $\tilde{\phi}_r$  at a set of nearby spatial points which is  $2\pi$  periodic in  $2\tilde{\phi}$ , and satisfies all the discrete lattice symmetries (i.e., translations, rotations, and parity). For example, terms of the form  $\cos(2l\tilde{\phi})$  for arbitrary integer  $l$  are allowed, although these will generically become less relevant with increasing  $l$ . As we shall see, for our “minimal” model of the RL phase, the most relevant perturbation is of the form

$$L_t = -t_c \sum_{\mathbf{r}} \sum_{j=1,2} \cos(2\tilde{\phi}_j \tilde{\phi}_r). \quad (161)$$

Before studying the perturbative stability to such operators, we try to get some physical intuition for the meaning of the operator  $e^{i\tilde{\phi}}$ . From the commutation relations in Eq. (156), the operator  $e^{i\tilde{\phi}_r}$  increases  $\tilde{n}_r$  by 1, and creates some sort of quasiparticle excitation on the spatial site  $\mathbf{r}$ . Since the perturbation in Eq. (160) changes the number  $\tilde{n}_r$  by  $\pm 2$ , the total number of these quasiparticles  $\tilde{n}_{\text{tot}} = \sum_{\mathbf{r}} \tilde{n}_r$  is *not* conserved, but the complex charge  $Q_c = e^{i\pi \tilde{n}_{\text{tot}}}$  is conserved. The perturbation in Eq. (161) can then be interpreted as a charge hopping process. To get some feel for the nature of the quasiparticle, it is instructive to introduce an external magnetic field  $B = \epsilon_{ij} \partial_i A_j$ , which enters into  $H_u$  above via the substitution  $\Delta_{xy} \tilde{\phi}_r \rightarrow \Delta_{xy} \tilde{\phi}_r - B_{\mathbf{r}+\mathbf{w}}$ . A spatially uniform field  $B$  can readily be removed from the Gaussian RL Lagrangian by letting

$$\tilde{\phi}_{\mathbf{r}} \rightarrow \tilde{\phi}_{\mathbf{r}} + Bxy. \quad (162)$$

But then  $B$  appears in the cosine perturbations, in “almost” a minimal coupling form. For example, one has the combination  $\partial_x \tilde{\phi} - 2A_x$  where we have chosen the gauge  $A_x = -By/2$  and  $A_y = Bx/2$ , suggesting that  $e^{i\tilde{\phi}}$  carries the Cooper pair charge. But the  $y$  derivative enters as  $\partial_y \tilde{\phi} + 2A_y$ —with the *wrong* sign. Thus, this quasiparticle is *not* a conventional electrically charged particle. Nevertheless, as we show below, condensation of the quasiparticle with  $\langle e^{i\tilde{\phi}} \rangle \neq 0$  does drive the RL phase into a superconducting state.

To evaluate the relevance of the various perturbations in the RL phase, requires diagonalizing the associated action  $S_{\text{RL}}$ . In momentum space one has

$$S_{\text{RL}} = \int_{k_\mu} \left[ \frac{D_{\phi\phi}}{2} |\tilde{\phi}|^2 + \frac{D_{aa}}{2} |a|^2 + D_{a\phi} a(k_\mu) \tilde{\phi}(-k_\mu) \right], \quad (163)$$

$$D_{\phi\phi} = \frac{1}{\pi^2 \kappa_r} \left[ \omega_n^2 + \frac{u_v \kappa_r |\mathcal{K}_x \mathcal{K}_y|^2}{\mathcal{K}^2} \right], \quad (164)$$

$$D_{aa} = \frac{1}{u_v} [\omega_n^2 + \tilde{v}_0^2 \mathcal{K}^2], \quad (165)$$

$$D_{a\phi} = \frac{\omega_n (\mathcal{K}_x^2 - \mathcal{K}_y^2) e^{i\mathbf{k}\cdot\mathbf{w}}}{2\pi\mathcal{K}}, \quad (166)$$

with  $k_\mu = (\mathbf{k}, \omega_n)$ . Evaluation of the two-point function of  $e^{2i\tilde{\phi}}$  then gives

$$\langle e^{i2\tilde{\phi}_{\mathbf{r}}(\tau)} e^{-i2\tilde{\phi}_0(0)} \rangle_{\text{RL}} \sim \delta_{\mathbf{r},0} |\tau|^{-2\Delta_c}, \quad (167)$$

with scaling dimension

$$\Delta_c = 2\pi \sqrt{\frac{\kappa_r \tilde{v}_0}{u_v v_+}}. \quad (168)$$

We note in passing that this power-law form of the charge operator ( $e^{2i\tilde{\phi}}$ ) two-point function in Eq. (167) differs from  $\exp(-\ln^2 \tau)$  behavior of the corresponding object in the exciton Bose liquid of Ref. 12, due to the long-range logarithmic interactions between vortices. Stability of the RL phase requires  $\Delta_c > 1$ . Evaluation of the two-point function of the charge hopping operator  $e^{2i\partial_y \tilde{\phi}}$  gives

$$\langle e^{i2\partial_y \tilde{\phi}_{\mathbf{r}}(\tau)} e^{-i2\partial_y \tilde{\phi}_0(0)} \rangle_{\text{RL}} \sim \frac{\delta_{y,0}}{(x^2 + v_{\text{rot}}^2 \tau^2)^{\Delta_c}}, \quad (169)$$

with  $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ , and with the *same* scaling dimension  $\Delta_c$  as above. However, since this two-point function decays algebraically in two (rather than one) space-time dimensions, the perturbation  $t_c$  is relevant for  $\Delta_c < 2$ .

When  $\Delta_c < 2$ , the charge hopping process will grow at low energies, and will destabilize the roton liquid phase. Not surprisingly, the resulting quantum ground state is superconducting. Indeed, the exponent  $\Delta_c$  in Eq. (168) above is in fact *identical* to the exponent characterizing the power law decay

of the Cooper pair propagator in Eq. (121), so that for small  $\Delta_c$  the roton liquid phase is already almost superconducting. Moreover, when the vortex core energy greatly exceeds the roton hopping strength  $u_v \gg \kappa_r$ , the Hamiltonian  $H_\theta$  in Eq. (88) is deep within its superconducting phase. This limit precisely corresponds to  $\Delta_c \ll 1$ , the limit where the charge hopping is strongly relevant. More directly, when the charge-hopping strength  $t_c$  grows large, the field  $\tilde{\phi}$  gets trapped at the minimum of the cosine potentials in Eqs. (160) and (161), and it is legitimate to expand the cosine potentials to quadratic order. Once massive, the expectation value  $\langle e^{i\tilde{\phi}} \rangle \neq 0$ —and the charge quasiparticle has condensed. Moreover, setting  $\tilde{\phi} = 0$  in Eq. (161) in the presence of an applied magnetic field will generate a term of the form

$$\mathcal{L}_t \sim \frac{t_c}{2} \vec{A}^2, \quad (170)$$

indicative of a Meissner response.

In the resulting superconducting phase, the rotons—gapless in the RL phase—become gapped. This follows upon expanding the cosine term in Eq. (160)  $L_\lambda = 2\lambda \tilde{\phi}^2$ , which leaves the roton liquid Lagrangian quadratic, and allow one to readily extract the modified roton dispersion. For  $k_x \rightarrow 0$  at fixed  $k_y$ , this gives

$$\omega_{\text{rot}}(\mathbf{k}) = \sqrt{v_{\text{rot}}^2 k_x^2 + m_{\text{rot}}^2}, \quad (171)$$

with the “roton mass gap” given by  $m_{\text{rot}} = 2\pi\sqrt{\kappa_r \lambda}$ . Since the product  $\Delta_c \Delta_v = \frac{1}{2}$ , it is not possible to have both the vortex-hopping and the charge-hopping terms simultaneously irrelevant.

## V. THE ROTON FERMI LIQUID PHASE

We now put the fermions back into the description of the roton-liquid. We first consider setting the explicit pairing term in the fermion Hamiltonian  $\mathcal{H}_f$  in Eq. (48) to zero:  $\Delta = 0$ . As in Sec. III, we will assume for the most part (with the exception of Sec. VIII B 3 in the discussion) that the equilibrium fermion density is equal to the charge density  $\rho_0$ . This choice naturally minimizes Coulomb energy and vortex kinetic energy, as discussed therein. As we shall see, in this way we will arrive at a description of a novel non-Fermi-liquid phase—the roton Fermi liquid—which supports a gapless Fermi surface of quasiparticles coexisting with a gapless set of roton modes. We then reintroduce a nonzero pairing term, and study the perturbative effects of  $\Delta$ . We argue that, when the scaling exponent that describes the decay of the off-diagonal order in the roton liquid is large enough  $\Delta_c > \Delta_c^* > 3/2$ , the explicit pairing term is perturbatively irrelevant, and the RFL phase with a full gapless Fermi surface is stable.

Even in the absence of the explicit pairing term—which couples the fermionic and vortex degrees of freedom in a highly nonlinear manner—the rotons and quasiparticles interact through (three-)current-current interactions “mediated” by the  $\beta_\mu$  and  $e_i$  fields. Although these interactions are long ranged for individual vortices, they are not for rotons,

which carry no net vorticity. Moreover, due to phase space restrictions we find that the residual short-ranged interactions asymptotically decouple at low energies. The resulting RFL phase supports both gapless charge and spin excitations with no broken spatial or internal symmetries, just as in a conventional Fermi liquid. But, due to the vortex sector of the theory, the RFL phase is demonstrably a non-Fermi liquid, with a gapless Bose surface of rotons and with ODQLRO in the Cooper pair field but no Meissner effect (see below). Moreover, the quasiparticles at the RFL Fermi surface are sharp (in the sense of the electron spectral function), but electrical currents are carried by the (quasi)condensate. Even with impurities present the resistivity vanishes as a power law of temperature in the RFL. The power law exponent  $\gamma$  varies continuously, but is greater than or equal to one. For  $\gamma < 1$ , the zero temperature RFL phase is unstable to a quantum confinement transition, which presumably drives the system into a conventional Fermi liquid phase.

To describe the RFL, we start with the general Lagrangian (68)–(71), and first make the same approximation as in the RL of expanding the roton-hopping term to quadratic order. That is, to leading order,  $\mathcal{L}_{\text{kin}} \approx \mathcal{L}_r^0$ , with

$$\begin{aligned} \mathcal{L}_r^0 = & \frac{\kappa_r}{2} \left[ \partial_{xy} \theta - \frac{1}{2} \{ \partial_x (a_y - \alpha_y) + \partial_y (a_x - \alpha_x) \} \right]^2 \\ & + \frac{\kappa_r}{8} [\epsilon_{ij} \partial_i (a_j - \alpha_j)]^2. \end{aligned} \quad (172)$$

As before for the RL, this approximation will be corrected perturbatively by charge and vortex hopping terms, which will *not* be expanded.

Turning to the fermionic sector, we assume for the moment that the power-law decay (ODQLRO) of the Cooper-pair field is sufficiently rapid that the pair field term  $\Delta_j$  can be neglected. We argue later this is correct for  $\Delta_c > \Delta_c^* > 3/2$ . This gives a nonanomalous fermionic Lagrange density  $\mathcal{L}_f$ , which we further presume is well described by Fermi liquid theory, again checking the correctness of this assumption perturbatively in the couplings to  $\beta_\mu$  and  $e_i$ . Hence we replace  $\mathcal{L}_f \approx \mathcal{L}_f^0 + \mathcal{L}_f^1$ , with (working for simplicity at zero temperature)

$$\mathcal{L}_f^0 = f_{\mathbf{r}\sigma}^\dagger (\partial_0 - \mu) f_{\mathbf{r}\sigma} - t \sum_j f_{\mathbf{r}+\hat{\mathbf{x}}_j\sigma}^\dagger f_{\mathbf{r}\sigma}, \quad (173)$$

$$\begin{aligned} \mathcal{L}_f^1 = & -i\beta_0(\mathbf{r}) f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma} - i \sum_j t\beta_j(\mathbf{r}) + t_e [\pi \bar{e}_j(\mathbf{r}) + A_j(\mathbf{r})] f_{\mathbf{r}+\hat{\mathbf{x}}_j\sigma}^\dagger f_{\mathbf{r}\sigma} \\ & + \sum_j \left[ \frac{t_s}{2} \beta_j(\mathbf{r})^2 + \frac{t_e}{2} [\beta_j(\mathbf{r}) + \pi \bar{e}_j(\mathbf{r}) + A_j(\mathbf{r})]^2 \right] f_{\mathbf{r}+\hat{\mathbf{x}}_j\sigma}^\dagger f_{\mathbf{r}\sigma}. \end{aligned} \quad (174)$$

Note that, to leading order, the fermionic dispersion is controlled by the sum of the two hopping amplitudes  $t = t_s + t_e$ . Here we have included explicitly the physical external vector potential  $A_j(\mathbf{r})$  in the electron hopping term. Some care needs to be taken when treating  $A_j(\mathbf{r})$ . The above form is correct provided  $A_j(\mathbf{r})$  is also coupled into  $e_j$  in the quadratic Hamiltonian in the canonical fashion  $\pi \bar{e}_j \rightarrow \pi \bar{e}_j + A_j$ , in the

roton/plasmon portion of the Hamiltonian. It is *not* correct if this canonical coupling is removed by shifting  $\bar{e}_j$ , which is the procedure needed to generate the  $A_\mu J_\mu$  coupling in Eq. (74). If the latter form of the Lagrangian is used, the vector potential should be removed from the electron hopping term. Either choice is correct if used consistently.

The full Lagrangian that we then use to access the RFL phase is given by

$$\mathcal{L}_{\text{RFL}} = \mathcal{L}_a + \mathcal{L}_0 + \mathcal{L}_r^0 + \mathcal{L}_{cs} + \mathcal{L}_f^0 + \mathcal{L}_f^1, \quad (175)$$

with the definition,

$$\mathcal{L}_0 = \frac{1}{2u_0} (\partial_0 \theta - a_0 + \alpha_0)^2. \quad (176)$$

The interaction terms between the fermions and the fields  $\beta_j$  and  $e_j$  in  $\mathcal{L}_f^1$  will be treated in the random phase approximation. Doing so, one arrives at a tractable, if horrendously algebraically complicated Lagrangian describing the RFL, which is quadratic in the fields  $\theta$ ,  $a_\mu$ ,  $e_j$ ,  $\alpha_\mu$ , and  $\beta_\mu$ . This makes calculations of nearly any physical quantity possible in the RFL.

Before turning to these properties, we verify (in the remainder of this section) the above claims that the coupling of fermions and vortices does not destabilize the RFL—i.e., it neither modifies the form of the low-energy roton excitations at the Bose surface nor the fermionic quasiparticles at the Fermi surface.

### A. Quasiparticle scattering by rotons

In this subsection, we show that the coupling of the electronic quasiparticles to the vortices does not destroy the Fermi surface. To do so, we will integrate out the vortex degrees of freedom to arrive at effective interactions amongst the quasiparticles. This procedure is somewhat gauge dependent. To provide a useful framework for the calculation of the electron spectral function in the following section, we will choose the gauge  $\vec{\nabla} \cdot \vec{\beta} = \pi \epsilon_{ij} \partial_i e_j$ , in which the  $f$ ,  $f^\dagger$  operators create fermionic quasiparticles with nonvanishing overlap with the bare electrons, without the need for any additional string operators. This is essentially equivalent to working in the electron formulation of Appendix D.

Since this gauge choice explicitly involves  $\epsilon_{ij} \partial_i e_j$ , we must employ a path integral representation in which the transverse component of the electric field  $e_j^t$  has not been integrated out. It is further convenient to fix the two remaining gauge choices according to  $\vec{\nabla} \cdot \vec{a} = a_0 = 0$ , and to integrate out the field  $a_0$  in the  $u_0 \rightarrow 0$  limit. The full RFL Lagrangian density (before imposing the constraint  $\beta_j^t = \pi e_j^t$ ) then takes the form

$$\mathcal{L}_{\text{RFL}} = \mathcal{L}_{\text{vort}} + \mathcal{L}_f^0(\beta_\mu) + \mathcal{L}_f^1(\beta_\mu), \quad (177)$$

with

$$\begin{aligned} \mathcal{L}_{\text{vort}} = & \frac{u_v}{2}(e_j^t)^2 + i e_j^t \partial_0 a_j^t + \frac{v_0^2}{2u_v}(\epsilon_{ij} \partial_i a_j^t - \pi \bar{\rho})^2 \\ & + \frac{i}{\pi} \epsilon_{ij} \alpha_j (\partial_j \beta_0 - \partial_0 \beta_j) + \mathcal{L}_\theta(\alpha_j - a_j^t), \end{aligned} \quad (178)$$

where

$$\begin{aligned} \mathcal{L}_\theta(\alpha_j) = & \frac{1}{2u_v}(\partial_0 \theta)^2 + \frac{\kappa_r}{8}(\epsilon_{ij} \partial_i \alpha_j)^2 \\ & + \frac{\kappa_r}{2} \left[ \Delta_{xy} \theta + \frac{1}{2}(\partial_x \alpha_y + \partial_y \alpha_x) \right]^2. \end{aligned} \quad (179)$$

To assess the perturbative effects of the vortices upon the electronic quasiparticles, we wish to integrate out the vortices perturbatively in the coupling of the gauge-field  $\beta_\mu$  to the fermions. For this we require the correlation functions of the  $\beta_\mu$  fields (neglecting the couplings inside  $\mathcal{L}_f$  and in particular to lowest order just  $\langle \beta_\mu \beta_\nu \rangle$ ). To obtain the latter, we add a source term to the Lagrangian

$$\mathcal{L}_{\text{vort}} \rightarrow \mathcal{L}_{\text{vort}} + i(\lambda_0 i \beta_0 + \vec{\lambda} \cdot \vec{\beta}). \quad (180)$$

Here we have included an extra factor of  $i$  with  $\beta_0$  to compensate for the factor of  $i$  present in the coupling of  $\beta_0$  to the fermion density. Upon fully integrating out the  $e^t$ ,  $\theta$ ,  $a^t$ ,  $\beta_\mu$ ,  $\alpha_j$  fields, the coefficient of  $\lambda_\mu \lambda_\nu$  in the effective action will give (half) the desired correlator. We perform the integration in two stages. First, imposing the constraint  $\beta^t = \pi e^t$ , we eliminate  $e^t$ , shift  $\alpha_j \rightarrow \alpha_j - (\partial_j \theta - a_j^t)$ , and integrate out the  $\theta$ ,  $a_j^t$ , and  $\beta_\mu$  fields. One obtains  $\mathcal{L}_{\text{vort}} \rightarrow \tilde{\mathcal{L}}_{\text{vort}}(\alpha_j, \lambda_\mu)$ , with

$$\begin{aligned} \tilde{\mathcal{L}}_{\text{vort}} = & \frac{v_0^2}{2u_v}(\epsilon_{ij} \partial_i \alpha_j + i \pi \tilde{\lambda}_0)^2 + \frac{\kappa_r}{4}[(\partial_x \alpha_y)^2 + (\partial_y \alpha_x)^2] \\ & + \frac{1}{2u_v}(\partial_0 \alpha_i - \pi \epsilon_{ij} \lambda_j)^2 - t_v \cos(\alpha_j), \end{aligned} \quad (181)$$

with  $\tilde{\lambda}_0 = \lambda_0 - i\bar{\rho}$ .

In Eq. (181) we have added back in the vortex-hopping term,  $\mathcal{L}_v = -t_v \cos(\alpha_j)$ , neglected in the RFL Lagrangian. In the RFL phase, the vortex-hopping is irrelevant, and scales to zero at low energies. If we put  $t_v = 0$ , the remaining integrations over  $\alpha_j$  are Gaussian and can be readily performed. We will return to the effects of nonzero vortex-hopping upon the fermions in Sec. VII A. The final effective action then takes the form

$$S_{\text{eff}}(\lambda_\mu) = -\frac{1}{2} \int_{\mathbf{k}, \omega_n} U_{\mu\nu}^{(0)}(\mathbf{k}, \omega_n) \lambda_\mu(\mathbf{k}, \omega_n) \lambda_\nu(-\mathbf{k}, -\omega_n). \quad (182)$$

Here

$$U_{\mu\nu}^{(0)}(\mathbf{k}, \omega_n) = \frac{\pi^2}{u_v(\omega_n^2 + \omega_{\text{pl}}^2)(\omega_n^2 + \omega_{\text{rot}}^2)} u_{\mu\nu}(\mathbf{k}, \omega_n) \quad (183)$$

specifies the  $\beta_\mu$  propagator  $\langle \beta_0 \beta_0 \rangle_0 = U_{00}^{(0)}$ ,  $\langle \beta_i \beta_j \rangle_0 = -U_{ij}^{(0)}$ ,  $\langle i \beta_0 \beta_j \rangle_0 = -U_{0j}^{(0)}$ , where the superscript zero reminds us that

this is the result to zeroth order in  $t_v = 0$ , and we have the definitions

$$u_{00} = v_0^2 \left[ \omega_n^4 + \frac{v_1^2 \mathcal{K}^2}{2} \omega_n^2 + \frac{v_1^4}{4} |\mathcal{K}_x \mathcal{K}_y|^2 \right], \quad (184)$$

$$u_{xx} = -v_1^2 [\bar{v}_2^0 |\mathcal{K}_x \mathcal{K}_y|^2 + v_+^2 |\mathcal{K}_x|^2 \omega_n^2], \quad (185)$$

$$u_{yy} = -v_1^2 [\bar{v}_2^0 |\mathcal{K}_x \mathcal{K}_y|^2 + v_+^2 |\mathcal{K}_y|^2 \omega_n^2], \quad (186)$$

$$u_{xy} = -v_0^2 \mathcal{K}_x \mathcal{K}_y \omega_n^2, \quad (187)$$

$$u_{0j} = -v_0^2 \mathcal{K}_x i \omega_n [2\omega_n^2 + v_1^2 (\mathcal{K}^2 - \mathcal{K}_j^2)]. \quad (188)$$

In the dc limit, these interactions simplify considerably, and one obtains the simple results

$$U_{00}^{(0)}(\mathbf{k}, \omega_n = 0) = \frac{\pi^2 \kappa_r}{4} \left( \frac{v_0}{\bar{v}_0} \right)^2, \quad (189)$$

$$U_{ij}^{(0)}(\mathbf{k}, \omega_n = 0) = -\frac{\pi^2}{u_v} \delta_{ij}, \quad (190)$$

and  $U_{0i}^{(0)}(\mathbf{k}, \omega_n = 0) = 0$ .

Since  $i\beta_0$  and  $\beta$  couple to the fermion density and current, respectively,  $\mathcal{S}_{\text{eff}}$  mediates an effective frequency and wave-vector-dependent interaction between fermions. Since, in the dc limit,  $-U_{00}^{(0)}(\mathbf{k}, \omega_n = 0)$  and  $U_{ii}^{(0)}(\mathbf{k}, \omega_n = 0)$  are finite and wave-vector independent, they describe local (on-site) repulsive quasiparticle density-density and attractive current-current interactions, respectively.

Generally, a repulsive density-density interaction between fermions will lead to Fermi-liquid corrections in the quasiparticle dynamics and thermodynamics, but will not destroy the Fermi surface. On the other hand, the current-current interaction for near-neighbor quasiparticle hopping can be rewritten in terms of antiferromagnetic, near-neighbor repulsive, and pairing interactions

$$H_J = J \sum_{\mathbf{r}, \mathbf{r}'} \left[ \mathbf{S}_{\mathbf{r}} \cdot \mathbf{S}_{\mathbf{r}'} + \frac{1}{4} n_{\mathbf{r}}^f n_{\mathbf{r}'}^f + 2 \bar{\Delta}_{\mathbf{r}}^f \Delta_{\mathbf{r}'}^f \right], \quad (191)$$

up to a shift of the fermion chemical potential, with  $\mathbf{S} \equiv f^\dagger (\vec{\sigma}/2) f$  (with  $\vec{\sigma}$  the vector of Pauli matrices)  $n^f = f^\dagger f$ ,  $\Delta^f = f_\uparrow f_\downarrow$ , and  $\bar{\Delta}^f = f_\downarrow^\dagger f_\uparrow^\dagger$ . Here,  $J \sim t_s^2 / u_v$  is inversely proportional to the vortex core energy. One expects that the antiferromagnetic interaction can lead to an a Cooper instability in the  $d$ -wave (or extended  $s$ -wave) channel. The repulsive pairing interaction does not favor an  $s$ -wave Cooper instability. Hence it seems possible that for small  $u_v$ , for which this  $J$  is large, a spontaneous quasiparticle pairing instability may occur, most probably of  $d$ -wave symmetry. Another possibility, which appears very natural for *extremely* small doping  $x \rightarrow 0^+$  (and  $u_v \rightarrow 0^+$ ), is that the antiferromagnetic interaction drives an antiferromagnetic spin-density-wave instability. We will discuss both possibilities in the discussion section.



### B. Roton scattering by quasiparticles

Having established that the Fermi surface remains intact (apart from a possible BCS-type pairing instability) in the presence of gapless rotons, we need to see how the fermions feed back and effect the roton modes. To this end, we will integrate out all fields except  $\theta$ , treating the fermions within the random phase approximation (RPA), to study the effects upon the roton dispersion. The RPA is complicated by the two distinct amplitudes  $t_s$  and  $t_e$ , describing spinon and electron hopping processes, the latter coupling to the electric field  $e_j$  as well as the  $\beta_j$  gauge field. We begin with the RFL Lagrangian in Eq. (175). It is convenient to first shift  $\alpha_\mu \rightarrow \alpha_\mu + a_\mu$  then integrate out  $a_0$  and take  $u_0 \rightarrow 0$ , which constrains  $\nabla \times \vec{\beta} = \pi \vec{\nabla} \cdot \vec{e}$  and  $\alpha_0 = -\partial_0 \theta$ . Choosing in addition  $\vec{\nabla} \cdot \vec{\beta} = -\pi \vec{\nabla} \times \vec{e}$ , we essentially return to the electron formulation, with  $\beta_i = -\pi \bar{e}_i$ . This choice is convenient, since the  $\beta_i$  and  $\pi \bar{e}_i$  fields present in the electron-hopping term precisely cancel, and only  $-\pi \bar{e}_i$  appears “similar to a gauge field” in the spinon-hopping term. The fermionic quasiparticles can then be integrated out in the random phase approximation (RPA). One finds  $S_f \rightarrow S_{\text{RPA}}$ , with

$$S_{\text{RPA}}(\beta_\mu) = \frac{1}{2} \int_{\mathbf{k}, \omega_n} \left[ \Pi_{00} |\beta_0(\mathbf{k}, \omega_n)|^2 + \frac{\pi^2 t_s^2}{t^2} \Pi_{11} |e_t(\mathbf{k}, \omega_n)|^2 + \pi^2 t_s \left( 1 - \frac{t_s}{t} \right) T_{\text{NN}} |e(\mathbf{k}, \omega_n)|^2 \right], \quad (192)$$

where  $T_{\text{NN}} = \langle f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}+\hat{x}\sigma} \rangle$  is the nearest-neighbor kinetic energy, and  $t = t_s + t_e$ . Here  $\Pi_{00}$  and  $\Pi_{11}$  are, respectively, the density-density and current-current response kernels (polarization bubbles + diamagnetic contribution in the case of  $\Pi_{11}$ ) that would obtain for the Fermi sea were a single gauge field minimally coupled to the fermions. These depend upon the band structure. We will not require particular expressions for these quantities, but will use the fact that both  $\Pi_{00}(\mathbf{k}, \omega_n)$  and  $\Pi_{11}(\mathbf{k}, \omega_n)$  are finite and generally nonvanishing at *fixed wave vector*  $|\mathbf{k}| > 0$  and  $\omega_n = 0$ . Since we will focus upon the low-energy rotons, which occur near the principle axes in the Brillouin zone, we have dropped terms in Eq. (192) that are small for  $\omega_n \sim k_x \ll k_y$  and  $\omega_n \sim k_y \ll k_x$  [e.g., due to the parity symmetry  $x \rightarrow -x$  of the 2D square lattice  $\Pi_{01}(k_x=0, k_y, \omega_n)$  vanishes]. We also note that Eq. (192) does not have the usual RPA form even in this limit, due to the nonminimal coupling form of the fermion Lagrangian (minimal coupling is restored only for  $t_s \rightarrow t$ ).

It may be helpful to keep in mind the forms for a circular Fermi surface (e.g., valid at small electron densities), where, at small wave vectors and frequencies, one has

$$\Pi_{00}(\mathbf{k}, \omega_n) \sim m^*, \quad (193)$$

where  $m^* \sim t^{-1}$  is the effective mass and

$$\Pi_{11}(\mathbf{k}, \omega_n) \sim \frac{1}{m^*} \left[ \mathbf{k}^2 + \frac{|\omega|}{v_F k} \right], \quad (194)$$

which is valid for  $|\omega| < v_F k$ . We stress, however, that our results do not depend upon these particular forms.

Since  $S_{\text{RPA}}$  is quadratic, one can perform the remaining integrals straightforwardly. We choose  $\vec{\nabla} \cdot \vec{a} = \vec{\nabla} \cdot \vec{\alpha} = 0$ , and integrate out  $a$ ,  $\beta_0$ , and  $e$ , to obtain

$$S = \int_{\mathbf{k}, \omega_n} \left\{ \frac{\omega_n^2}{2\bar{u}_v} \alpha^2 + \frac{\kappa_r 4v_0^2 + \bar{v}_1^2}{8\bar{v}_1^2} \alpha^2 + \frac{\omega_n^2 \mathcal{K}^2}{\bar{u}_v} \theta^2 + \frac{\kappa_r}{2} \left[ \mathcal{K}_x \mathcal{K}_y \theta + \frac{(\mathcal{K}_x^2 - \mathcal{K}_y^2)}{2\mathcal{K}} \alpha \right]^2 \right\}, \quad (195)$$

where  $\bar{u}_v = u_v + \pi^2 t_s (1 - t_s/t) T_{\text{NN}}$ ,  $\bar{u}_v = \bar{u}_v + \pi^2 t_s^2 / t^2 \Pi_{11}$ , and  $\bar{v}_1^2 = v_1^2 + \kappa_r \pi^2 \Pi_{00} v_0^2$ . Without loss of generality, we focus upon the branch of rotons with  $\omega_n \sim k_x \ll k_y \sim O(1)$ , for which the first term in Eq. (195) is negligible, and the remaining  $\alpha$  integral can be carried out to obtain finally the effective action

$$S_{\text{rot}} = \frac{1}{2} \int_{\mathbf{k}, \omega_n}' \frac{\bar{\mathcal{K}}_2^y}{\bar{u}_v} [\omega_n^2 + v_{\text{rot}}^2(k_y) k_x^2] \theta^2, \quad (196)$$

with

$$v_{\text{rot}}^2(k_y) = v_1^2 \frac{\bar{u}_v v_0^2 + \frac{1}{4} \bar{v}_1^2}{u_v v_0^2 + \frac{1}{2} \bar{v}_1^2}. \quad (197)$$

We have thereby shown that even a gapless Fermi sea *does not* lead to a qualitative change in the gapless Bose surface of rotons. Due to the  $k_y$  dependence of  $\Pi_{00}$  and  $\Pi_{11}$  (implicit in  $\bar{v}_1$  and  $\bar{u}_v$ ), the roton velocity is now seen to depend upon  $k_y$ . Additional “direct” quasiparticle-quasiparticle interactions (not mediated by the rotons) would in any case similarly renormalize  $v_{\text{rot}}$ . But the location of the Bose surface and the qualitative low-energy dispersion of the roton modes are seen to be unaffected by the fermions. Together with the earlier demonstration of the stability of the Fermi surface, this result establishes the RFL phase as a stable 2D non-Fermi liquid with *both* gapless charge and spin excitations.

## VI. INSTABILITIES OF THE ROTON FERMION LIQUID

As for the RL, the RFL has potential instabilities to superconducting and Fermi liquid states, driven by effects/terms neglected in the previous subsection. We address each in turn now.

### A. Landau–Fermi-liquid instability

The arguments of Sec. IV A for determining the relevance or irrelevance of the neglected vortex hopping term within the simpler RL continue to hold for the full RFL, provided the proper renormalized roton liquid parameters are employed. In particular, the vortex hopping term continues to be described by a (1+1)-dimensional scaling dimension  $\Delta_v$ , which is, however, renormalized by the statistical interactions with the fermions, to wit

$$\Delta_v = \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \frac{\bar{u}_v(k_y)}{v_{\text{rot}}(k_y)}, \quad (198)$$

with  $v_{\text{rot}}(k_y)$  given by Eq. (197). The same arguments thus continue to apply, and the vortex hopping term is irrelevant

for  $\Delta_v > 2$ . For  $\Delta_v < 2$ , we expect an instability to a state with proliferated vortices. The dual Meissner effect for this vortex condensate confines particles with nonzero gauge charge, as discussed in Sec. IV A. Coming from the RFL, the natural expectation is then that the system becomes a Fermi liquid. This hypothesis is fleshed out in Appendix A.

## B. Superconducting instabilities

As for the RL, the RFL can also be unstable to a superconducting state. However, the inclusion of the fermionic quasiparticles opens new routes to superconductivity from the RFL. We explore each of these in turn.

### 1. Charge hopping

As for the vortex-hopping considered above, the arguments for the relevance of the charge hopping in Sec. IV B 2 for the RL continue to apply with only a renormalization of the roton-liquid parameters. In particular, the scaling dimension defined from the charge hopping (169) is modified to

$$\Delta_c = \pi \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} |\mathcal{K}_y(k_y)|^2 \frac{v_{\text{rot}}(k_y)}{\bar{u}_v(k_y)}. \quad (199)$$

With this modification, the charge-hopping processes become relevant for  $\Delta_c < 2$ , as before. It should be noted that, including the renormalizations due to scattering by fermions  $\Delta_c \Delta_s \neq 1/2$ , owing to the  $k_y$  dependence of the  $v_{\text{rot}}$  and  $\bar{u}_v$ .

As we established in Sec. IV B 2 by employing the plaquette duality transformation, when the charge hopping processes are relevant the rotons become gapped out and the system exhibits a Meissner effect. Here, we briefly comment on the corresponding fate of the fermionic quasiparticles, which are gapless across the Fermi surface in the RFL phase. The relevance of the charge hopping in the plaquette dualized representation, indicates that it is not legitimate to expand the cosine term that enters into the roton hopping Lagrangian  $\mathcal{L}_r$ . Rather, in the original vortex representation, the state corresponds to a ‘‘vortex vacuum,’’ and the properties of the state can be accessed by taking all of the vortex hopping processes small,  $t_v, t_{2v}, \kappa_r \ll u_v$ . At zeroth order in the vortex kinetic terms, the full Hamiltonian of the U(1) vortex-spinon field theory in Eq. (43) is independent of  $\alpha_j$ , so that  $\beta_j$  commutes with  $\hat{H}$  and naively can be taken as  $c$  numbers. A simple choice consistent with the condition  $N_r=0$  is  $\beta_j=0$ , and in the vortex vacuum one also has that  $e_j^l=0$ . In this case, the full fermionic Hamiltonian describing the quasiparticles in Eq. (48) reduces to the Bogoliubov–de Gennes form, apart from the coupling to  $e_j^l$  which describes the Doppler shift couplings to the superflow. In particular, the string operator which enters into the explicit pairing term equals unity  $\mathcal{S}_r^2=1$ , so that the fermionic quasiparticles are paired. We denote by  $|0\rangle_f$  the ground state of  $\hat{H}_f$  with  $\beta_j=e_j=0$ , which is easily found by filling the Bogoliubov–de Gennes levels below the Fermi energy. Our naive ground state is then

$$|0\rangle_0 = |0\rangle_f \otimes |\beta_j=0\rangle \otimes |N_r=0\rangle. \quad (200)$$

In the U(1) sector, unfortunately, we must take somewhat more care, since the condition  $\beta_j=0$  is in fact inconsistent

with the gauge constraint  $\mathcal{G}_f(\Lambda_r)=1$ . We can, however, project into the U(1) subspace using the operator

$$\hat{P} = \prod_{\mathbf{r}} \delta \left( f_{r\sigma}^\dagger f_{r\sigma} - \frac{1}{\pi} \epsilon_{ij} \partial_i \alpha_j(\mathbf{r}-\mathbf{w}) \right). \quad (201)$$

Since  $[\hat{P}, \hat{H}] = [\hat{P}, \mathcal{G}_v(\chi_r)] = 0$ , the state

$$|0\rangle = \hat{P}|0\rangle_0 \quad (202)$$

satisfies all gauge constraints and remains an eigenstate of  $H$  (with zero kinetic terms). This establishes that the resulting superconducting state is an ordinary BCS-type superconductor with paired electrons.

### 2. BCS instability

As we saw in Section V, the coupling of the spinons to the gapless rotons in the RFL phase leads to Heisenberg exchange and pair field interactions between the fermions with strength  $J \sim t_s^2/u_v$ . In the physically interesting limit  $t \approx t_s \gg t_e$ , the quasiparticles move primarily as spinons, i.e., without any associated electrical charge, and hence do not experience a long-ranged Coulomb repulsion. Thus one may imagine that the above interactions could lead to a BCS pairing instability at ‘‘high’’ energies (still below the quasiparticle Fermi energy to make the Fermi liquid description appropriate, but quantitatively large compared to, e.g., more conventional BCS critical temperatures) of order  $J$ . Here we explore the properties of the resulting phase which emerges when the fermions pair spontaneously and condense. To this end, we focus on the fluctuations of the phase,  $\Phi$  of the fermion pair field  $\langle f_\uparrow f_\downarrow \rangle = \Delta_f e^{i\Phi}$ . Keeping  $\Delta_f$  fixed, and working once more in the electron gauge  $\beta_i = -\pi \bar{e}_i$ , we integrate out the fermions entering in  $\mathcal{L}_{\text{RFL}}$  to generate an effective action for  $\Phi$ ,  $\beta_0$ , and  $\bar{e}_i$ . Specifically, one obtains  $\mathcal{L}_f^0 + \mathcal{L}_f^1 \rightarrow \mathcal{L}_\Phi$ , with

$$\begin{aligned} \mathcal{L}_\Phi = & \frac{g_f}{2} (\partial_0 \Phi - 2\beta_0)^2 + \frac{\rho_s^f t_s}{2} (\partial_i \Phi + 2\pi \bar{e}_i)^2 + \frac{\rho_s^f t_e}{2} (\partial_i \Phi - 2A_i)^2 \\ & - \alpha \frac{k_B T}{t \Delta_f} (t \partial_i \Phi + 2\pi t_s \bar{e}_i - 2t_e A_i)^2. \end{aligned} \quad (203)$$

Here  $g_f$  is of order the density of states at the Fermi surface  $A_i$  is (time independent for simplicity) external vector potential, and  $\rho_s^f$  is the fermion pair (‘‘superfluid’’) density. The final term represents the low but nonzero temperature corrections to the superfluid density appropriate to the case of  $d$ -wave pairing. Here  $\alpha$  is a nonuniversal constant of order one representing the effects of Fermi liquid corrections, or equivalently, high-energy renormalizations of the ‘‘Doppler-shift’’ coupling constant.

The excitations and response functions of the system can then be obtained from the RFL Lagrangian by shifting  $\alpha_\mu \rightarrow \alpha_\mu + a_\mu$  and then integrating out  $a_0$ ,  $a_j$ , and  $\alpha_0$ . Having made the replacement  $\mathcal{L}_f \rightarrow \mathcal{L}_\Phi$  one thereby obtains  $\mathcal{L}_{\text{RFL}} \rightarrow \mathcal{L}_{\text{eff}}$ , with an effective Lagrangian given by  $\mathcal{L}_{\text{eff}} = \hat{\mathcal{L}}_{\text{rot}} + \mathcal{L}_\Phi$ , where

$$\begin{aligned} \hat{\mathcal{L}}_{\text{rot}} = & \frac{u_v}{2} \left( e_i + \frac{1}{\pi} \epsilon_{ij} A_j \right)^2 + i \partial_0 e_i (\partial_i \theta + \alpha_i) + \frac{u_v}{2 \pi^2 v_0^2} \beta_0^2 \\ & + \frac{i}{\pi} \beta_0 \epsilon_{ij} \partial_i \alpha_j + \frac{\kappa_r}{8} (\epsilon_{ij} \partial_i \alpha_j)^2 \\ & + \frac{\kappa_r}{2} \left[ \Delta_{xy} \theta + \frac{1}{2} (\partial_x \alpha_y + \partial_y \alpha_x) \right]^2. \end{aligned} \quad (204)$$

First, it is instructive to consider the transverse electromagnetic response, in particular consider a static, transverse external gauge field  $\partial_0 A_i = \vec{\nabla} \cdot \vec{A} = 0$ . Since the external gauge field is at zero frequency  $\alpha_j$ ,  $\beta_0$ ,  $\theta$ , and  $e_i$  are decoupled from  $A_i$  and  $e_i$  in this limit, and we may thus neglect all but the first term in Eq. (204). In addition,  $\Phi$  is decoupled from  $e_i$  ( $\bar{e}_i$ ) as well, so we may simplify the effective Lagrangian to

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \frac{u_v}{2} \left( e_i + \frac{A_i}{\pi} \right)^2 + 2 t_e \rho_s^f A_i^2 + 2 \pi^2 t_s \rho_s^f e_i^2 \\ & - \alpha \frac{k_B T}{t \Delta_f} (2 \pi t_s e_i - 2 t_e A_i)^2. \end{aligned} \quad (205)$$

Integrating over  $e_i$  then gives the superfluid stiffness  $K_s$  as the coefficient of  $2 A_i^2$  (corresponding to the pair-field phase stiffness) in the effective action

$$K_s = \frac{\rho_s^f (t u_v + 4 \pi^2 t_e t_s \rho_s^f)}{u_v + 4 \pi^2 t_s \rho_s^f} - \frac{(t u_v + 4 \pi^2 t_e t_s \rho_s^f)^2}{(u_v + 4 \pi^2 t_s \rho_s^f)^2} \frac{2 \alpha k_B T}{t \Delta_f} \quad (206)$$

to linear order in temperature. Note that, for  $\rho_s^f \neq 0$ , the system is a true charge superfluid, displaying a Meissner effect.

We would next like to establish the fate of the roton excitations. To do these, we let  $A_i = 0$ , specialize to  $T = 0$ , and consider the appropriate limit  $\omega_n \sim k_x \ll k_y \sim O(1)$ , integrating out fields to obtain an effective action for  $\theta$  alone. Integrating out  $\Phi$ , one obtains in this limit

$$\mathcal{L}_\Phi \rightarrow 2 g_f \beta_0^2 + 2 \pi^2 \rho_s^f t_s e_i^2, \quad (207)$$

using the above conditions on  $\omega_n$  and  $\mathbf{k}$ , and  $\partial_0 \beta_0 \ll \epsilon_{ij} \partial_i e_j$ , which follows in this limit. Further taking the gauge  $\vec{\nabla} \cdot \vec{a} = 0$ , one sees that  $e_i$  couples only to  $\partial_0 \alpha$  in Eq. (204), and thus generates only negligible terms at low frequencies and may be dropped. We therefore need only the effective action for  $\alpha = \alpha_i$ ,  $\theta$ ,  $\beta_0$ , and  $e_i$ ,  $S_{\text{eff}} = \int_{\mathbf{k}, \omega_n} S_{\text{eff}}$ , which in this limit becomes

$$\begin{aligned} S_{\text{eff}} = & \frac{u_v + 4 \pi^2 \rho_s^f t_s}{2} e_i^2 + \frac{u_v + 4 \pi^2 v_0^2 g_f}{2 \pi^2 v_0^2} \beta_0^2 - |\mathcal{K}_y| \omega_n e_i \theta \\ & + \frac{i |\mathcal{K}_y|}{\pi} \beta_0 \alpha + \frac{\kappa_r}{8} \mathcal{K}_y^2 \alpha^2 + \frac{\kappa_r \mathcal{K}_y^2}{2} \left| k_x \theta + \frac{\text{sgn}(k_y)}{2} \alpha \right|^2. \end{aligned} \quad (208)$$

Performing the integration over  $\alpha$ ,  $\beta_0$ , and  $e_i$ , one obtains the final effective action for  $\theta$

$$s_{\text{eff}}(\theta) = \frac{1}{2} \frac{\mathcal{K}_y^2}{u_v + 4 \pi^2 \rho_s^f t_s} (\omega_n^2 + \tilde{v}_{\text{rot}}^2 k_x^2) |\theta|^2, \quad (209)$$

where

$$\tilde{v}_{\text{rot}}^2 = v_1^2 \frac{u_v + 4 \pi^2 \rho_s^f t_s}{u_v} \frac{u_v \left( v_0^2 + \frac{v_1^2}{4} \right) + \pi^2 g_f v_0^2 v_1^2}{u_v \left( v_0^2 + \frac{v_1^2}{2} \right) + 2 \pi^2 g_f v_0^2 v_1^2}. \quad (210)$$

Thus, despite the superconductivity induced by quasiparticle pairing, the gapless roton excitations survive, with some quantitative modifications to their velocity and correlations.

As we shall explore further in the concluding sections, in the limit of a very small ‘‘bare’’ vortex core energy appropriate in the underdoped limit  $u_v \sim x \rightarrow 0$ , there is a large energy scale for pairing  $J \sim t_s^2 / u_v$ . It is then natural to assume  $t \rho_s^f \approx t(1-x) \gg u_v$ . If we presume that the fermionic kinetic energy is predominantly due to ‘‘spinon hopping’’  $t \approx t_s \gg t_e$ , then one has

$$K_s \approx \frac{u_v}{4 \pi^2} + t_e \rho_s^f - \left( \frac{u_v}{4 \pi^2} + t_e \rho_s^f \right)^2 \frac{2 \alpha k_B T}{t \Delta_f (\rho_s^f)^2}. \quad (211)$$

Thus, despite the large bare superfluid stiffness coming from the BCS pairing of the quasiparticles, the renormalized stiffness is small, set by the bare vortex core energy  $K_s = u_v / 4 \pi^2 \sim x$  or the electron hopping  $t_e$  (presumed small compared to  $t_s$ ). It is the renormalized stiffness which determines the vortex core energy, and sets the scale for the finite temperature Kosterlitz–Thouless superconducting transition  $T_{\text{KT}} \sim u_v \sim x$ . In this way, one can understand the large separation in energy scales between the pseudogap line at scale  $J$  and the superconducting transition temperature  $T_{\text{KT}} \sim x$ . Unfortunately (since it is in apparent conflict with the small amount of experimental data for this quantity), along with this small superfluid stiffness, one obtains a small linear temperature derivative  $\partial K_s / \partial T|_{T=0}$ , due to the same mechanism. This is similar to results for the U(1) gauge theory for the  $t$ - $j$  model.

To complete the analysis, one should consider the effects of the heretofore neglected explicit pairing term in this novel ‘‘rotonic’’ superconductor. In particular, one may imagine that, once the fermionic pair field has condensed, it may induce true off-diagonal long-range order in the rotonic sector through the proximity effect. Naively, ODLRO appears incompatible with gapless rotons, so one may expect the explicit pairing term to induce a gap in the roton spectrum. While this is possible, it is easy to see that it is not inevitable. To see this, note that, in the rotonic superconductor, the fermionic pair field in the explicit pairing term may be replaced simply by its mean-field value  $\Delta_1 c_{\mathbf{r}+\hat{x}_i, \sigma} \epsilon_{\sigma\sigma'} c_{\mathbf{r}\sigma'} = \Delta_2 c_{\mathbf{r}+\hat{x}_i, \sigma} \epsilon_{\sigma\sigma'} c_{\mathbf{r}\sigma'} = \Delta_f$ . This leads, in the roton sector, to an additional term

$$H_\Delta = - \Delta_f \sum_{j\mathbf{r}} (B_{\mathbf{r}, \mathbf{r}+\hat{x}_j}^\dagger + \text{H.c.}). \quad (212)$$

Indeed, Eq. (212), since it embodies a linear coupling to the bosonic pair field  $B_{\mathbf{r}, \mathbf{r}+\hat{x}_j}$ , will induce ODLRO in the  $B_r$  op-

erators. However, this need not itself be a mechanism to induce a roton gap. Noting that from Sec. III C, the boson pair field correlators are power law in form, we expect that for  $\Delta_c$  sufficiently large, perturbation theory in  $\Delta_f$  will be regular and convergent, and the gapless rotons will be preserved. Due to the anisotropic structure of these correlators, we cannot reliably determine the relevance or irrelevance of  $\Delta_f$  by simple power-counting arguments. We note that for  $\Delta_c > 3/2$ , the induced ODQLRO is expected to be linear in  $\Delta_f$ , i.e., the bosonic pair-field susceptibility is finite. However, the criterion for irrelevance is probably more stringent, e.g.,  $\Delta_c > 3$ . To determine the precise value  $\Delta_c^*$  such that for  $\Delta_c > \Delta_c^*$ , the explicit pairing term remains irrelevant after condensation of quasiparticle pairs, would require a more careful analysis of the structure of the perturbation theory in  $\Delta_f$ . We leave this for braver souls, and content ourselves with the observation that there is a range of stability to this perturbation. We note that, however, since the RL itself even without the pairing term is always unstable to either vortex or charge hopping, the rotonic superconductor, with true gapless rotons, exists at best as an intermediate energy scale crossover. Nevertheless, provided the inevitable roton gap is small (i.e., for small  $\Delta_f, t_c, t_v$ ), we expect the gap onset will only slightly modify the values for the superfluid stiffness and its linear temperature derivative determined above.

### 3. Explicit pairing term

The final potential instability of the RFL we consider is from the explicit (spinon) pairing term  $\Delta$ . Recall from Sec. II C 1,

$$\mathcal{H}_\Delta = \sum_j e^{i\beta_j(\mathbf{r})} \Delta_j [\mathcal{S}_\mathbf{r}]^2 f_{\mathbf{r}+\hat{\mathbf{x}}_j \sigma} \epsilon_{\sigma\sigma'} f_{\mathbf{r}\sigma'} + \text{H.c.} \quad (213)$$

$$= \sum_j \Delta_j B_{\mathbf{r}, \mathbf{r}+\hat{\mathbf{x}}_j}^\dagger c_{\mathbf{r}+\hat{\mathbf{x}}_j \sigma} \epsilon_{\sigma\sigma'} c_{\mathbf{r}\sigma'} + \text{H.c.}, \quad (214)$$

where the last line is written in electron variables or, equivalently, in the electron gauge with  $\beta_j = -\pi \bar{e}_j$ . This representation is convenient because it eliminates all unphysical gauge fluctuations, which, although they do not appear in any physical properties, may enter inadvertently through approximations. Had we considered instead a Hamiltonian with  $s$ -wave pairing, we would have had

$$\mathcal{H}_\Delta^{s\text{-wave}} = \Delta^s [\mathcal{S}_\mathbf{r}]^2 f_{\mathbf{r}\sigma} \epsilon_{\sigma\sigma'} f_{\mathbf{r}\sigma'} + \text{H.c.} = \Delta^s B_{\mathbf{r}}^\dagger c_{\mathbf{r}\sigma} \epsilon_{\sigma\sigma'} c_{\mathbf{r}\sigma'} + \text{H.c.} \quad (215)$$

Since we have already determined the correlations of the boson pair field operator  $B_\mathbf{r}$  (which exhibits ODQLRO), in Sec. III C, all the operators appearing in Eq. (215) have well understood properties at this stage, and so we will discuss this case for simplicity. For the physically more interesting scenario of  $d$ -wave pairing, we require instead the behavior of the bond pair field correlations. As discussed in Sec. III C 3, this behavior is qualitatively identical to that of the local pair field, with a possible renormalization of  $\Delta_c$ . Hence we believe that the results we obtain for local ( $s$ -wave) pairing—in particular that the pairing term is irrelevant for sufficiently large  $\Delta_c$ —carry over to the  $d$ -wave case, pro-

vided a possible (and for the moment unknown)  $O(1)$  modification of  $\Delta_c$  is made in the relations.

We would like to determine the “relevance” of  $\mathcal{H}_\Delta$  in the renormalization group (RG) sense, i.e., whether its presence destabilizes the low-energy properties of the RFL. Unfortunately, due to the extremely anisotropic nature of the rotonic spectrum, and the very different nature of the low energy electronic quasiparticle states at the Fermi surface, we do not know how to formulate a proper RG transformation. However, we do note that correlations of  $\mathcal{H}_\Delta$  decay as power laws in space and imaginary time in the theory with  $\Delta=0$ , since then these correlations factor into  $G^{\text{CP}}$  (with power law behavior described in Sec. III C) and the fermion pair field correlator, which has the power law form characteristic of a Fermi liquid. Clearly, for sufficiently large  $\Delta_c$ , the Cooper pair propagator  $G^{\text{CP}}(\mathbf{r}, \tau)$  will decay sufficiently rapidly that perturbation theory in  $\Delta$  does not generate any (primitive) singularities. However, determining the critical  $\Delta_c^*$  above which this occurs is beyond the scope of the current study. A simple argument clearly bounds  $\Delta_c^* > 3/2$ . In particular, one may attempt to integrate out the vortices perturbatively in  $\mathcal{H}_\Delta$  in a cumulant expansion. The first nontrivial term in this expansion occurs at second order, and generates an attractive fermion pair-field to pair-field interaction whose vertex is simply the Fourier transform of  $G^{\text{CP}}(\mathbf{r}, \tau)$ . For  $\Delta_c < 3/2$ , a simple scaling analysis indicates that this Fourier transform is divergent at  $\mathbf{q} = \omega_n = 0$ . Thus for such values of  $\Delta_c$ , this attractive Cooper channel interaction will overwhelm any other repulsive interaction that might be present at low energies, leading to a Cooper instability and pairing. This analysis, however, neglects higher cumulant terms which are certainly present in integrating out the vortices, and which are presumably crucial in determining the ultimate limits of stability of the RFL. Nevertheless, and this is all we shall require at present, it is clear that  $\Delta_c^*$  exists and is not infinite, so that a nonvanishing region of stability also exists. When  $\Delta_c < \Delta_c^*$  the explicit pairing term will be relevant, destabilizing the RFL phase, presumably driving it into a conventional superconducting phase with paired electrons and gapped rotons. Given our lack of knowledge of  $\Delta_c^*$ , the additional uncertainty in the decay exponent of ODQLRO is not particularly damaging.

## VII. PROPERTIES OF THE RFL PHASE

Having established the existence of the RFL phase, we now discuss some of its properties. Since the rotons are effectively decoupled from the fermions at low energies, much of the physics of the RFL phase follows directly from the results we established for the charge sector in Secs. III and IV. Specifically, one expects *three* different gapless excitations in the RFL phase—a gapless longitudinal plasmon, a Bose surface of gapless rotons, and a set of particle-hole excitations across the Fermi surface. The *spin* physics in the RFL phase is then qualitatively similar to that of a conventional normal metal. Also, similar to a metal, the RFL has ODQLRO in the Cooper-pair field. Unlike an ordinary metal, however, this power law off-diagonal order exists with two distinct unrelated powers, one originating from the two-



particle excitations of the Fermi sea, and the other from the gapless rotons. Because the rotons and electronic quasiparticles exist as nearly independent excitations in the RFL, one would expect a sharp electron spectral function despite the critical rotons. While this is true, as we demonstrate below in detail, the quasiparticles do scatter appreciably and in a singular manner from the gapless rotons at particular hot spots on the Fermi surface. Nevertheless, even at these hot spots, the decay rate vanishes more rapidly than the electron's energy.

Despite the existence of such long lived electronlike quasiparticle excitations, the electrical transport in the RFL phase is strikingly non-Drude-like, as we now demonstrate. Specifically, in the presence of impurities the electrical conductivity at low temperatures is dominated by the quasicondensate in the charge sector, diverging at low temperatures as  $\sigma \sim T^{-\gamma}$  with  $\gamma > 1$ . But as we shall see, the *Hall conductivity* behaves very differently, being dominated by the electronlike quasiparticles.

### A. Electrical conductivity

Here we evaluate the electrical resistance in the RFL phase. As in Sec. IV A, it will be necessary to include the effects of the vortex hopping term, even though vortex hopping is technically irrelevant when  $\gamma > 1$ . The RFL phase exhibits the same “emergent symmetry” as in the roton liquid—the number of vortices on every row and column being independently conserved—so inclusion of vortex-hopping is necessary to generate *any* dissipative resistance whatsoever. In order to access the Hall conductivity we apply a uniform external magnetic field. We choose a gauge with  $A_0=0$ , and set

$$A_j(\mathbf{r}, \tau) = A_j^B(\mathbf{r}) + \tilde{A}_j(\tau), \quad (216)$$

with  $\epsilon_{ij}\partial_i A_j^B = B$  the external magnetic field and  $\tilde{A}_j$  a time-dependent source term used to extract the conductivity tensor. As might be expected it is important to include the effects of elastic scattering from impurities.

It is convenient to employ, as in Sec. VI B 2, in which the fermions have been integrated out, their effects felt only through an additional contribution to the effective action. In particular, we choose, as in Sec. VI B 2, the electron gauge, and write

$$S_{\text{eff}} = \int_{\tau} \sum_{\mathbf{r}} [\hat{\mathcal{L}}_{\text{rot}} - t_v \cos(\partial_i \theta + \alpha_i)] + S_f(e_i, \beta_0, \tilde{A}_i), \quad (217)$$

with  $\hat{\mathcal{L}}_{\text{rot}}$  given in Eq. (204). Here  $S_f$  represents the fermionic terms in the action. We proceed by shifting  $\tilde{e}_j \rightarrow \tilde{e}_j - A_j^B/\pi$ , which takes  $A_j \rightarrow \tilde{A}_j$  in  $\hat{\mathcal{L}}_{\text{rot}}$  without any further changes since  $A_j^B$  is time independent. This has the effect for the fermions of making the magnetic field appear uniformly in both electron- and spinon-hopping terms. Integrating out the fermions then perturbatively in  $\tilde{e}_i$ ,  $\beta_0$ , and  $\tilde{A}_i$  effectively replaces  $S_f \rightarrow \tilde{S}_f = \int_{\mathbf{k}\omega_n} S_f$ , where

$$s_f = \frac{\Pi_{00}}{2} |\beta_0(\mathbf{k}, \omega_n)|^2 + \frac{\pi^2 t_s}{2} T_{\text{NN}} |\tilde{e}_j|^2 + \frac{t_e}{2} T_{\text{NN}} |\tilde{A}_j|^2 + \frac{\tilde{\Pi}_{ij}}{2t^2} (\pi t_s \tilde{e}_i + t_e \tilde{A}_i)_{\mathbf{k}, \omega_n} (\pi t_s \tilde{e}_j + t_e \tilde{A}_j)_{-(\mathbf{k}\omega_n)}. \quad (218)$$

In Eq. (218), both  $\Pi_{00}$  and  $\tilde{\Pi}_{ij}$  are functions of  $\mathbf{k}$  and  $\omega_n$  (and  $B$  through  $A_j^B$  which appears in the electron-hopping term), and we have neglected a cross-term  $\Pi_{0i}$  between  $\beta_0$  and the spatial gauge fields, which is negligible in all the limits of interest. We will require the behavior of  $\tilde{\Pi}_{ij}$  in two regimes. Since the external fields are spatially uniform, we will need  $\tilde{\Pi}(\mathbf{k}=\mathbf{0}, \omega_n)$  at low frequencies

$$\tilde{\Pi}_{ij}(\mathbf{k}=\mathbf{0}, \omega_n) \approx \sigma_{xx}^f |\omega_n| \delta_{ij} + \sigma_{xy}^f \omega_n \epsilon_{ij}, \quad (219)$$

where  $\sigma_{ij}^f$  is the conductivity tensor for the fermions. Roton fluctuations are dominated in contrast by  $\omega_n \sim k_x \ll k_y \sim O(1)$  (or the same with  $k_x \leftrightarrow k_y$ ). In this limit,  $\tilde{\Pi}_{ij}$  becomes a nontrivial function of the  $O(1)$  component of the wave vector, but has the useful property (due to square reflection symmetry) of decoupling in the longitudinal and transverse basis

$$\tilde{\Pi}_{ij}(\mathbf{k}, \omega_n) \approx \tilde{\Pi}_t \left( \delta_{ij} - \frac{\mathcal{K}_i \mathcal{K}_j^*}{\mathcal{K}^2} \right) + \tilde{\Pi}_l \frac{\mathcal{K}_i \mathcal{K}_j^*}{\mathcal{K}^2}. \quad (220)$$

In this limit, we note that  $\tilde{\Pi}_t$  is related to  $\Pi_{11}$  of Sec. V B by  $\Pi_{11} = \tilde{\Pi}_t + T_{\text{NN}}$ .

As discussed above, the conductivity is finite only once the vortex-hopping is included to break the row/column symmetries of the RFL. Hence to extract the conductivity, we must compute the effective action for  $\tilde{A}_j$  to  $O(t_v^2)$ , which gives the first nontrivial correction. This requires only Gaussian integrals, with no further approximations. However, for ease of presentation, it is convenient, analogously to Sec. IV A 1, to take the simplifying limit  $u_v/v_0^2 + \Pi_{00} \ll 1/\kappa_r$ . In this limit the fluctuations of  $\beta_0$  are extremely strong, which in turn strongly suppresses the fluctuations of  $\alpha_i(\mathbf{k}, \omega_n)$  except at  $\mathbf{k}=\mathbf{0}$ . Furthermore, choosing the gauge  $\vec{\nabla} \cdot \vec{\alpha} = 0$ , we may thereby take  $\alpha_i(\mathbf{r}, \tau)$  to be a function of  $\tau$  only. Doing so, we may drop all spatial derivatives of  $\alpha_j$  in  $\hat{\mathcal{L}}_{\text{rot}}$ . Furthermore, since fluctuations of  $\alpha_j$  are only temporal, it can be accurately treated in an RPA fashion.

To carry out the RPA calculation, we first perform the integral over  $e_i$ , which gives an effective action in terms of the remaining  $\theta$ ,  $\alpha_i$ , and  $\tilde{A}_i$  fields  $S_{\text{eff}} \rightarrow S'_{\text{eff}} = S'_1(\tilde{A}_i) + S'_2(\alpha_i, \tilde{A}_i) + S'_3(\theta, \alpha_i)$ , with

$$S'_1 = L^2 \int_{\omega_n} \frac{1}{2} \left\{ (\eta t_s + t_e) T_{\text{NN}} \delta_{ij} + \frac{1}{t^2} (t_e^2 + \eta^2 t_s^2 + 2\eta t_e t_s) \times \tilde{\Pi}_{ij}(\mathbf{k}=\mathbf{0}, \omega_n) \right\} \tilde{A}_i(-\omega_n) \tilde{A}_j(\omega_n), \quad (221)$$

$$S'_2 = L^2 \int_{\omega_n} \left\{ \frac{\eta \omega_n^2}{2u_v} \left[ \delta_{ij} + \frac{\eta \pi^2 t_s^2}{u_v t^2} \tilde{\Pi}_{ij}(\mathbf{k}=\mathbf{0}, \omega_n) \right] \right. \\ \left. \times \alpha_i(-\omega_n) \alpha_j(\omega_n) - \frac{\eta \omega_n}{\pi} \epsilon_{ij} \alpha_i(-\omega_n) \tilde{A}_j(\omega_n) \right\}, \quad (222)$$

$$S'_3 = \int_{\mathbf{k}\omega_n} \frac{1}{2} \left[ \frac{\omega_n^2 \mathcal{K}^2}{\bar{u}_v} + \kappa_r |\mathcal{K}_x \mathcal{K}_y|^2 \right] |\theta(\mathbf{k}, \omega_n)|^2 \\ - t_v \sum_{\tau} \int_{\tau} \cos(\partial_i \theta + \alpha_i), \quad (223)$$

where  $\eta = u_v / (u_v + \pi^2 t_s T_{\text{NN}})$ , and  $\bar{u} = u_v + \pi^2 t_s T_{\text{NN}} + \pi^2 t_s^2 \tilde{\Pi}_{\tau} / t^2 = \bar{u}_v + \pi^2 t_s^2 \tilde{\Pi}_{11} / t^2$  as in Sec. V B. In Eqs. (221) and (222),  $L^2$  is the system volume (number of sites in the square lattice), arising since  $\alpha_i, \tilde{A}_i$  are spatially constant. In obtaining Eqs. (221) and (222), we expanded to linear order in  $\tilde{\Pi}_{ij}(\mathbf{k}=\mathbf{0}, \omega_n)$  since we are interested ultimately in the low-frequency limit, and  $\tilde{\Pi}_{ij}$  is linear in frequency. In Eq. (223), we used the fact that  $\theta$  coupled to only to the longitudinal part of  $e_i$  and hence only to  $\tilde{\Pi}_{\tau}$ . Note that the quadratic part of Eq. (223) reproduces precisely Eqs. (196) and (197), in the limit  $v_0 \gg \bar{v}_1$ , as assumed herein.

We now can integrate out  $\theta$  to  $O(t_v^2)$ . This proceeds identically as in Sec. IV A 1, with  $a_j$  replaced by  $-\alpha_j$ , and with the renormalized roton liquid parameters  $u_v \rightarrow \bar{u}_v$ . Hence we obtain the correction  $S'_3(\theta, \alpha_i) \rightarrow S''_3(\alpha_j)$ , with

$$S''_3 = \frac{t_v^2}{4} L^2 \int_{\omega_n} \tilde{\mathcal{R}}(\omega_n) |\alpha_j(\omega_n)|^2, \quad (224)$$

with  $\tilde{\mathcal{R}}(\omega_n) \sim -|\omega_n|^{1+\gamma}$  as in Eq. (141) of Sec. IV A 1, but  $\gamma = 2\Delta_v - 3$  with the renormalized  $\Delta_v$  given in Eq. (198).

With this replacement, the remaining quadratic integral over  $\alpha_i$  can be easily performed to determine the physical response kernel  $S'_1 + S'_2 + S''_3 \rightarrow S_{\text{resp}}(\tilde{A}_j)$ , with

$$S_{\text{resp}} = L^2 \int_{\omega_n} \frac{1}{2} \tilde{A}_i(-\omega_n) \Pi_{ij}^{\text{RFL}}(i\omega_n) \tilde{A}_j(\omega_n), \quad (225)$$

where

$$\Pi_{ij}^{\text{RFL}}(i\omega_n) \approx \tilde{\Pi}_{ij}^f + \tilde{\Pi}_{ij}^{\text{rot}} \quad (226)$$

and

$$\tilde{\Pi}_{ij}^f = t_e T_{\text{NN}} \delta_{ij} + \frac{t_e^2 + 2\eta t_e t_s}{t^2} \tilde{\Pi}_{ij}(\mathbf{k}=\mathbf{0}, \omega_n), \quad (227)$$

$$\tilde{\Pi}_{ij}^{\text{rot}} = \left[ \frac{u_v}{\pi^2} - \frac{u_v^2 t_v^2 \tilde{\mathcal{R}}(\omega_n)}{2\pi^2 \omega_n^2} \right] \delta_{ij}. \quad (228)$$

The conductivity is obtained from the Kubo formula as

$$\sigma_{ij}(\omega) = \left[ \frac{\Pi_{ij}^{\text{RFL}}(i\omega_n) - t_e T_{\text{NN}} \delta_{ij}}{\omega_n} \right]_{i\omega_n \rightarrow \omega + i\delta}. \quad (229)$$

As in Sec. IV A 1, we see that the rotonic contribution to the conductivity is not perturbative (at low frequencies) in  $t_v$ , so to capture the expected behavior, we replace it by

$$\sigma_{ij}^{\text{rot}} = \left[ \frac{\tilde{\Pi}_{ij}^{\text{rot}}}{\omega_n} \right]_{i\omega_n \rightarrow \omega + i\delta} \rightarrow \frac{\delta_{ij}}{-i\omega \frac{\pi^2}{u_v} + i \frac{\pi^2 t_v^2}{2} \tilde{\mathcal{R}}_{\text{ret}}(\omega) / \omega}. \quad (230)$$

This gives

$$\sigma_{ij}(\omega, T) = \sigma_{ij}^{\text{rot}}(\omega, T) + \frac{t_e^2 + 2\eta t_e t_s}{t^2} \sigma_{ij}^f(\omega, T). \quad (231)$$

Notice that from Eq. (231), the conductivity is the sum of separate fermion and roton contributions, and that the only effects of the fermions upon the rotonic piece is to modify the exponent  $\gamma$  (or  $\Delta_v$ ) implicit in  $\tilde{\mathcal{R}}(\omega)$ . Moreover, the fermionic contribution vanishes for  $t_e=0$ , as expected on physical grounds since the spinon-hopping term does not transport charge.

Consider first the dissipative diagonal dc (sheet) resistance  $R(T) = \text{Re}[\sigma_{xx}^{-1}]$ , which at low temperatures is completely dominated by the rotonic contribution, given as in Eq. (150),

$$R(T) = c_{\gamma} \frac{(\pi t_v)^2}{v_{\text{rot}}^2} \left[ \frac{\pi T}{v_{\text{rot}}} \right]^{\gamma}, \quad (232)$$

except with  $v_{\text{rot}}$  and the exponent  $\gamma$  renormalized by the interactions with the fermions. We thereby arrive at the important conclusion that the resistance in the RFL phase varies as  $R(T) \sim T^{\gamma}$ , with  $\gamma > 1$ . When  $\gamma < 1$  the RFL phase is unstable to confinement into a Fermi liquid phase, and right at the quantum confinement transition the resistance is linear in temperature.

The Hall conductivity and hence Hall angle on the other hand, are dominated by the fermionic quasiparticles. Specifically,  $\sigma_{xy} \approx \sigma_{xy}^f$ , and in the dc limit the fermionic Hall conductivity can be approximately obtained from a Drude expression  $\sigma_{xy}^f \approx \omega_c \tau_f (n_f e^2 \tau_f) / m \sim \omega_c \tau_f^2$ , where  $\omega_c = eB/m$  is the cyclotron frequency. The fermionic scattering time  $\tau_f$  has a variety of contributions, from elastic impurity scattering to interactions with rotons, considered in the next section, and hence may (or may not) have temperature dependence of its own—in contrast to the diagonal conductivity  $\sigma_{xx} \sim T^{-\gamma}$ , which diverges as  $T \rightarrow 0$  due to the rotonic contribution. These considerations suggest that the cotangent of the Hall angle, defined in terms of the resistivity tensor  $\rho_{ij}$  as  $\cot(\Theta_H) \equiv \rho_{xx} / \rho_{xy}$ , will vary as,

$$\cot(\Theta_H) \sim \frac{\sigma_{xx}}{\sigma_{xy}} \sim \frac{T^{-\gamma}}{\omega_c \tau_f^2} \sim \frac{1}{\omega_c T^{\gamma} \tau_f^2}. \quad (233)$$

The complete absence of a roton contribution to the Hall conductivity  $\sigma_{xy}^{\text{RFL}}$  is a consequence of the magnetic field independence of the roton liquid Lagrangian  $\mathcal{L}_{\text{RL}}$ , either in

its original or dual forms (90) and (159), respectively. But as discussed in Sec. IV B, the dual representation of  $\mathcal{L}_{\text{RL}}$  allows for additional terms involving cosines of the dual field  $\phi$ , which are present due to the underlying discreteness of the vortex number operator. While being irrelevant in the roton liquid phase, these neglected terms *do depend* on the external magnetic field and if retained will lead to a nonvanishing roton contribution to the Hall conductivity. But this contribution is expected to vanish rapidly at low temperatures. If on the other hand, these terms are *relevant* and drive a superconducting instability at low temperature, they will likely contribute significantly to the Hall conductivity. A careful analysis of the Hall response *above* the superconducting transition temperature in this situation will be left for future work.

### B. Electron lifetime

Although the electron spectral function is expected to be sharp at zero temperature right on the Fermi surface in the ideal RFL fixed-point theory, at finite energies (or temperatures) one expects the electrons to scatter off the rotors through interactions neglected in the RFL, causing a decay. Here, we consider two contributions to this electron decay rate at lowest order in the perturbations to the RFL. We consider two scattering mechanisms. First, scattering due to coupling of the quasiparticle current to boson currents through the  $\beta_\mu$  and  $e_i$  terms present in the fermionic Hamiltonian. Because singular bosonic current fluctuations are primarily induced by vortex-hopping, nontrivial contributions to the fermion lifetime through this mechanism occur first at  $O(t_v^2)$ . Second, we consider scattering of quasiparticles due to superconducting fluctuations, i.e., fermion decay mediated by the “Josephson coupling”  $\Delta$  to the bosonic pair field. Since we expect  $\Delta \ll t_v$  on physical grounds (see Sec. VIII B 2), this  $O(\Delta^2)$  contribution is naively much smaller than the former one. However, depending upon the parameters of the RFL, this need not be the case. This point will be returned to in the discussion.

As discussed above, to compute the spectral function it is simplest to work in the gauge  $\beta^l = \pi e^l$ , in which we may assume the electron operator  $c_{r\sigma} \sim f_{r\sigma}$ , and hence study

$$\mathcal{G}_f(\mathbf{r}, \tau) = -\langle T_{\neq} f_{r\sigma}(\tau) f_{0\sigma}^\dagger(0) \rangle. \quad (234)$$

In other gauges, it would be necessary to include string factors as indicated in Eq. (55).

Neglecting the fluctuating  $\beta_\mu$  fields, one has  $\mathcal{G}_0(\mathbf{k}, \omega_n) = (i\omega_n - \epsilon_{\mathbf{k}})^{-1}$ , with dispersion  $\epsilon_{\mathbf{k}} = -2t \cos(k_j) - \mu_s$ . Both fluctuations of the  $\beta_\mu$  gauge fields, which mediate retarded interactions amongst the quasiparticles, as well as the explicit interactions in the pairing term, will induce a self-energy  $\Sigma(\mathbf{k}, \omega_n)$ , defined by

$$\mathcal{G}_0(\mathbf{k}, \omega_n) = \frac{1}{i\omega_n - \epsilon_{\mathbf{k}} - \Sigma(\mathbf{k}, \omega_n)}. \quad (235)$$

We will compute the lowest-order corrections to the electron self-energy, obtaining a single-particle lifetime from the imaginary part of its retarded continuation  $\Sigma_{\text{ret}}(\mathbf{k}, \omega)$

$= \Sigma(\mathbf{k}, \omega_n \rightarrow -i\omega + 0^+)$  in the usual way. At the level of this discussion, the two types of interactions act in parallel to scatter quasiparticles, so

$$\Sigma(\mathbf{k}, \omega) = \Sigma_{\text{CF}}(\mathbf{k}, \omega) + \Sigma_{\text{SF}}(\mathbf{k}, \omega). \quad (236)$$

We will focus upon the imaginary part  $\Sigma''(\mathbf{k}, \omega)$ , which describes broadening of the electron spectral function  $1/\tau_f(T) = \Sigma''(\mathbf{k}, 0; T)$ . We have

$$\tau_f^{-1} = (\tau_f^{\text{CF}})^{-1} + (\tau_f^{\text{SF}})^{-1}. \quad (237)$$

In principle this single-particle “lifetime” is distinct from the momentum scattering rate which is relevant in discussions of transport quantities. However, we will use the behavior obtained for  $1/\tau_f$  as a crude guide to the quasiparticle transport as well, leaving a more careful treatment for future study.

#### 1. Scattering mediated by vortex hopping–enhanced current fluctuations

Taking into account quadratic fluctuations of the  $\beta_\mu$  fields, the leading self-energy correction due to current fluctuations is

$$\begin{aligned} \Sigma_{\text{CF}}(\mathbf{k}, i\omega_n) &= \sum_{\mu\nu} \int_{\mathbf{q}, \omega'_n} v_{F\mu} v_{F\nu} \mathcal{G}_0(\mathbf{k} - \mathbf{q}, \omega_n - \omega'_n) U_{\mu\nu}(\mathbf{q}, \omega'_n) \\ &= \sum_{\mu\nu} \int_{\mathbf{q}, \omega'_n} v_{F\mu} v_{F\nu} \frac{1}{i(\omega_n - \omega'_n) - \epsilon_{\mathbf{k}-\mathbf{q}}} U_{\mu\nu}(\mathbf{q}, \omega'_n), \end{aligned} \quad (238)$$

with the definitions  $v_{Fj} = \partial \epsilon_{\mathbf{k}} / \partial k_j$ ,  $v_{F0} = 1$ . Here we have written the self-energy in terms of the full  $\beta$ -gauge-field correlator  $U_{\mu\nu}$  to all orders in  $t_v$ , rather than restricting to  $t_v = 0$  as we did in obtaining Eq. (183).

Introducing a spectral representation

$$U_{\mu\nu}(\mathbf{k}, \omega_n) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{U''_{\mu\nu}(\mathbf{q}, \omega)}{\omega - i\omega_n}, \quad (239)$$

with  $U''_{\mu\nu}(\omega) = \text{Im } U_{\mu\nu}^{\text{ret}}(\omega) = \text{Im } U_{\mu\nu}(\omega_n \rightarrow -i\omega + 0^+)$ , allows one to analytically continue to obtain the (retarded) self-energy

$$\Sigma_{\text{ret}}(\mathbf{k}, \omega) = \Sigma(\mathbf{k}, \omega_n) \Big|_{i\omega_n \rightarrow \omega + i0^+}. \quad (240)$$

For positive frequencies,  $\omega > 0$ , the imaginary part,  $\Sigma'' = \text{Im } \Sigma_{\text{ret}}$  is given by

$$\begin{aligned} \Sigma''_{\text{CF}}(\mathbf{k}, \omega) &= \sum_{\mu\nu} \int_{\mathbf{q}} v_{F\mu} v_{F\nu} U''_{\mu\nu}(\mathbf{q}, \omega - \epsilon_{\mathbf{k}+\mathbf{q}}) \Theta(\epsilon_{\mathbf{k}+\mathbf{q}}) \\ &\quad \times \Theta(\omega - \epsilon_{\mathbf{k}+\mathbf{q}}). \end{aligned} \quad (241)$$

The self-energy can now be evaluated by considering progressive orders in  $t_v$ . To zeroth order, the expressions for  $U_{\mu\nu}^{(0)}(\mathbf{k}, \omega_n)$  may be taken from Eqs. (183)–(188). We note that, because they come from the quadratic RL Lagrangian, they contain only simple poles. Furthermore, they are explicitly real functions of  $i\omega_n$ , so that, upon analytic continuation, their retarded correlator has zero imaginary part. Since  $U_{\mu\nu}^{(0)''} = 0$ , one thus has  $\Sigma_{\text{CF}}^{(0)''}(\mathbf{k}, \omega) = 0$ .

Thus there is no broadening of the quasiparticle peak at zeroth order in the vortex-hopping. The first nontrivial contribution to the quasiparticle lifetime occurs through the  $O(t_v^2)$  corrections to the gauge field propagator  $U_{\mu\nu} = U_{\mu\nu}^{(0)} + t_v^2 U_{\mu\nu}^{(2)} + \dots$ . This correction is obtained by integrating out  $\alpha_j$  to second order in  $t_v$  starting from Eq. (181) to obtain  $S_{\text{eff}}(\lambda_\mu)$  to  $O(t_v^2)$  using the cumulant expansion. We shift  $\alpha_i(\mathbf{r}, \tau) \rightarrow \alpha_i(\mathbf{r}, \tau) - v_i(\mathbf{r}, \tau)$ , to eliminate the linear terms in  $\alpha_i$  in the Lagrangian, with  $v_i$  linear in  $\lambda_\mu$  given explicitly by

$$\begin{aligned} v_x(\mathbf{k}, \omega_n) &= -\frac{\pi}{(\omega_n^2 + \omega_{\text{pl}}^2)(\omega_n^2 + \omega_{\text{rot}}^2)} [v_0^2 \mathcal{K}_x \mathcal{K}_y (v_0^2 \mathcal{K}_x \lambda_0 + i\omega_n \lambda_x) \\ &\quad - (\omega_n^2 + v_+^2 |\mathcal{K}_x|^2) (v_0^2 \mathcal{K}_y \lambda_0 + i\omega_n \lambda_y)], \\ v_y(\mathbf{k}, \omega_n) &= \frac{\pi}{(\omega_n^2 + \omega_{\text{pl}}^2)(\omega_n^2 + \omega_{\text{rot}}^2)} [v_0^2 \mathcal{K}_x \mathcal{K}_y (v_0^2 \mathcal{K}_y \lambda_0 + i\omega_n \lambda_y) \\ &\quad - (\omega_n^2 + v_+^2 |\mathcal{K}_y|^2) (v_0^2 \mathcal{K}_x \lambda_0 + i\omega_n \lambda_x)]. \end{aligned} \quad (242)$$

After this shift the correction to the effective action can be formally written

$$\begin{aligned} S_{\text{eff}}^{(2)} &= -\frac{t_v^2}{2} \sum_{ij} \sum_{\mathbf{r}, \mathbf{r}'} \int_{\tau, \tau'} \langle \cos[\alpha_i(\mathbf{r}, \tau) - v_i(\mathbf{r}, \tau)] \\ &\quad \times \cos[\alpha_j(\mathbf{r}, \tau) - v_j(\mathbf{r}, \tau)] \rangle_\alpha, \end{aligned} \quad (243)$$

where the subscript  $\alpha$  indicates a Gaussian average over  $\alpha_i$  with respect to the quadratic terms in Eq. (181). Since  $v_i$  is linear in  $\lambda_\mu$ , this correction is not quadratic in  $\lambda_\mu$ , indicating that the  $\beta_\mu$  fluctuations are not Gaussian. To evaluate the leading order self-energy correction, however, we require only the two-point function of  $\beta_\mu$ , and hence may expand to quadratic order in  $v_i$ . One finds

$$S_{\text{eff}}^{(2)} = -\frac{t_v^2}{4} \int_{\mathbf{k}, \omega_n} (|v_x(\mathbf{k}, \omega_n)|^2 \tilde{\mathcal{R}}(k_y, \omega_n) + |v_y(\mathbf{k}, \omega_n)|^2 \tilde{\mathcal{R}}(k_x, \omega_n)). \quad (244)$$

Here  $\tilde{\mathcal{R}}(k, \omega_n)$  is the two-point function of the vortex hopping operator considered in Sec. IV A, i.e.,

$$\begin{aligned} \tilde{\mathcal{R}}(k, \omega_n) &= \sum_x \int_\tau [1 - \cos(kx + \omega_n \tau)] \langle e^{i\alpha_y(x, y, \tau)} e^{-i\alpha_y(0, y, 0)} \rangle_\alpha \\ &\sim \frac{A(\Delta_v)}{\sin \pi(\Delta_v - 1)} (\omega_n^2 + v_{\text{rot}}^2 k^2)^{\Delta_v - 1}, \end{aligned} \quad (245)$$

where the latter behavior gives the leading nonanalytic term for small  $\omega_n$ ,  $k$ , with  $A(\Delta_v)$  a constant prefactor. An analytic quadratic term is also present (and larger than the given form for  $\Delta_v > 2$ ), but does not contribute to the imaginary part of the self-energy for the same reasons described above for the  $t_v = 0$  terms.

Next, we must analytically continue to obtain the  $U_{\mu\nu}^{(2)''}(\mathbf{k}, \omega)$ . From Eq. (242), one immediately sees that  $v_i^{\text{ret}}(\mathbf{k}, \omega)$  is purely real. Hence, the imaginary part comes entirely from the analytic continuation of  $\tilde{\mathcal{R}}(k_i, \omega_n)$ . One has

$$\mathcal{R}''(k_i, \omega) \sim A(\Delta_v) (\omega^2 - v_{\text{rot}}^2 k_i^2)^{\Delta_v - 1} \Theta(|\omega| - v_{\text{rot}} |k_i|) \text{sgn}(\omega). \quad (246)$$

From inspection of Eq. (244), one thereby sees the imaginary part of the retarded gauge-field correlator  $U_{\mu\nu}^{(2)''}(\mathbf{k}, \omega)$  is the sum of two terms, which are nonzero only for  $|\omega| > v_{\text{rot}} |k_x|$  or  $\omega > v_{\text{rot}} |k_y|$ , respectively. Thus the momentum integrals in the expression for  $\Sigma_{\text{CF}}''$  in Eq. (241), are constrained not only by  $0 < \epsilon_{\mathbf{k}+\mathbf{q}} < \omega$  but also either  $\epsilon_{\mathbf{k}+\mathbf{q}} < \omega - v_{\text{rot}} |q_x|$  or  $\epsilon_{\mathbf{k}+\mathbf{q}} < \omega - v_{\text{rot}} |q_y|$ , for the two terms, respectively.

We focus on the self-energy for momenta exactly on the Fermi surface  $|\mathbf{k}| = k_F$ . In this case, the constraints clearly require small  $\mathbf{q}$  for small  $\omega$ . For such small wave vectors, one may approximate  $\epsilon_{\mathbf{k}+\mathbf{q}} \approx \vec{v}_F \cdot \vec{q} + q^2/2m^*$ . For a generic point on the Fermi surface (taken for simplicity in the quadrant with  $k_x, k_y > 0$ , for which  $\vec{v}_F$  makes an angle  $\theta_F \neq 0, \pi/2$  to the  $x$  axis, the  $q^2$  term is negligible, and one has  $\epsilon_{\mathbf{k}+\mathbf{q}} \approx v_F (q_x \cos \theta_F + q_y \sin \theta_F)$ . Applying the above constraints, one finds that both  $q_x$  and  $q_y$  are integrated over a small bounded region in which both are  $O(\omega)$ . Thus for such generic points on the Fermi surface, we should consider the limit of  $U_{\mu\nu}^{(2)''}(\mathbf{q}, \omega - \epsilon_{\mathbf{k}+\mathbf{q}})$  in which all arguments are  $O(\omega)$ . In this limit, inspection of Eqs. (242) and (245) shows that the correlator satisfies a scaling form

$$U_{\mu\nu}^{(2)''}(\mathbf{q}, \omega - \epsilon_{\mathbf{k}+\mathbf{q}}) \sim \omega^{2(\Delta_v - 2)} \mathcal{U}(q_x/\omega, q_y/\omega), \quad (247)$$

with a well-behaved limit as  $\mathbf{q} \rightarrow 0$ . Inserting this into Eq. (241) and rescaling the momentum integrals by  $\omega$ , one sees that  $\Sigma_{\text{CF}}''(\mathbf{k}_F, \omega) \sim \omega^{2(\Delta_v - 1)}$  for  $\theta_F \neq 0, \pi$ . Since  $\Delta_v \geq 2$  in the stable region of the RFL, this dependence is always as weak or weaker than the ordinary  $\omega^2$  scattering rate due to Coulomb interactions in a Fermi liquid.

The special cases when  $\theta_F = 0, \pi/2$  require separate consideration. For these values, the Fermi velocity is along one of the principal axes of the square lattice, and the linear approximation for  $\epsilon_{\mathbf{k}+\mathbf{q}}$  is inadequate. Taking for concreteness  $\theta_F = 0$ , we have instead  $\epsilon_{\mathbf{k}+\mathbf{q}} \approx v_F q_x + q_y^2/2m^*$ , and it is not obviously consistent to neglect the  $q_y^2$  term. For the first term [involving  $\tilde{\mathcal{R}}''(q_y, \omega - \epsilon_{\mathbf{k}+\mathbf{q}})$ ], the constraints reduce to  $0 < v_F q_x + q_y^2/2m^* < \omega - v_{\text{rot}} |q_x|$ . This is approximately solved for small  $\omega$  by  $0 < q_y < (\omega - v_{\text{rot}} |q_x|)/v_F$  and  $|q_x| < \omega/v_F$ . Thus both  $q_x, q_y$  are again bounded and  $O(\omega)$ , so the above Fermi-liquid-like scaling applies.

For the second term [involving  $\tilde{\mathcal{R}}''(q_x, \omega - \epsilon_{\mathbf{k}+\mathbf{q}})$ ], the constraints reduce to  $0 < v_F q_x + q_y^2/2m^* < \omega - v_{\text{rot}} |q_x|$ . This is solved by taking  $-\omega/v_{\text{rot}} < q_x < \omega/(v_{\text{rot}} + v_F)$  and  $\max(0, -v_F q_x) < q_y^2/2m^* < \omega - v_F q_x - v_{\text{rot}} |q_x|$ . Hence for this term,  $q_x$  is  $O(\omega)$  while  $q_y$  is  $O(\sqrt{\omega})$ . Since with this scaling  $|q_y| \gg \omega \sim |q_x|$  one has the significant simplification

$$v_y(\mathbf{q}, \omega) \sim \frac{\pi v_0^2 v_1^2}{2v_+^2} \frac{q_x \lambda_0}{v_{\text{rot}}^2 q_x^2 - \omega^2} \quad \text{for } |q_y| \gg \omega \sim |q_x|. \quad (248)$$

Since only  $\lambda_0$  appears, in this limit,  $U_{00}^{(2)''} \gg U_{ii}^{(2)''}, U_{0i}^{(2)''}$ , i.e., the fluctuations of  $\beta_0$  are much stronger than those of  $\beta_i$ . One may therefore approximate



$$U_{\mu\nu}^{(2)''}(\mathbf{q}, \omega)_{\theta_F=0} \sim \frac{A(\Delta_v)\pi^2 t_v^2 v_0^4 v_1^4}{2 v_+^4} q_x^2 (\omega^2 - v_{\text{rot}}^2 q_x^2)^{\Delta_v-3} \times \Theta(|\omega| - v_{\text{rot}}|q_x|) \text{sgn } \omega \delta_{\mu 0} \delta_{\nu 0}. \quad (249)$$

Inserting this into Eq. (241), one may integrate over  $q_y$  (to yield a constant multiplying  $\sqrt{|q_x|}$  and rescale  $q_x \rightarrow \omega q_x$ ) to obtain the result

$$\Sigma_{\text{CF}}''(\mathbf{k}_F, \omega) \sim \omega^{2\Delta_v-5/2} \quad \text{for } \theta_F=0, \frac{\pi}{2}, \dots \quad (250)$$

At the end point of the RFL, for which  $\Delta_v=2$ , this gives the anomalous lifetime  $\Sigma''(\mathbf{k}_F, \omega) \sim \omega^{3/2}$  at these special hot spots for which the Fermi velocity is parallel to one of the principal axes of the square lattice. For the conventional Fermi surface believed to apply to the cuprates, this corresponds to the points on the Fermi surface closest to  $(\pi, 0)$ ,  $(0, \pi)$ .

Comparing the scattering rate at these hot spots to elsewhere on the Fermi surface, one finds

$$\frac{\Sigma_{\text{CF;hot}}''(\omega)}{\Sigma_{\text{CF;typ}}''(\omega)} \sim \sqrt{\frac{m^* v_F^2}{\hbar \omega}}. \quad (251)$$

Since the Fermi surface in the cuprates is particularly flat near  $(\pi, 0)$ , the effective mass would be large  $m^* \gg m_e$ , and a significantly enhanced scattering rate at such hot spots would be expected in the RFL phase.

In parameter regimes where the RFL phase is unstable (via charge hopping) at low temperatures to a conventional superconductor, the enhanced scattering at the hot spots will be significantly suppressed upon cooling below the transition temperature. Indeed, the low-energy roton excitations are gapped out in the superconducting phase, and at temperatures well below  $T_c$  will not be appreciably thermally excited. Unimpeded by the rotons, the electron lifetime will be greatly enhanced relative to that in the normal state, particularly at the hot spots on the Fermi surface with tangents along the  $\hat{x}$  or  $\hat{y}$  axes, for example, at wave vectors near  $(\pi, 0)$  in the cuprates.

## 2. Scattering mediated by superconducting fluctuations, i.e., boson-fermion pair exchange

A second mechanism of fermion decay is by the ‘‘pairing term’’  $\mathcal{H}_\Delta$  in Eq. (213). We suppose  $|\Delta|$  is small, and so consider the first perturbative contribution to the quasiparticle lifetime at  $O(\Delta^2)$ . In general, for  $d$ -wave pairing, this takes the form

$$\Sigma_{\text{SF}}''(\mathbf{k}, \omega_n) = \sum_{i,j=1}^2 \Delta_i \Delta_j \int_{\mathbf{q}\Omega_n} \frac{e^{ik_i} + e^{i(q_i-k_i)}}{2} \frac{e^{-ik_j} + e^{-i(q_j-k_j)}}{2} \times \frac{G_{ij}^{\text{CP}}(\mathbf{q}, \Omega_n)}{-i(\Omega_n - \omega_n) + \epsilon_{\mathbf{q}-\mathbf{k}}}, \quad (252)$$

where

$$G_{ij}^{\text{CP}}(\mathbf{q}, \Omega_n) = \sum_{\mathbf{r}} \int_{\tau} G_{ij}^{\text{CP}}(\mathbf{r}, \tau) e^{i\mathbf{q}\cdot\mathbf{r} - i\Omega_n \tau}, \quad (253)$$

with  $G_{ij}^{\text{CP}}(\mathbf{r}, \tau)$  from Eq. (109).

Our uncertainties in the details of the Cooper pair propagator (originating from ambiguities in the string geometry) do not allow a thorough calculation of the resulting self-energy. However, as we will now show, we can obtain a rough understanding of its scaling properties by some simple approximations, which we believe do not significantly effect the results. In particular, we will assume that the lifetime is controlled by the small wave vector  $|q| \ll \pi$  portion of the Cooper pair propagator. In this regime, we expect  $G_{ij}^{\text{CP}}(\mathbf{q}, \Omega_n) \approx G^{\text{CP}}(\mathbf{q}, \Omega_n)$ , the Fourier transform of the *local* Cooper pair propagator studied in depth in Sec. III C. Making this approximation, one has  $|q| \ll |k|$  for  $\mathbf{k}$  near the quasiparticle Fermi surface, and one may write

$$\Sigma_{\text{SF}}(\mathbf{k}, \omega_n) \approx |\Delta_{\mathbf{k}}|^2 \int_{\mathbf{q}\Omega_n} \frac{G^{\text{CP}}(\mathbf{q}, \Omega_n)}{-i(\Omega_n - \omega_n) + \epsilon_{\mathbf{q}-\mathbf{k}}}, \quad (254)$$

with  $\Delta_{\mathbf{k}} = |\Delta|(\cos k_x - \cos k_y)$ . Note that this form immediately implies that scattering due to this mechanism is strongly suppressed upon approaching the nodal regions of the Fermi surface.

A detailed analysis is now possible based on a spectral representation of  $G^{\text{CP}}$ , as in Sec. VII B 1.<sup>17</sup> Unlike the above case, however [since the power law decay of  $G^{\text{CP}}(\mathbf{r}, \tau)$  is approximately isotropic], a much simpler scaling analysis suffices to obtain the qualitative behavior of the lifetime. In particular, the scaling form of Eq. (127) implies that

$$G^{\text{CP}}(\mathbf{q}, \Omega_n) \sim |\Omega_n|^{2\Delta_c-3} \tilde{\mathcal{G}}(\mathbf{q}/\Omega_n), \quad (255)$$

up to nonsingular additive corrections, for small  $|q|$  and  $\Omega_n$ . Furthermore, for small  $\mathbf{q}$  and  $\mathbf{k}$  on the Fermi surface, one may write  $\epsilon_{\mathbf{q}-\mathbf{k}} \approx v_F q_{\parallel} + q_{\perp}^2/2m$ , where  $q_{\parallel}$ ,  $q_{\perp}$  are the components of  $\mathbf{q}$  parallel and perpendicular to the local Fermi velocity at  $\mathbf{k}$ , and  $v_F$  and  $m$  are the magnitude of the local Fermi velocity and effective mass. The singularity in the integrand in Eq. (254) is then cut off by the external frequency  $\omega_n$ , and rescaling  $\mathbf{q} \rightarrow \mathbf{q}\Omega_n$ , one sees that the effective mass term is negligible, and obtains by power counting

$$\Sigma_{\text{SF}}(\mathbf{k}, \omega_n) \sim |\Delta_{\mathbf{k}}|^2 |\omega_n|^{2\Delta_c-1}, \quad (256)$$

again with possible analytic and subdominant corrections. Upon analytic continuation to obtain the lifetime, only nonanalytic terms contribute, and we expect

$$\Sigma_{\text{SF}}''(\mathbf{k}, \omega) \sim |\Delta_{\mathbf{k}}|^2 |\omega|^{2\Delta_c-1} \text{sgn}(\omega), \quad (257)$$

for  $\mathbf{k}$  on the Fermi surface. This result can be verified by more detailed calculations using the spectral representation of  $G^{\text{CP}}$ .<sup>17</sup> Moreover, we expect that for  $k_B T \gg \omega$ ,  $\omega$  can be replaced by  $k_B T$  in this formula.

As expected, for sufficiently large  $\Delta_c$ , this lifetime vanishes rapidly at low energies, and in particular for  $\Delta_c > 3/2$ , this contribution is subdominant to the usual Fermi liquid one. However, in the regime with  $\Delta_c < \Delta_v$ , this is never the case ( $\Delta_c < 1/\sqrt{2}$ ), at least within our simple model without

dramatic corrections to the RL exponents. Indeed, for  $\Delta_c < 1$ , as supposed herein, this scattering rate is much *larger* than the quasiparticle energy at low frequency. Taken literally, this implies increasingly incoherent behavior away from the nodes as temperature is lowered. Near the nodes, the amplitude of this strong scattering contribution vanishes rapidly, similar to the idea of cold spots proposed by Ioffe and Millis.<sup>18</sup>

### VIII. DISCUSSION

We close with a discussion of some theoretical issues concerning the vortex-quasiparticle formulation of interacting electrons that we have been employing throughout. In particular, we contrast our approach with the earlier  $Z_2$  formulation<sup>5</sup> and mention the connection with more standard and more microscopic formulations of correlated electrons. We then address the possible relevance of the roton Fermi liquid phase to the cuprate phase diagram and the associated experimental phenomenology.

#### A. Theoretical issues

Since the discovery of the cuprate superconductors, the many attempts to reformulate theories of two-dimensional strongly correlated electron in terms of new collective or composite degrees of freedom, have been fueled on the one hand by related theoretical successes and on the other by cuprate phenomenology. The remarkable and successful development of *bosonization*<sup>19</sup> both as a reformulation of interacting one-dimensional electron models in terms of bosonic fields and as a means to extract qualitatively new physics outside of the Fermi liquid paradigm, played an important role in a number of early theoretical approaches to the cuprates.<sup>13</sup> The equally impressive successes of the *composite boson* and *composite fermion* approaches in the fractional quantum Hall effect<sup>20</sup> were also influential in early high- $T_c$  theories, most notably the *anyon* theories.<sup>21</sup> Gauge theories of the Heisenberg and  $t$ - $J$  models were notable early attempts to reformulate 2D interacting electrons in terms of “spin-charge” separated variables<sup>22,23</sup>—electrically neutral fermionic spin one-half spinons and charge  $e$  bosonic holons—and were motivated both by analogies with 1D bosonization and by resonating valence bond ideas.<sup>13</sup> More recent approaches to 2D spin-charge separation have highlighted the connections with superconductivity,<sup>4,5</sup> by developing a formulation in terms of vortices, Bogoliubov quasiparticles, and plasmons: the three basic collective excitations of a 2D superconductor. These latter theories were primarily attempting to access the pseudogap regime by approaching from the superconducting phase<sup>4</sup>—focusing on low-energy physics where appreciable pairing correlations were manifest.

#### 1. Vortex-fermion formulation

In this paper, although we are employing a formulation with the same field content— $hc/2e$  vortices, fermionic quasiparticles, and collective plasmons—we are advocating a rather different philosophy. In particular, we wish to use these same fields to describe higher-energy physics in re-

gimes where the physics is decidedly non-BCS-like even at short distances, most importantly the cuprate normal state near optimal doping. The philosophy of this approach is very similar to that of the  $Z_2$  gauge theory proposal of a fractionalized underdoped normal state. Indeed, as demonstrated in Sec. II C and Appendixes B and C, our U(1) formulation is completely (unitarily) equivalent to a  $Z_2$  gauge theory. However, because the unitary transformation relating the two formulations is nontrivial, the U(1) formulation is (much) more convenient for the types of manipulations and approximations we employ here, largely because the U(1) gauge fields are continuous variables. The price paid for the use of the U(1) formulation is that the “spinon pairing term” of the  $Z_2$  gauge theory appears nonlocal in the U(1) Hamiltonian.

Fortunately, this nonlocality and its consequences are readily understood. In particular, although the U(1) vortex-quasiparticle Hamiltonian is microscopically equivalent to a  $Z_2$  gauge theory, it is *also* equivalent (as described in Appendix D) to a theory of electronic (i.e., charge  $e$ ) quasiparticles coupled to charge  $2e$  bosons (with the latter described in dual vortex variables). The “two-fluid” point of view of this vortex-electron formulation is convenient for understanding the qualitative properties of the RFL phase and its descendants, although it should be emphasized that the current-current couplings [embodied by the gauge fields of the U(1) formulation] between the two fluids are not expected to be weak. In the vortex-electron language, the nonlocal term simply represents coherent Josephson coupling of electron pairs and the bosons. The nonlocality of the pairing term arises simply from the nonlocal representation of boson operators in the usual U(1) boson-vortex duality. The crucial feature which allows us to handle the pairing term despite its nonlocality is the strength of vortex fluctuations in the bosonic roton liquid, which persists even with these strong current-current couplings. The resulting power-law decay of bosonic pair-field correlations (ODQLRO) opens up a regime, corresponding to the RFL phase, in which the pairing term is irrelevant and can be treated perturbatively. We note in passing that, although it is not relevant for the cuprates, the above discussion makes clear that a RL phase (for which fermions need never be introduced) should be possible in a purely bosonic model, which would be interesting in and of itself.

A second important feature of the present formulation is the retention of the lattice-scale structure. Appropriate to the cuprates, we have carefully defined the theory on a 2D square lattice which we wish to identify with the microscopic copper lattice. Since our theory obviously does not similarly retain physics on atomic *energy* scales, our starting bare Hamiltonian should be viewed as a *low energy* but not spatially coarse-grained effective theory (or at most an effective theory coarse-grained spatially only insofar as to remove, e.g., the  $O p$  orbitals). It is important to emphasize that the very existence of the roton Fermi liquid phase *requires* the presence of a square lattice. In the RFL ground state there is an infinite set of dynamically generated symmetries, corresponding to a conserved number of vortices on every row and column of the lattice. If we study the same model on a different lattice, say the triangular lattice, the quantum

ground state analogous to the RFL phase (obtained, again, by taking the roton hopping amplitude  $\kappa_r \rightarrow \infty$ ) is highly unstable, being destroyed by the presence of an arbitrarily small single vortex hopping amplitude.

More generally, the importance of employing the dual vortex-quasiparticle field theory reformulation cannot be overemphasized. The novel and unusual RFL phase emerges quite simply when one takes a large amplitude for the roton hopping term, and the fixed-point theory is also quite simple consisting of a Fermi sea of quasiparticles minimally coupled to an electric field with Gaussian dynamics. It is very difficult to see how one could adequately describe such a phase working with bare electron operators. Indeed, we do not yet understand the simpler task of constructing a microscopic (undualized) boson model which enters the RL phase, even without the complications of fermions. However, previous experience with dual vortex formulations for bosonic and electronic systems strongly argues that the RL and RFL theories properly describe physically accessible (albeit microscopically unknown) models.

An unfortunate drawback of the present formulation, however, is the apparent lack of *direct* connection between the starting lattice Hamiltonian and any microscopic electron model. In particular, it is presently unclear what microscopic electron physics could be responsible for generating such a large roton hopping term. In the absence of a microscopic foundation, one must resort to developing phenomenological implications of the roton Fermi liquid phase and comparing with the experimentally observed cuprate phenomenology. We take a preliminary look in this direction in Sec. VIII B below.

## 2. Related approaches

It is instructive to contrast the RFL phase with earlier notions of spin-charge separation. Anderson's original picture of the cuprate normal state at optimal doping consisted of gas of spinons with a Fermi surface coexisting and somehow weakly interacting with a gas of holons.<sup>1,13</sup> In the early gauge theory implementations of this picture, both the spinons and holons carried a U(1) [or SU(2)] gauge charge.<sup>22,23</sup> Mean-field phase diagrams were obtained by pairing the spinons and/or condensing the holons, and a pseudogap regime with *d*-wave paired spinons yet uncondensed holons was predicted<sup>22,23</sup>—several years before experiment (of course the *d*-wave nature of the superconducting state was also predicted by other approaches<sup>24</sup>). Within this approach, the normal state at optimal doping was viewed as an incoherent gas of uncondensed holons strongly interacting with unpaired spinons. Some efforts were made to use the neglected gauge-field fluctuations to stop the holons condensing at inappropriately high-energy scales, with some success.<sup>25</sup> A serious worry is that these gauge fluctuations, ignored at the mean-field level, will almost certainly drive confinement<sup>26</sup> (gluing the spinons and the holons together) and thus at low temperatures invalidate the assumed stability of the initial mean-field saddle point.

A few early papers emphasized that spinon pairing would break the continuous U(1) [or SU(2)] gauge symmetry down to a discrete  $Z_2$  subgroup,<sup>7,27</sup> and this might be a way to

avoid confinement. These ideas were later considerably amplified,<sup>5</sup> and the stability of genuinely spin-charge separated ground states established.<sup>28,29</sup> Such ground states are exotic electrical insulators which support fractionalized excitations carrying separately the spin and charge of the electron. Each fragment carries a  $Z_2$  gauge charge, but in contrast to the U(1) gauge theory saddle points, the  $Z_2$  spinons are electrically neutral and do not contribute additively to the resistance. Such fractionalized insulators necessarily support an additional  $Z_2$  vortexlike excitation: the vison.<sup>8</sup>

Within the present formulation, these  $Z_2$  fractionalized insulators can be readily accessed by condensing *pairs of vortices*,<sup>4</sup> rather than condensing rotons. Since vortex pairs only have statistical interactions with charged particles and not the spinons, the resulting pair-vortex condensate is an electrical *insulator* with deconfined spinon, charge-on, and vison excitations<sup>4,5</sup>—dramatically different from the roton Fermi liquid. Recent experiments on very underdoped cuprate samples have failed to find evidence of a gapped vison,<sup>10,11</sup> suggesting that the pseudogap phase is not fractionalized. But a negative result in these vison detection experiments does not preclude the RFL phase, which supports mobile single vortices even at very low temperatures. A different approach supposing unbound and mobile single vortices is the QED<sub>3</sub> theory of Ref. 14.

At the most basic level, a roton is a small pattern of swirling electrical currents. Much recent attention has focused on the possibility of a phase in the underdoped cuprates with nonvanishing orbital currents,<sup>30</sup> counter-rotating about elementary plaquettes on the two sublattices of the square lattice. Such a phase was initially encountered as a mean-field state within a slave-fermion gauge theory approach<sup>31</sup>—the so-called “staggered-flux phase,” but has been resurrected as the “*d*-density wave” ordered state of a Fermi liquid.<sup>30</sup> In either case, such a phase within the present formulation would be described as a “vortex–antivortex lattice”—a checkerboard configuration on the plaquettes of the 2D square lattice. Ground states with *short-ranged* “orbital antiferromagnetic” order have also been suggested recently.<sup>32</sup> Surprisingly, the roton liquid phase also has appreciable short-ranged orbital order. A “snapshot view” of the orbital current correlations in the RL phase can be obtained by examining the vortex-density structure function. For simplicity, consider the charge sector of the RFL theory (the RL) in the limit of large plasmon velocity  $v_0 \rightarrow \infty$ , which suppresses the longitudinal charge density fluctuations. The transverse electrical currents are then described by the Hamiltonian  $H_{\text{rot}}$  in Eq. (89) with  $\tilde{a}_j \rightarrow 0$ . The vortex-density structure function  $S_{\text{NN}}$  can be readily obtained from this Gaussian theory:

$$S_{\text{NN}}(\mathbf{k}) \equiv \langle |\hat{N}(\mathbf{k})|^2 \rangle = \frac{1}{2} \sqrt{\frac{\kappa_r}{u_v}} |\mathcal{K}_x(\mathbf{k}) \mathcal{K}_y(\mathbf{k})| \sqrt{\mathcal{K}^2(\mathbf{k})}, \quad (258)$$

with  $|\mathcal{K}_j(\mathbf{k})| = 2|\sin(k_j/2)|$  and  $\mathcal{K}^2(\mathbf{k}) = \sum_j |\mathcal{K}_j(\mathbf{k})|^2$  as before. The vortex structure function is analytic except on the  $k_x = 0$  and  $k_y = 0$  axes and is *maximum* at  $\mathbf{k} = (\pi, \pi)$ , indicative of short-ranged orbital antiferromagnetism. As such, the RFL/RL phase can perhaps be viewed as a quantum-melted



staggered-flux (or vortex-antivortex) state, with only residual short-ranged orbital current correlations—the rapid motion of the rotons being responsible for the melting. If the fermions are paired, the amplitude for the basic roton-hopping process which generates the structure function above vanishes upon approaching half filling, as is apparent from Eq. (84). A preliminary analysis<sup>12</sup> suggests that the further neighbor roton hopping processes which do survive at half filling, lead to a vortex structure function which vanishes as  $|\mathbf{k} - (\pi, \pi)|$  for wave vectors near  $(\pi, \pi)$ .

### B. Cuprates and the RFL phase?

In applying BCS theory to low-temperature superconductors, one implicitly assumes that the normal state above  $T_c$  is adequately described by Fermi liquid theory. Within a modern renormalization group viewpoint,<sup>33</sup> this is tantamount to presuming that the effective Hamiltonian valid below atomic energy scales (of say 10 eV) is sufficiently “close” (in an abstract space of Hamiltonians) to the Fermi liquid fixed-point Hamiltonian (actually an invariant or “fixed” manifold of Hamiltonians characterized by the marginal Fermi-liquid parameters). In practical terms, “close” means that under a renormalization group transformation which scales down in energies, the renormalized Hamiltonian arrives at the Fermi liquid fixed point on energy scales which are still well above  $T_c$ . BCS theory then describes the universal crossover flow between the Fermi liquid fixed point (which is marginally unstable to an attractive interaction in the Cooper channel) and the superconducting fixed point which characterizes the universal low-energy properties of the superconducting state well below  $T_c$ . Four orders of magnitude between 10 eV and  $T_c$  gives the RG flows plenty of “time” to accomplish the first step to the Fermi liquid fixed point, and is the ultimate reason behind the amazing quantitative success of BCS theory.

#### 1. Assumptions underlying the RFL approach

In what follows, our working hypothesis is that the effective electron Hamiltonian on the 10 eV scale appropriate for the 2D copper-oxygen planes at doping levels within and nearby the superconducting “dome,” is close to the roton Fermi liquid “fixed manifold” of Hamiltonians (characterized by Fermi/Bose liquid parameters for the quasiparticles/rotons). More specifically, we presume that when renormalized down to the scale of say one half of an eV, the effective Hamiltonian can be well approximated by a Hamiltonian *on the RFL fixed manifold*—up to *small* perturbations. The important small perturbations are those that are *relevant* over appreciable portions of the fixed manifold. As established in Secs. IV and V, there are three such perturbations: (i) Single vortex-hopping, (ii) charge hopping, and (iii) attractive quasiparticle interactions in the Cooper channel. When relevant, these three processes destabilize the RFL phase and cause the RG flows to cross over to different fixed points which determine the asymptotic low-temperature behavior. From our calculations, we find that at least one, and often more of these three perturbations is always relevant, regardless of the roton-liquid parameters. Hence the RFL should be regarded

as a *critical*, and usually *multicritical* phase, rather than a stable one. For these three processes, the resulting quantum ground states are, respectively, (i) a (“confined” and) conventional Fermi liquid phase, (ii) a conventional superconducting phase with singlet  $d_{x^2-y^2}$  pairing and gapless nodal Bogoliubov quasiparticles, and (iii) a “rotonic superconductor” with the properties of a conventional superconductor but coexisting gapless roton excitations. The rotonic superconductor, as described in Sec. VI B 2, has further potential instabilities driven by either single vortex hopping and charge hopping/explicit pairing. Both perturbations, if relevant, will generate an energy gap for the rotons. It seems likely that the domains of relevance of these two perturbations overlap, so that there no regime of true stability of the rotonic superconductor. The true ground state of the system in this regime is then a conventional superconductor, and its rotonic nature is evidenced only as an intermediate energy crossover.

#### 2. Effective parameters

In order to construct a phase diagram within this scenario, it is necessary to specify the various parameters (e.g., Bose and Fermi liquid parameters) of the RFL model as a function of the doping level,  $x$ . Because the RFL is multicritical, we cannot rely upon “universality” to validate *ad hoc* requirements of smallness of perturbations, as might be the case, e.g., for renormalized perturbations around a stable fixed point. Instead, we will make some assumptions based (partly) on physics. First, our main assumption is the basic validity at high energies of the roton dynamics, and of Fermi liquidlike quasiparticles. Second, we assume that superconductivity is never strong, i.e., always occurs below the rotonic/Fermi liquid energy scales. Mathematically, these two assumptions are encompassed in the inequalities

$$\kappa_r, u_v, t_s \gg t_v \gg t_c, \Delta_j. \quad (259)$$

Reading from left to right, this corresponds to the first and second assumptions above. In practice, we can at best hope for a factor of a few between, e.g.,  $t_v$  and  $\kappa_r$ , so the “ $\gg$ ” symbols above should not be taken too strongly. Although it is not important to our discussion, it is natural to assume that  $v_0 \sim \kappa_r, u_v, t_s$  (since Coulombic energies at the lattice scale are comparable to the electronic bandwidth). A third assumption, which is not needed for consistency of the approach, but seems desirable empirically, is that the fermion dynamics is primarily by spinon rather than electron hopping  $t_s \gg t_e$ . Many of the parameters of the RFL phase can be fixed empirically from the observed behavior of the cuprates on or above the eV scale. For example, the  $k$ -space location of the quasiparticle Fermi surface can be chosen to coincide with the electron Fermi surface as measured via angle-resolved photoemission spectroscopy (ARPES).<sup>34</sup> The value of other parameters, such as the bare velocity  $v_0$  which appears in  $\mathcal{L}_a$  and sets the scale of the plasmon velocity, can be roughly estimated from the basic electronic energy scales, and in any case does not greatly effect the relevance/irrelevance of the three important perturbations. For our basic lattice Hamiltonian introduced in Sec. II, the two most important parameters characterizing the RFL phase are the vortex core energy  $u_v$  and the roton hopping strength  $\kappa_r$ . Indeed, as shown in



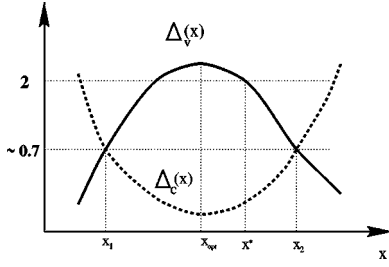


FIG. 2. Proposed values for the vortex and charge hopping scaling dimensions  $\Delta_v(x)$  and  $\Delta_c(x)$  as a function of the doping  $x$ . These hopping perturbations are relevant at the RFL fixed point when their  $\Delta < 2$ . Within the doping range  $x_1 < x < x_2$ , the charge hopping is more relevant and the RFL phase is unstable to superconductivity.

Sec. IV, the scaling dimensions which determine the relevance of the vortex and charge hopping perturbations at the RFL fixed point, denoted  $\Delta_v$  and  $\Delta_c$ , respectively, depended sensitively on the ratio  $u_v/\kappa_r$ . For example, ignoring renormalizations from the gapless fermions we found

$$\Delta_v = \frac{1}{2\Delta_c} = \frac{1}{4\pi} \sqrt{\frac{u_v v_{\pm}}{\kappa_r \tilde{v}_0}}, \quad (260)$$

with  $v_{\pm}^2 = \tilde{v}_0^2 + \frac{1}{4}u_v\kappa_r = v_0^2 + \frac{1}{2}u_v\kappa_r$ .

Rather than specifying the doping dependence of  $u_v$ ,  $\kappa_r$ ,  $v_0$  and the Bose/Fermi-liquid parameters, though, it is simpler and suffices for our purposes to specify the final  $x$  dependence of  $\Delta_v$  and  $\Delta_c$ . Shown in Fig. 2 is a proposed form for  $\Delta_{v/c}(x)$ , primarily chosen to fit the gross features of the cuprate phase diagram. For example, since the RFL phase is strongly unstable to the Fermi liquid phase when  $\Delta_v \ll 2$ , we have taken  $\Delta_v$  decreasing to small values in the strongly overdoped limit. On the other hand, to account for the observed *non-Fermi-liquid* behavior in the normal state near *optimal* doping,<sup>13</sup> requires that we take  $\Delta_v \gg 2$  in that regime. On the underdoped side of the dome we can use the observed linear  $x$  dependence of the superfluid density to guess the behavior of  $\Delta_v$  for small  $x$ . In this regime the (renormalized) vortex core energy in the superconducting state is presumably tracking the transition temperature, varying linearly with  $x$ . It seems plausible that the bare vortex core energy  $u_v$  in the lattice Hamiltonian, while perhaps significantly larger, also tracks this  $x$  dependence. This implies  $\Delta_v \sim \sqrt{u_v} \sim \sqrt{x}$  as depicted in the figure. Moreover, to recover (conventional) insulating behavior when  $x \rightarrow 0$ , requires that vortex condensation (rather than charge hopping) be more relevant in this limit, i.e.,  $\Delta_v < \Delta_c$  (see below).

### 3. Phase diagram

Under the above assumptions, we now discuss in some detail the resulting phase diagram and predicted behaviors. Consider first the ground states upon varying  $x$ . In the extreme overdoped limit with  $\Delta_v \ll 2$ , vortex hopping will be a strongly relevant perturbation at the RFL fixed point. The vortices will condense at  $T=0$ , leading to a conventional Fermi liquid ground state. Upon decreasing  $x$  there comes a special doping value ( $x_2$  in Fig. 2), where  $\Delta_c$  becomes smaller than  $\Delta_v$ . At  $x=x_2$  the RFL phase is unstable to both

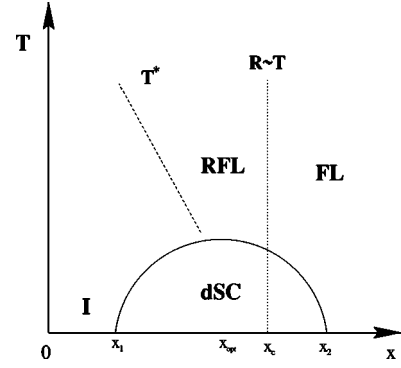


FIG. 3. Schematic phase diagram that follows from the doping dependence of  $\Delta_{v,c}$  shown in Fig. 2. With increasing doping  $x$ , the ground state evolves from a charge ordered insulator ( $I$ ), into a superconductor (dSC), and then back into a Fermi liquid (FL). The normal state behavior near optimal doping is controlled by the roton Fermi-liquid (RFL) ground state. Below  $T^*$ , the quasiparticles pair. A quantum phase transition between the RFL and FL ground states occurs at  $x_c$  but is preempted by superconductivity. At  $x=x_c$ , a normal state resistance *linear* in temperature is predicted.

vortex and charge hopping processes, since both  $\Delta_v = \Delta_c = 1/\sqrt{2} < 2$ . It seems reasonable to assume that in this situation, the more strongly relevant process ultimately dominates at low energies. This implies that  $x_2$  demarcates the boundary between a Fermi liquid and a superconducting ground state, as illustrated in Fig. 3. Upon further decreasing  $x$ , the charge hopping becomes even more strongly relevant, tending to increase  $T_c$  until it reaches a maximum at “optimal doping,” denoted  $x_{opt}$  in Figs. 2 and 3. As one further decreases  $x$ ,  $T_c$  should start decreasing. But as shown in Sec. V A, with decreasing vortex core energy  $u_v \sim x$ , the enhanced vortex density fluctuations generate an increasing antiferromagnetic exchange interaction  $J \sim t_s^2/u_v$ . This *attractive* interaction between fermions will mediate quasiparticle pairing, with a pairing energy scale growing rapidly as  $x \rightarrow 0$ . It is natural to associate this “quasiparticle pairing” temperature scale with the crossover temperature into the pseudogap regime,<sup>13</sup> shown as  $T^*$  in Fig. 3. As discussed in Sec. VI B 2, under the assumption that quasiparticle kinetic energy arises primarily through spinon-hopping  $t_s \gg t_e$ , the superfluid density associated with the quasiparticle pairing is small, so that potential true superconductivity as a consequence of this pairing is suppressed to a low or zero temperature.

Since  $\Delta_v < \Delta_c$  for doping levels with  $x < x_1$ , vortex-hopping should again dominate over charge hopping. This is the same condition which we argued leads to the Fermi-liquid state for  $x > x_2$  above. However, the physics for small  $x$  is more complex, owing to the strong antiferromagnetic interactions and proximity to the commensurate filling  $x=0$  at which antiferromagnetic order is probable. In the U(1) vortex-quasiparticle formalism of this paper, this difference arises from the freedom to choose (as  $\Delta_j \rightarrow 0$ ) the fermion density to minimize the total (free) energy of the system. In the majority of this paper we have taken  $n_f = \rho_0$ , in order to minimize the vortex kinetic energy. However, if antiferromagnetic quasiparticle interactions are large, and  $x \ll 1$ , another possibility arises. To optimally benefit from the antifer-

romagnetic interactions, one may instead choose the fermion density commensurate  $n_f=1$ , for which the Fermi surface is optimally nested and the quasiparticles can become fully gapped, gaining the maximal “condensation energy” from antiferromagnetic ordering. The cost of this choice is some loss of vortex kinetic energy as the vortex motion becomes somewhat frustrated by the resulting dual flux  $\pi x$ . Although we assume vortex energy scales are large, this flux itself is small for small  $x$ , so eventually as  $x \rightarrow 0$  this rise in kinetic energy becomes smaller than the lowering of fermionic energy due to antiferromagnetism and such a choice becomes favorable. The ultimate physics of the remaining vortices in this limit is difficult to analyze reliably, but it seems evident that for  $\Delta_v < \Delta_c$ , the more relevant vortex-hopping will lead to an insulating state. Since the quasiparticles are then gapped, the remaining uncompensated flux  $\pi x$  is expected to lead to incommensurate charge ordering when the vortices condense—the dual analog of the Abrikosov lattice.

#### 4. The RFL normal state

Having established the phase diagram which follows from the assumed doping dependence of  $\Delta_v(x)$ , we turn to a discussion of the normal state properties above  $T_c$ . Under our working assumption, in the energy range above  $T_c$  we can ignore the small (but ultimately relevant) charge hopping perturbation, and use the RFL fixed point Hamiltonian to describe the physics of the normal state. First consider the special doping value  $x=x_c$ , where  $\Delta_v=2$  on the overdoped side of the superconducting dome (see Figs. 2 and 3). For  $x < x_c$  the vortex hopping strength  $t_v$  decreases upon scaling down in energy, so that it can be treated perturbatively. As shown in Sec. VI A, at second order in the vortex hopping, the electrical conductivity is additive in the roton and quasiparticle conductivities. With impurities present the quasiparticle conductivity  $\sigma^f$  will saturate to low temperatures, whereas  $\sigma_{\text{rot}} \sim T^{-\gamma}$  (with  $\gamma=2\Delta_v-3$ ) diverges as  $T \rightarrow 0$ . The low-temperature normal state electrical resistance is thus predicted to behave as a power law

$$R(T) \sim t_v^2 T^\gamma, \quad (261)$$

with a *vanishing residual resistivity*. Moreover, right at  $x=x_c$ , since  $\gamma=1$  a *linear* temperature dependence is predicted. Notably, this transport behavior is due to the presence of the quasiconsensate in the RFL phase, and as such is completely independent of the single-particle scattering rate.

The Hall conductivity, however, will be largely determined by the fermionic quasiparticle contribution, as detailed in Sec. VII. Specifically, the cotangent of the Hall angle  $\cot(\Theta_H) \equiv \rho_{xx}/\rho_{xy}$ , was found to vary as  $\cot(\Theta_H) \sim 1/(T^\gamma \tau_f^2)$  in the RFL phase, with  $\tau_f^{-1}$  the spinon momentum relaxation rate. Moreover, at the hot spots on the Fermi surface  $\tau_f^{-1} \sim T^{\gamma+1/2}$ , due to scattering off the gapless rotons. Under the assumption that these same processes dominate the temperature dependence of the quasiparticle *transport* scattering rate, we deduce that

$$\cot(\Theta_H) \sim \frac{1}{T^\gamma \tau_f^2} \sim T^{1+\gamma}, \quad (262)$$

and right at  $x=x_c$ , a quadratic dependence  $\cot(\Theta_H) \sim T^2$ . In striking contrast to conventional Drude theory which predicts

$\cot(\Theta_H) \sim \tau^{-1} \sim R$  (with  $\tau$  the electron’s momentum relaxation time), in the RFL phase the cotangent of the Hall angle varies with a *different power* of temperature than for the electrical resistance  $R \sim T^\gamma$ . This non-Drude behavior is consistent with the electrical transport generally observed in the optimally doped cuprates,<sup>35–37</sup> where  $R \sim T$  and  $\cot(\Theta_H) \sim T^2$ .

Consider next the thermal conductivity  $\kappa$  near optimal doping within the RFL normal state. One of the most important defining characteristics of a conventional Fermi liquid is the Wiedemann–Franz law—the universal low-temperature ratio of thermal and electrical conductivities  $L \equiv \kappa/\sigma T$ . In a Fermi liquid, electronlike Landau quasiparticles carry both the conserved charge and the heat, and since the energy of the individual quasiparticles becomes conserved as  $T \rightarrow 0$ , the Lorenz ratio is universal,  $L^{\text{FL}} = L_0 = \pi^2 k_B^2 / 3e^2$ . In contrast, the electrical conductivity is infinite in a superconductor, but the condensate is ineffective at carrying heat so that the Lorenz ratio vanishes,  $L^{\text{SC}} = 0$ . Within the RFL phase, heat can also be transported by the single fermion excitations, with a contribution to the thermal conductivity linear in temperature:  $\kappa_s = L_0 \sigma^f T$ . At low enough temperatures this will dominate over the phonon contribution  $\kappa_{\text{phon}} \sim T^3$ . But the roton excitations, which have a quasi-one-dimensional dispersion at low energies, will presumably also contribute a linear temperature dependence  $\kappa_{\text{rot}} \sim T$ . Thus, the *total* thermal conductivity in the RFL phase is expected to vanish linearly in temperature  $\kappa \sim T$ . But since the roton contribution to the *electrical* conductivity diverges as  $T \rightarrow 0$ , the RFL phase is predicted to have a vanishing Lorenz ratio

$$L^{\text{RFL}} = \frac{\kappa}{\sigma T} \sim T^\gamma. \quad (263)$$

The quasiconsensate in the RFL phase is much more effective at transporting charge than heat, much as in a superconductor. Electron doped cuprates near optimal doping, when placed in a strong magnetic field to quench the superconductivity, do exhibit a small Lorenz ratio at low temperatures<sup>38</sup>  $L \approx L_0/5$ . But extracting the zero field Lorenz ratio is problematic, since above  $T_c$  the phonon contribution to  $\kappa$  is non-negligible.

It is instructive to consider the electrical resistance also in the *underdoped regime*, particularly upon cooling below the fermion pairing temperature. Above this crossover line the predicted electrical resistance varies with a power of temperature  $R \sim T^\gamma$ . Since, as we assume, the fermion’s kinetic energy comes primarily in the form of spinon hopping  $t \approx t_s \gg t_e$ , the resulting superfluid density is, however, very small (see Sec. VI B 2), and phase coherent superconductivity does not result, at least not in this temperature range. Nevertheless, one would expect a dramatic increase in the fermion conductivity  $\sigma^f$  upon cooling through  $T^*$ —much as seen in superconducting thin films upon cooling through the materials bulk transition temperature. Since the conductivity is additive in the roton and spinon contributions  $R(T)^{-1} \sim \sigma_{\text{rot}}(T) + \sigma^f(T)$ , a large and rapid increase in  $\sigma^f(T)$  should be detectable as a drop in the electrical resistance relative to the “critical” power law form, i.e.,

$$\frac{R(T)}{T^\gamma} \sim \frac{1}{1 + cT^\gamma \sigma^f(T)}. \quad (264)$$

This behavior is generally consistent with that seen in the underdoped cuprates,<sup>13,37</sup> provided we take  $\gamma \approx 1$ .

### 5. Entering the superconductor

We finally discuss the predicted behavior upon cooling from the RFL normal state into the superconducting phase. The main change occurs in the spectrum of roton excitations, which become gapped inside the superconductor. The roton gap  $\Delta_{\text{rot}}$  should be manifest in optical measurements, since the optical conductivity will drop rapidly for low frequencies  $\omega < \Delta_{\text{rot}}$ . As in BCS theory, the ratio of the (roton) gap to the superconducting transition temperature  $2\Delta_{\text{rot}}/T_c$  should be of order one. This ratio is determined by the RG crossover flow between the RFL and superconducting fixed points. It seems likely that this ratio will be *nonuniversal*, depending on the marginal Bose liquid parameters characterizing the RFL fixed point (this is distinct from changes in this ratio in strong coupling Eliashberg theories, since here variations in the ratio are due to marginal parameters of the RFL fixed manifold even at arbitrarily weak coupling). Nevertheless, it would be instructive to compute this ratio for the simple near-neighbor RFL model we have been studying throughout, and to analyze the behavior of the optical conductivity above the gap.

Another important consequence of gapped rotons below  $T_c$ , is that the electron lifetime should rapidly increase upon cooling into the superconducting phase. With reduced scattering from the rotons, the ARPES linewidth should narrow, most dramatically near the normal state hot spots (with tangents parallel to the  $x$  or  $y$  axes). This behavior is consistent with the ARPES data in the cuprates,<sup>34</sup> which upon cooling into the superconductor does show a significant narrowing of the quasiparticle peak, particularly so at the Fermi surface crossing near momentum  $(\pi, 0)$ .

Despite these preliminary encouraging similarities between various properties of the RFL phase and the cuprate phenomenology, much more work is certainly needed before one can establish whether this exotic non-Fermi-liquid ground state might actually underlie the physics of the high-temperature superconductors. We have already emphasized the strengths of this proposition, but there are, of course, some experimental features which seem challenging to explain from this point of view. The ‘‘quasiparticle charge,’’ i.e., temperature derivative of the superfluid density  $\partial K_s / \partial T|_{T=0}$  appears, based upon a small number of experimental data points, to be large and roughly independent of doping  $x$ , in apparent conflict with the RFL prediction. The linear temperature dependence of the electron lifetime  $1/\tau_f \sim k_B T / \hbar$  observed for nodal quasiparticles near optimal doping in ARPES also does not seem natural in the RFL. However, it seems likely that it may be possible to explain a small number of such deviations from the *most naive* RFL predictions by more detailed considerations. Further investigations of the RFL proposal should confront other experimental probes, such as interlayer transport and tunneling. Toward this end, it will be necessary to generalize the present approach to three

dimensions. More detailed predictions, such as for the optical conductivity upon entering the superconductor and the thermal Hall effect in the RFL normal state, might also be helpful in this regard. It would, of course, be most appealing to identify a more convincing experiment for the roton Fermi-liquid state, analogous to the vison-trapping experiment<sup>8</sup> for detecting 2D spin-charge separation. However, given the critical nature of the RFL, with copious gapless excitations with varieties of quantum numbers, finding such an incontrovertible experimental signature may be difficult.

### ACKNOWLEDGMENTS

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### APPENDIX A: FERMIL LIQUID PHASE

We expect the Fermi liquid phase to occur upon complete proliferation and unbinding of vortices. To obtain the Fermi liquid in our formulation, we therefore in this appendix consider the limit of large vortex hopping  $t_v \rightarrow \infty$  and small vortex energy  $u_0 \rightarrow 0$ . We have previously demonstrated the equivalence of the U(1) gauge theory to a  $Z_2$  gauge theory, and it is this latter formulation which is most convenient in this limit.

To observe the Fermi liquid, we analyze the  $Z_2$  gauge theory in its Hamiltonian form. Although this limit is very straightforward, based on previous work on  $Z_2$  gauge theory, it is instructive to go through it in some detail here, in order to observe the effects of the spinon pairing term. The Hamiltonian density can be separated into pure gauge, charge, and spin parts  $\mathcal{H} = \mathcal{H}_h + \mathcal{H}_c^{Z_2} + \mathcal{H}_s^{Z_2}$ , with

$$\mathcal{H}_h = -h \sum_j \sigma_j^x(\mathbf{r}), \quad (A1)$$

where  $\sigma_j^x$  is the usual Pauli matrix in the space of states on link in the  $j$  direction coming from site  $\mathbf{r}$  (and hence anti-commutes with  $\sigma_j^z$ ). In the charge sector

$$\mathcal{H}_c^{Z_2} = -2t_c \sum_j \sigma_j^z(\mathbf{r}) \cos(\phi_{\mathbf{r}+\hat{\mathbf{x}}_j} - \phi_{\mathbf{r}}) + u_c n_{\mathbf{r}} [n_{\mathbf{r}} - 1], \quad (A2)$$

where  $n_{\mathbf{r}}$  is the number operator conjugate to  $\phi_{\mathbf{r}}$ , satisfying  $[\phi_{\mathbf{r}}, n_{\mathbf{r}'}] = i\delta_{\mathbf{r}, \mathbf{r}'}$ . For simplicity, in the spin sector we consider a local  $s$ -wave pair field instead of a  $d$ -wave one. This is not essential, but simplifies the presentation and still addresses the essential issue of the relevance of spinon pairing in the Fermi liquid. Hence,



$$\begin{aligned} \mathcal{H}_s^{Z_2} = & -t_s \sigma_j^z(\mathbf{r}) [f_{\mathbf{r}+\hat{\mathbf{x}}_j}^\dagger f_{\mathbf{r}\sigma} + \text{H.c.}] + \epsilon_0 f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma} \\ & + \Delta (f_{\mathbf{r}\uparrow} f_{\mathbf{r}\downarrow} + \text{H.c.}) + u_f (f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma})^2. \end{aligned} \quad (\text{A3})$$

Note that we have added a local on-site energy  $\epsilon_0$  and interaction  $u_f$ , allowed in general by symmetry. The Hamiltonian commutes with the gauge generators

$$G_{\mathbf{r}} = (-1)^{n_{\mathbf{r}} + \sum_j f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma}} \prod_j \sigma_j^x(\mathbf{r}) \sigma_j^x(\mathbf{r} - \hat{\mathbf{x}}_j). \quad (\text{A4})$$

We require  $G_{\mathbf{r}} = 1$  to enforce gauge invariance.

In the analysis of the large vortex hopping limit above, we obtained the action for a  $Z_2$  gauge theory with zero kinetic term. This corresponds in the Hamiltonian to large  $h$  (in particular we will take  $h \gg t_c, t_s$ ). In this limit  $\sigma_j^x \approx 1$ , and the gauge constraint becomes

$$(-1)^{n_{\mathbf{r}} + \sum_j f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma}} = 1, \quad (\text{A5})$$

i.e., requiring an even number of bosons and fermions on each site. Further, for large  $h$ , the chargon and spinon hopping terms are strongly suppressed, and can be considered perturbatively. At zeroth order in  $t_c, t_s$ , then, the charge and spin sectors are decoupled at each site and decoupled also from one another except by the gauge constraint. In the charge sector, for  $u_c > 0$ , it is energetically favorable to have only either zero or one charge-on per site  $n_{\mathbf{r}} = 0, 1$ . If  $n_{\mathbf{r}} = 0$ , then we must have either zero or two spinons per site. Note that in this subspace, even for small  $\Delta$ , these two local spinon singlet states are nondegenerate: the energy of the two-spinon state differs from the zero-spinon state by the energy  $\epsilon_0 + 4u_f$ . With nonzero  $\Delta$ , one obtains as eigenstates simply two different linear combination of these two states on each site. The lower energy of the two will be realized in the ground state, and the upper-energy state has no physical significance. Physically, the lower-energy state, which is neutral ( $n_{\mathbf{r}} = 0$ ) and is a spin singlet, corresponds to the local vacuum, i.e., a site with no electron on it. If  $n_{\mathbf{r}} = 1$ , then we must have one spinon on this site, which may have either spin orientation. This state has thus the quantum numbers of a physical electron. Fixing the total charge of the system  $Q = \sum_{\mathbf{r}} n_{\mathbf{r}} \neq 0$  will require some number of electrons in the system. At zeroth order these are localized, but at  $O(t_c t_s / h)$ , the electrons acquire a hopping between sites, and one obtains a system clearly in a Fermi liquid phase (it is not non-interacting, since one has a hardcore constraint in the limit considered).

The above considerations can be applied for zero or non-zero  $\Delta$ , and there are no qualitative differences in the results (the detailed nature of the vacuum state depends smoothly on  $\Delta$ , leading to a weak dependence of the effective electron-hopping on the ratio  $\Delta / u_f$ ) in either case. This strongly suggests that  $\Delta$  is not a relevant (in the renormalization group sense) perturbation in the Fermi liquid phase. This notion can be confirmed more formally by considering the limit of very weak  $\Delta \ll u_f$ , in which it may be treated perturbatively. At zeroth order in this perturbation theory, the vacuum state (on a single site) is just the state with zero spinons. Formally, the perturbative relevance of  $\Delta$  is determined by the behavior of the two-point function of the pair-field operator, e.g.,

$$C_{\Delta}(\tau) = \langle f_{\mathbf{r}\uparrow}(\tau) f_{\mathbf{r}\downarrow}(\tau) f_{\mathbf{r}\uparrow}^\dagger(0) f_{\mathbf{r}\downarrow}^\dagger(0) \rangle. \quad (\text{A6})$$

Since the pair-field operator creates a site doubly occupied by spinons, the energy of the intermediate states encountered in the imaginary time evolution from 0 to  $\tau$  is increased by  $u_f$ , so that the spinon pair-field correlator decays exponentially,  $C_{\Delta}(\tau) \sim e^{-4u_f \tau}$ . This indicates that the spinon pair field is strongly irrelevant (formally with infinite scaling dimension). The importance of this observation in the context of this paper is that it provides an example in which the spinon pair field—which naively has a special significance because it alone violates spinon number conservation—is irrelevant. This irrelevance is a consequence of strong vorticity fluctuations, which bind (confine) charge to the spinons to form electrons. Since charge is conserved, electron number *must* be conserved in the resulting effective theory. Similar (but not quite so large) vorticity fluctuations in the RFL have the effect of rendering the spinon pair field irrelevant.

## APPENDIX B: ENSLAVING THE $Z_2$ GAUGE FIELDS

To enslave the  $Z_2$  gauge fields, we employ two sequential unitary transformations  $U_{12} = U_1 U_2$ , with

$$U_1 = \prod_{\mathbf{r}} \left( \prod_{x'=0}^{\infty} \sigma_1^z(\mathbf{r} + x' \hat{\mathbf{x}}) \right)^{n_{\mathbf{r}}^f} = \prod_{\mathbf{r}} [\sigma_1^z(\mathbf{r})]^{\sum_{x'=0}^{\infty} n_{\mathbf{r}-x' \hat{\mathbf{x}}}^f}, \quad (\text{B1})$$

$$U_2 = \prod_{\mathbf{r}} \left( \prod_{x'=0}^{\infty} \bar{\sigma}_1^x(\mathbf{r} + x' \hat{\mathbf{x}}) \right)^{N_{\mathbf{r}}} = \prod_{\mathbf{r}} [\bar{\sigma}_1^x(\mathbf{r})]^{\sum_{x'=0}^{\infty} N_{\mathbf{r}-x' \hat{\mathbf{x}}}}. \quad (\text{B2})$$

The two operators  $U_1$  and  $U_2$  are mutually commuting. Roughly,  $U_1$  transforms to a gauge in which  $\sigma_1^z = 1$ , and  $U_2$  transforms to a gauge with  $\bar{\sigma}_1^x = 1$ . More precisely, applying the first unitary transformation  $H_{\text{pl}}$  and  $H_N$  are invariant, while the fermion Hamiltonian transforms to

$$U_1^\dagger H_f^{Z_2} U_1 = H_f^{Z_2} |_{\sigma_j^z(\mathbf{r}) \rightarrow \sigma_j^z, \text{slave}(\mathbf{r}; N)}, \quad (\text{B3})$$

where

$$\begin{aligned} \sigma_1^z, \text{slave}(\mathbf{r}) &= 1, \\ \sigma_2^z, \text{slave}(\mathbf{r}) &= \prod_{x'=0}^{\infty} \prod_{\square(\mathbf{r}+\mathbf{w}+x' \hat{\mathbf{x}})} \sigma^z = \prod_{x'=0}^{\infty} (-1)^{N_{\mathbf{r}+\mathbf{w}+x' \hat{\mathbf{x}}}}. \end{aligned} \quad (\text{B4})$$

The vortex kinetic terms also transform

$$U_1^\dagger H_{\text{kin}}^{Z_2} U_1 = H_{\text{kin}}^{Z_2} |_{\bar{\sigma}_j^x(\mathbf{r}) \rightarrow \bar{\sigma}_j^x, \text{slave}(\mathbf{r}; N)}, \quad (\text{B5})$$

with

$$\begin{aligned} \bar{\sigma}_1^x, \text{slave}(\mathbf{r}) &= 1, \\ \bar{\sigma}_2^x, \text{slave}(\mathbf{r}) &= \prod_{x'=0}^{\infty} (-1)^{n_{\mathbf{r}-x' \hat{\mathbf{x}}}^f}. \end{aligned} \quad (\text{B6})$$

Simultaneously, the first constraint is rendered trivial



$$C_{\mathbf{r}}^{1,\text{slave}} = U_1^\dagger \tilde{C}_{\mathbf{r}}^1 U_1 = \prod_{\square(\mathbf{r})} \bar{\sigma}^x = 1. \quad (\text{B7})$$

Further transformation with  $U_2$  removes  $\bar{\sigma}_j^x$  from  $H_{\text{kin}}^{Z_2}$  and trivializes the remaining constraint, i.e.,

$$C_{\mathbf{r}}^{2,\text{slave}} = U_2^\dagger \tilde{C}_{\mathbf{r}}^2 U_2 = \prod_{\square(\mathbf{r})} \sigma^z = 1. \quad (\text{B8})$$

The final transformed Hamiltonian no longer involves any dynamical gauge fields [whose state is uniquely specified by Eqs. (B7) and (B8)], and is simply given by

$$H_{Z_2}^{\text{slave}} = U_{12}^\dagger \tilde{H}_{Z_2} U_{12} = \tilde{H}_{Z_2} |_{\sigma_j^\mu \rightarrow \sigma_j^{\mu,\text{slave}}}. \quad (\text{B9})$$

The electron destruction operator in the  $Z_2$  vortex-spinon theory  $\tilde{c}_{\mathbf{r}\sigma} = \tilde{b}_{\mathbf{r}} f_{\mathbf{r}\sigma}$  with  $\tilde{b}_{\mathbf{r}}$  in Eq. (40), transforms upon enslaving in an identical fashion

$$U_{12}^\dagger \tilde{c}_{\mathbf{r}\sigma} U_{12} = \tilde{c}_{\mathbf{r}\sigma} |_{\sigma_j^z \rightarrow \sigma_j^{z,\text{slave}}}. \quad (\text{B10})$$

### APPENDIX C: ENSLAVING THE U(1) GAUGE THEORY

As for the  $Z_2$  case, to enslave the gauge fields in the U(1) formulation we apply two sequential unitary transformations  $U_{ab} = U_a U_b$ , with

$$U_a = e^{i\sum_{\mathbf{r},\mathbf{r}'} N_{\mathbf{r}} V(\mathbf{r}-\mathbf{r}') (\tilde{\nabla} \cdot \tilde{\alpha})(\mathbf{r}')} e^{i\sum_{\mathbf{r},\mathbf{r}'} n_{\mathbf{r}}^f V(\mathbf{r}-\mathbf{r}') (\tilde{\nabla} \cdot \tilde{\beta})(\mathbf{r}')}, \quad (\text{C1})$$

where  $\nabla^2 V(\mathbf{r}-\mathbf{r}') = \delta_{\mathbf{r},\mathbf{r}'}$  and

$$U_b = e^{(i/2)\sum_{\mathbf{r},\mathbf{r}'} n_{\mathbf{r}}^f \Theta(\mathbf{r}-\mathbf{r}') N_{\mathbf{r}'}}. \quad (\text{C2})$$

where the lattice Laplacian is

$$\nabla^2 f(\mathbf{r}) = \sum_j \partial_j^2 f(\mathbf{r} - \hat{\mathbf{x}}_j) = \sum_j f(\mathbf{r} + \hat{\mathbf{x}}_j) + f(\mathbf{r} - \hat{\mathbf{x}}_j) - 2f(\mathbf{r}). \quad (\text{C3})$$

Here we have introduced a ‘‘angle’’ function  $\Theta(\mathbf{r}-\mathbf{r}')$ , to be determined later. The two unitary transformations commute with one another. Again, both  $H_{\text{pl}}$  and  $H_N$  are invariant commuting with  $U_a$  and  $U_b$ , whereas under application of  $U_a$  the vortex kinetic energy transforms to

$$U_a^\dagger H_{\text{kin}} U_a = H_{\text{kin}} |_{e^{i\alpha_j} \rightarrow e^{i\alpha_j^{\text{slave}}}}. \quad (\text{C4})$$

Here,  $\alpha_j^{\text{slave}}$  is the transverse part of  $\alpha_j$  and  $\alpha_j^{\text{slave}}$  satisfies

$$\epsilon_{ij} \partial_i \alpha_j^{\text{slave}}(\mathbf{r} - \mathbf{w}) = \pi n_{\mathbf{r}}^f, \quad \tilde{\nabla} \cdot \tilde{\alpha}^{\text{slave}} = 0. \quad (\text{C5})$$

Similarly, the spinon and electron hopping Hamiltonians transform to

$$U_a^\dagger H_{s/e} U_a = H_{s/e} |_{e^{i\beta_j} \rightarrow e^{i\beta_j^{\text{slave}}}}. \quad (\text{C6})$$

with  $\beta_j^{\text{slave}}$  the transverse part of  $\beta_j$  and

$$\epsilon_{ij} \partial_i \beta_j^{\text{slave}}(\mathbf{r} - \mathbf{w}) = \pi N_{\mathbf{r}}, \quad \tilde{\nabla} \cdot \tilde{\beta}^{\text{slave}} = 0. \quad (\text{C7})$$

Notice that the longitudinal parts of  $\alpha$  and  $\beta$  have been eliminated, and the remaining transverse pieces commute

with one another and can be treated as  $c$  numbers. Simultaneously, the two U(1) constraints are rendered trivial,

$$\mathcal{G}_f^{\text{slave}} = U_a^\dagger \mathcal{G}_f U_a = e^{-(i/\pi)\sum_{\mathbf{r}} \Lambda_{\mathbf{r}} \epsilon_{ij} \partial_i \alpha_j^{\text{slave}}(\mathbf{r}-\mathbf{w})} = 1, \quad (\text{C8})$$

$$\mathcal{G}_v^{\text{slave}} = U_a^\dagger \mathcal{G}_v U_a = e^{(i/\pi)\sum_{\mathbf{r}} \chi_{\mathbf{r}} \epsilon_{ij} \partial_i \beta_j^{\text{slave}}(\mathbf{r}-\mathbf{w})} = 1, \quad (\text{C9})$$

implying that  $\alpha_j^{\text{slave}} = \beta_j^{\text{slave}} = 0$ , and fully eliminating both gauge fields from the full transformed Hamiltonian

$$U_a^\dagger H U_a = H |_{e^{i\alpha_j}, e^{i\beta_j} \rightarrow e^{i\alpha_j^{\text{slave}}}, e^{i\beta_j^{\text{slave}}}}. \quad (\text{C10})$$

Further transformation with  $U_b$ , which is essentially a nonsingular gauge transformation of both the vortices and spinons, modifies the enslaved gauge fields (so that they vanish on the horizontal bonds and are integer multiples of  $\pi$  on the vertical bonds) while leaving the gauge fluxes invariant. Specifically, we require

$$\partial_x \Theta(\mathbf{r}) = 2\tilde{\beta}_x(\mathbf{r}), \quad \forall \mathbf{r}, \quad (\text{C11})$$

$$\partial_y \Theta(\mathbf{r}) = 2\tilde{\beta}_y(\mathbf{r}), \quad \forall \mathbf{r} \neq x\hat{\mathbf{x}} + \bar{\mathbf{w}}, \quad x \geq 0. \quad (\text{C12})$$

Here  $\tilde{\beta}_j(\mathbf{r})$  is the enslaved gauge field configuration for a vortex located at  $\mathbf{r}=0$  [i.e., determined from Eqs. (C7) with  $N_{\mathbf{r}} = \delta_{\mathbf{r},0}$ ]. The transverseness of  $\tilde{\beta}_j$  implies then

$$\nabla^2 \Theta(\mathbf{r} + \bar{\mathbf{w}}) = \psi_x(\delta_{y,0} - \delta_{y,-1}), \quad (\text{C13})$$

with an unknown function  $\psi_x$  such that  $\psi_x = 0$  for  $x < 0$ . Taking a line sum around the origin requires then

$$\partial_y \Theta(x\hat{\mathbf{x}} + \bar{\mathbf{w}}) = -2\pi + 2\tilde{\beta}_y(x\hat{\mathbf{x}} + \bar{\mathbf{w}}), \quad (\text{C14})$$

for  $x \geq 0$ . Equations (C13) and (C14) are the lattice analog of Laplace’s equation and the condition that  $\Theta$  jumps by  $2\pi$  across the positive  $x$  axis. These conditions determine  $\psi_x$  and hence  $\Theta$ . After some algebra, the solution is expressible as a Fourier integral

$$\Theta(\mathbf{r}) = \int_{\mathbf{k}} \Theta(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (\text{C15})$$

where

$$\Theta(\mathbf{k}) = -\frac{2\pi e^{i\mathbf{k}\cdot\bar{\mathbf{w}}}}{\mathcal{K}^2 F(k_x)} \left[ \frac{\mathcal{K}_y^*}{\mathcal{K}_x^*} - \mathcal{K}_x^* \mathcal{K}_y^* I(k_x) \right], \quad (\text{C16})$$

with

$$F(k_x) = 1 - \frac{|\sin(k_x/2)|}{\sqrt{\sin^2(k_x/2) + 1}}, \quad (\text{C17})$$

$$I(k_x) = \frac{1}{4 \sin(k_x/2) \sqrt{\sin^2(k_x/2) + 1}}. \quad (\text{C18})$$

For large arguments, the asymptotic behavior can be obtained,

$$\Theta(\mathbf{r}) \sim \arctan[y/x], \quad (\text{C19})$$

for  $\sqrt{x^2+y^2} \gg 1$ , with the arctan defined on the interval  $[0, 2\pi]$ . Hence  $\Theta(\mathbf{r})$  gives a proper lattice version of the continuum angle function.

With this definition, one finds

$$H^{\text{slave}} = U_{ab}^\dagger H U_{ab} = H_{e^{i\alpha_j}, e^{i\beta_j} \rightarrow \bar{\sigma}_j^{\text{slave}}, \sigma_j^{\text{slave}}}, \quad (\text{C20})$$

with  $\bar{\sigma}_j^{\text{slave}}(\mathbf{r}; n^f)$  and  $\sigma_j^{\text{slave}}(\mathbf{r}; N)$  as defined in Eqs. (B6) and (B4), respectively. Remarkably, the enslaved U(1) Hamiltonian is *identical* to the enslaved  $Z_2$  Hamiltonian in Eq. (B9);  $H^{\text{slave}} \equiv H_{Z_2}^{\text{slave}}$ .

We have thereby established the formal equivalence between the  $Z_2$  and U(1) formulations of the vortex-spinon field theory—the unitarily transformed enslaved versions of the original Hamiltonians  $\tilde{H}_{Z_2}$  and  $H$  in Eqs. (28) and (43) are identical to one another. Finally, we can verify that the enslaved versions of the electron operators in the U(1) and  $Z_2$  formulations also coincide. From the definition of the electron operator in the U(1) formulation  $c_{\mathbf{r}\sigma}$  in Eq. (55), one can readily show that

$$U_{ab}^\dagger c_{\mathbf{r}\sigma} U_{ab} = c_{\mathbf{r}\sigma} |_{e^{i\beta_j} \rightarrow \sigma_j^{\text{slave}}}. \quad (\text{C21})$$

With the analogous expression for the enslaved  $Z_2$  electron operator in Eq. (B10), and upon comparing the defining expressions of the electron operators in the  $Z_2$  and U(1) formulations in Eqs. (41) and (55), respectively, one thereby establishes the desired formal equivalence  $U_{12}^\dagger \tilde{c}_{\mathbf{r}\sigma} U_{12} \equiv U_{ab}^\dagger c_{\mathbf{r}\sigma} U_{ab}$ .

#### APPENDIX D: THE VORTEX-ELECTRON FORMULATION

In this appendix we briefly discuss a third Hamiltonian formulation of the vortex-fermion field theory. The vortex-electron Hamiltonian will be expressed in terms of “electron operators,” or more correctly operators which create excitations having a nonvanishing overlap with the bare electron. In order to transform to this formulation, we start with the enslaved version of the U(1) vortex-spinon Hamiltonian as obtained in Appendix C:

$$U_a^\dagger H U_a = H_{e^{i\alpha_j}, e^{i\beta_j} \rightarrow e^{i\alpha_j^{\text{slave}}}, e^{i\beta_j^{\text{slave}}}}, \quad (\text{D1})$$

where  $\alpha_j^{\text{slave}}$  and  $\beta_j^{\text{slave}}$  satisfy

$$\epsilon_{ij} \partial_i \alpha_j^{\text{slave}}(\mathbf{r} - \mathbf{w}) = \pi n_{\mathbf{r}}^f, \quad \vec{\nabla} \cdot \vec{\alpha}^{\text{slave}} = 0, \quad (\text{D2})$$

$$\epsilon_{ij} \partial_i \beta_j^{\text{slave}}(\mathbf{r} - \mathbf{w}) = \pi N_{\mathbf{r}}, \quad \vec{\nabla} \cdot \vec{\beta}^{\text{slave}} = 0. \quad (\text{D3})$$

Now consider the unitary transformation

$$U_{el} = e^{(i/2) \sum_{\mathbf{r}, \mathbf{r}'} n_{\mathbf{r}}^f V(\mathbf{r} - \mathbf{r}') \epsilon_{ij} \partial_i e_j(\mathbf{r}' - \mathbf{w})}, \quad (\text{D4})$$

where again  $\nabla^2 V(\mathbf{r} - \mathbf{r}') = \delta_{\mathbf{r}, \mathbf{r}'}$ . As is apparent from Eqs. (39) and (59) this transformation takes one from the spinon operator to the electron operator

$$U_{el}^\dagger f_{\mathbf{r}\sigma} U_{el} = S_\phi(\mathbf{r}) f_{\mathbf{r}\sigma} = c_{\mathbf{r}\sigma}. \quad (\text{D5})$$

Here  $S_\phi(\mathbf{r}) = \prod_{\mathbf{r}'} e^{-i\pi e_j^f}$  is defined in Eq. (39) and the last equality follows from Eq. (59) in the enslaved gauge with

purely transverse gauge field  $\beta^f = 0$ . The electrical charge density in the vortex-spinon formulation transforms to include the *electron* density

$$U_{el}^\dagger \epsilon_{ij} \partial_i a_j(\mathbf{r} - \mathbf{w}) U_{el} = \epsilon_{ij} \partial_i a_j(\mathbf{r} - \mathbf{w}) + \pi c_{\mathbf{r}\sigma}^\dagger c_{\mathbf{r}\sigma}. \quad (\text{D6})$$

The full Hamiltonian density within the vortex-electron formulation is readily obtained from the enslaved vortex-spinon Hamiltonian:  $\mathcal{H}_{ve} = U_{el}^\dagger U_a^\dagger \mathcal{H} U_a U_{el}$ . It can be compactly expressed as

$$\begin{aligned} \mathcal{H}_{ve} = & \frac{u_v}{2} \sum_j e_j^2 + \frac{v_0^2}{2u_v} [\epsilon_{ij} \partial_i a_j + c_{\mathbf{r}\sigma}^\dagger c_{\mathbf{r}\sigma} - \pi \rho_0]^2 + \mathcal{H}_v(a_j) \\ & + \mathcal{H}_r(a_j) + \mathcal{H}_f, \end{aligned} \quad (\text{D7})$$

where  $\mathcal{H}_v$  and  $\mathcal{H}_r$  denote the vortex and roton hopping terms, respectively, and are given explicitly as

$$\mathcal{H}_v(a_j) = -t_v \sum_j \cos(a_j), \quad (\text{D8})$$

$$\mathcal{H}_r(a_j) = -\frac{\kappa_r}{2} \sum_i \cos(\epsilon_{ij} \partial_i a_j). \quad (\text{D9})$$

The fermionic Hamiltonian is

$$\begin{aligned} \mathcal{H}_f = & - \sum_j t_e c_{\mathbf{r}+\hat{x}_j, \sigma}^\dagger c_{\mathbf{r}\sigma} - \sum_j e^{i\pi \bar{e}_j(\mathbf{r})} [t_s c_{\mathbf{r}+\hat{x}_j, \sigma}^\dagger c_{\mathbf{r}\sigma} \\ & + \Delta_j B_{\mathbf{r}}^\dagger c_{\mathbf{r}+\hat{x}_j, \sigma} \epsilon_{\sigma\sigma'} c_{\mathbf{r}\sigma'} + \text{H.c.}]. \end{aligned} \quad (\text{D10})$$

The electric field appearing in the spinon hopping term yields the same physical effects as the gauge field in the U(1) formulation. Specifically, when the electron hops from one site to a neighboring site, the factor  $e^{i\pi \bar{e}}$  which shifts the gauge-field  $a_j$  by  $\pi$ , effectively hops a compensating charge-on in the opposite direction. In addition, this minimal coupling form encodes the requisite minus sign when a spinon is hopped around a vortex and vice versa. The full Hamiltonian must be supplemented with the constraint that  $\vec{\nabla} \cdot \vec{e} = N$ , with integer  $N$ . We emphasize that the total spin

$$\vec{S} = \frac{1}{2} \sum_{\mathbf{r}} c_{\mathbf{r}\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} c_{\mathbf{r}\sigma'} \quad (\text{D11})$$

and electric charge

$$Q = \sum_{\mathbf{r}} \left[ \frac{1}{\pi} \epsilon_{ij} \partial_i a_j(\mathbf{r} - \mathbf{w}) + c_{\mathbf{r}\sigma}^\dagger c_{\mathbf{r}\sigma} \right] \quad (\text{D12})$$

are conserved, commuting with  $\mathcal{H}_{ve}$ .

It is of course also possible to pass to a Euclidean path integral representation of the partition function associated with the above vortex-electron Hamiltonian. Specifically, the corresponding Euclidean Lagrangian can be readily expressed as

$$\mathcal{L}_{ve} = ie_j \partial_0 a_j + c_{\mathbf{r}}^\dagger \partial_0 c_{\mathbf{r}} + ia_0 (\vec{\nabla} \cdot \vec{e} - N_r) + \mathcal{H}_{ve}, \quad (\text{D13})$$

where the time component of the gauge field  $a_0(\mathbf{r})$  lives on the sites of the dual lattice. In the partition function, the vortex number  $N_r$  is a continuous field running over the real

numbers, but the integration only contributes when  $N$  is an integer. To see why, it is instructive to let  $a_\mu \rightarrow a_\mu - \partial_\mu \theta$ , and to integrate the vortex phase variable  $\theta$  over the reals. Since the Hamiltonian is  $2\pi$  periodic in  $\theta$ , upon splitting the integration as  $\partial_0 \theta = 2\pi l + \partial_0 \tilde{\theta}$  with  $\tilde{\theta} = [0, 2\pi]$ , the summation of  $\exp(i2\pi l N)$  over integer  $l$  vanishes unless the vortex number  $N$  is an integer.

To obtain a more tractable representation of the Lagrangian, we introduce a Hubbard–Stratonovich field  $e_0(\mathbf{r})$  to decouple the Coulomb interaction term above. Here  $e_0$  has the physical meaning of a dynamical electrostatic potential. In this way the full Euclidean Lagrangian can be conveniently decomposed as the sum of a bosonic charge sector and a fermionic spin and charge carrying sector  $\mathcal{L}_{ve} = \mathcal{L}_c + \mathcal{L}_f$ . The full bosonic sector is given by

$$\mathcal{L}_c = \frac{u_v}{2} \left[ e_j^2 + \frac{1}{\pi^2 v_0^2} e_0^2 \right] + ie_j (\partial_0 a_j - \partial_j a_0) - \frac{i}{\pi} e_0 \epsilon_{ij} \partial_i a_j + i(\partial_0 \theta - a_0) N + \mathcal{L}_v + \mathcal{L}_r, \quad (\text{D14})$$

with vortex and roton hopping terms

$$\mathcal{L}_v = -t_v \sum_j \cos(\partial_j \theta - a_j), \quad (\text{D15})$$

$$\mathcal{L}_r = -\frac{\kappa_r}{2} [\cos(\Delta_{xy} \theta - \partial_x a_y) + (x \leftrightarrow y)]. \quad (\text{D16})$$

The Lagrangian density in the fermionic sector is

$$\mathcal{L}_f = c_r^\dagger (\partial_0 - ie_0) c_r + \mathcal{H}_f + ie_0 \rho_0. \quad (\text{D17})$$

In addition to the global symmetries corresponding to spin and charge conservation, the full Lagrangian has a local gauge symmetry, being invariant under

$$\begin{aligned} \theta_r &\rightarrow \theta_r + \Theta_r, \\ a_\mu(\mathbf{r}) &\rightarrow a_\mu(\mathbf{r}) + \partial_\mu \Theta_r, \end{aligned} \quad (\text{D18})$$

with  $\Theta_r$  an arbitrary function of space and imaginary time. Because of this gauge invariance we are free to choose an appropriate gauge.

## APPENDIX E: POLARIZATION TENSOR

In this appendix, we calculate the corrections to the polarization tensor  $\Pi_{ij}$  at  $O(t_v^2)$  including fluctuations of both  $\tilde{a}$  and  $\theta$ , for the general case  $v_0 < \infty$ . Integrating out all dynamical fields to  $O(t_v^2)$ , one finds that the effective action as a functional of  $A_j$  takes the form  $S_A^{\text{eff}} = S_A^0 + S_A^{(2)}$ , where

$$S_A^{(2)} = -\frac{t_v^2}{2} e^{S_A^0} \sum_{\mathbf{r}, \mathbf{r}'} \int_{\tau\tau'} \langle C_i(\mathbf{r}, \tau) C_j(\mathbf{r}', \tau') e^{-S_A} \rangle_{A=0}, \quad (\text{E1})$$

with the shorthand notation  $C_i(\mathbf{r}, \tau) = \cos(\partial_i \theta - \tilde{a}_i)_{\mathbf{r}\tau}$  and with the  $\langle \cdots \rangle_{A=0}$  indicating a Gaussian average with respect to the RL action. This can be written as

$$S_A^{(2)} = -\frac{t_v^2}{4} e^{S_A^0} (e^{\langle \Gamma_x^2 \rangle_{A=0}} + e^{\langle \Gamma_y^2 \rangle_{A=0}}), \quad (\text{E2})$$

with

$$\begin{aligned} \Gamma_x = \int_{\mathbf{k}, \omega_n} \left[ \left( \psi_{\mathbf{k}\omega_n} \frac{\mathcal{K}_x^*}{\mathcal{K}} + \frac{i\omega_n \mathcal{K}_j^* A_j}{\pi \mathcal{K}} \right) a(\mathbf{k}, \omega_n) \right. \\ \left. - \psi_{\mathbf{k}\omega_n} \mathcal{K}_y \theta(\mathbf{k}, \omega_n) \right], \end{aligned} \quad (\text{E3})$$

$\psi_{\mathbf{k}\omega_n} = e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_n\tau)} - e^{i(\mathbf{k}\cdot\mathbf{r}' - \omega_n\tau')}$ , and  $\Gamma_y$  obtained from  $\Gamma_x$  by  $x \leftrightarrow y$ . Evaluating the expectation value gives

$$\begin{aligned} S_A^{(2)} \sim -t_v^2 \sum_{xx', yy'} \int_{\tau\tau'} \frac{1}{[x^2 + v_{\text{rot}}^2 (\tau - \tau')^2]^{\Delta_v}} \exp \int_{\mathbf{k}\omega_n} \frac{i\omega_n u_v}{\pi \mathcal{K}^2} \\ \times \psi_{-\mathbf{k}, -\omega_n} (\tilde{\mathcal{K}}_y G_{12} + \tilde{\mathcal{K}}_x G_{11}) \tilde{\mathcal{K}}_j A_j(-\mathbf{k}, -\omega_n) + x \leftrightarrow y. \end{aligned} \quad (\text{E4})$$

Since we are interested in the polarization tensor, we may expand the exponential in Eq. (E4) to  $O(A^2)$  to obtain  $\Pi_{ij} = \Pi_{ij}^0 + \Pi_{ij}^{(2)}$ , with

$$\begin{aligned} \Pi_{ij}^{(2)} \sim -\frac{t_v^2 \omega_n^2 u_v^2}{2\mathcal{K}^4} (\omega_n^2 + v_{\text{rot}}^2 k_x^2)^{\Delta_v - 1} |\tilde{\mathcal{K}}_y G_{12} + \tilde{\mathcal{K}}_x G_{11}|^2 \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}_j \\ + (x \leftrightarrow y). \end{aligned} \quad (\text{E5})$$

In the limit of interest for the conductivity  $|\mathbf{k}| \rightarrow 0$  at fixed frequency  $G_{11} \gg G_{12}$ , and we obtain Eq. (153) of the main text.

<sup>1</sup>P. W. Anderson, *Science* **235**, 1196 (1987).

<sup>2</sup>S. Kivelson, D. S. Rokhsar, and J. Sethna, *Phys. Rev. B* **35**, 8865 (1987).

<sup>3</sup>S. Sachdev, *Ann. Phys. (N.Y.)* **303**, 226 (2003).

<sup>4</sup>L. Balents, M. P. A. Fisher, and C. Nayak, *Int. J. Mod. Phys. B* **12**, 1033 (1998); *Phys. Rev. B* **60**, 1654 (1999); **61**, 6307 (2000).

<sup>5</sup>T. Senthil and M. P. A. Fisher, *Phys. Rev. B* **62**, 7850 (2000).

<sup>6</sup>For early papers on the topological vortex in spin-charge separated states see, N. Read and B. Chakraborty, *Phys. Rev. B* **40**,

7133 (1989); S. Kivelson, *ibid.* **39**, 259 (1989).

<sup>7</sup>X. G. Wen, *Phys. Rev. B* **44**, 2664 (1991).

<sup>8</sup>T. Senthil and M. P. A. Fisher, *Phys. Rev. Lett.* **86**, 292 (2001).

<sup>9</sup>T. Senthil and M. P. A. Fisher, *Phys. Rev. B* **63**, 134521 (2001); **64**, 214511 (2001).

<sup>10</sup>J. C. Wynn, D. A. Bonn, B. W. Gardner, Y. J. Lin, R. X. Liang, W. N. Hardy, J. R. Kirtley, and K. A. Moler, *Phys. Rev. Lett.* **87**, 197002 (2001).

<sup>11</sup>D. A. Bonn, J. C. Wynn, B. W. Gardner, Y. J. Lin, R. Liang, W. N. Hardy, J. R. Kirtley, and K. A. Moler, *Nature (London)* **414**,

- 887 (2001).
- <sup>12</sup>A. Paramekanti, L. Balents, and M. P. A. Fisher, *Phys. Rev. B* **66**, 054526 (2002).
- <sup>13</sup>See P. W. Anderson, *The Theory of Superconductivity in the High- $T_c$  Cuprates* (Princeton University Press, Princeton, 1997), and references therein.
- <sup>14</sup>See, e.g., M. Franz, Z. Tesanovic, and O. Vafek, *Phys. Rev. B* **66**, 054535 (2002); I. F. Herbut, *Phys. Rev. B* **66**, 094504 (2002).
- <sup>15</sup>For a discussion of the dual vortex-field theory for 2D bosons see, M. P. A. Fisher and D. H. Lee, *Phys. Rev. B* **39**, 2756 (1989); X. G. Wen and A. Zee, *Int. J. Mod. Phys. B* **4**, 37 (1990).
- <sup>16</sup>J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, *Phys. Rev. B* **16**, 1217 (1977).
- <sup>17</sup>L. Balents and M. P. A. Fisher (unpublished).
- <sup>18</sup>L. B. Ioffe and A. J. Millis, *Phys. Rev. B* **58**, 11 631 (1998).
- <sup>19</sup>For various approaches to bosonization see, for example, V. Emery, in *Highly Conducting One-dimensional Solids*, edited by J. Devreese, R. Evrard, and V. Van Doren (Plenum, New York, 1979), p. 247; A. W. W. Ludwig, *Int. J. Mod. Phys. B* **8**, 347 (1994); R. Shankar, *Acta Phys. Pol. B* **26**, 1835 (1995); M. P. A. Fisher and L. I. Glazman, in *Mesoscopic Electron Transport*, edited by L. L. Sohn, L. P. Kouwenhoven, and G. Schon, Vol. 345 of NATO Advanced Studies Institute, Series E: Applied Sciences (Kluwer Academic, Dordrecht, 1997), p. 331.
- <sup>20</sup>See, *Perspectives in Quantum Hall Effects*, edited by S. Das Sarma and A. Pinczuk (Wiley, New York, 1997).
- <sup>21</sup>R. B. Laughlin, *Phys. Rev. Lett.* **60**, 2677 (1988); **61**, 379 (1988).
- <sup>22</sup>G. Baskaran, Z. Zou, and P. W. Anderson, *Solid State Commun.* **63**, 873 (1987); G. Baskaran and P. W. Anderson, *Phys. Rev. B* **37**, 580 (1988).
- <sup>23</sup>I. Affleck and J. B. Marston, *Phys. Rev. B* **37**, 3774 (1988); L. Ioffe and A. Larkin, *ibid.* **39**, 8988 (1989); P. A. Lee, N. Nagaosa, T.-K. Ng, and X. G. Wen, *ibid.* **57**, 6003 (1998).
- <sup>24</sup>D. J. Scalapino, E. Loh, Jr., and J. E. Hirsch, *Phys. Rev. B* **34**, 8190 (1986).
- <sup>25</sup>D. H. Kim, D. K. K. Lee, and P. A. Lee, *Phys. Rev. Lett.* **76**, 4801 (1996); *Phys. Rev. B* **55**, 591 (1997).
- <sup>26</sup>A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic Publishers, London, 1987).
- <sup>27</sup>N. Read and S. Sachdev, *Phys. Rev. Lett.* **66**, 1773 (1991); S. Sachdev and N. Read, *Int. J. Mod. Phys. B* **5**, 219 (1991).
- <sup>28</sup>R. Moessner and S. L. Sondhi, *Phys. Rev. Lett.* **86**, 1881 (2001); C. Nayak and K. Shtengel, *Phys. Rev. B* **64**, 064422 (2001); L. Balents, M. P. A. Fisher, and S. M. Girvin, *ibid.* **65**, 224412 (2002).
- <sup>29</sup>O. I. Motrunich and T. Senthil, *Phys. Rev. Lett.* **89**, 277004 (2002); T. Senthil and O. I. Motrunich, *Phys. Rev. B* **66**, 205104 (2002).
- <sup>30</sup>S. Chakravarty, R. B. Laughlin, D. K. Morr, and C. Nayak, *Phys. Rev. B* **63**, 094503 (2001); S. Tewari, H.-Y. Kee, C. Nayak, and S. Chakravarty, *ibid.* **64**, 224516 (2001).
- <sup>31</sup>See F. C. Zhang, C. Gros, T. M. Rice, and H. Shiba, *Science Technol. (USA)* **1**, 36 (1988).
- <sup>32</sup>D. A. Ivanov, P. A. Lee, and X.-G. Wen, *Phys. Rev. Lett.* **84**, 3958 (2000).
- <sup>33</sup>R. Shankar, *Rev. Mod. Phys.* **66**, 129 (1994).
- <sup>34</sup>See, A. Damascelli, Z.-X. Shen, and Z. Hussain, *Rev. Mod. Phys.* **75**, 473 (2003), and references therein.
- <sup>35</sup>S. Martin, A. T. Fiory, R. M. Fleming, L. F. Schneemeyer, and J. V. Waszczak, *Phys. Rev. B* **41**, 846 (1990); S. J. Hagen, T. W. Jing, Z. Z. Wang, J. Horvath, and N. P. Ong, *ibid.* **37**, 7928 (1988).
- <sup>36</sup>J. M. Harris, Y. F. Yan, and N. P. Ong, *Phys. Rev. B* **46**, 14 293 (1992).
- <sup>37</sup>J. M. Harris, H. Wu, N. P. Ong, R. L. Meng, and C. W. Chu, *Phys. Rev. B* **50**, 3246 (1994).
- <sup>38</sup>See, R. W. Hill, C. Proust, L. Taillefer, P. Fournier, and R. L. Greene, *Nature (London)* **414**, 711 (2001).