Resonantly enhanced quantum decay: A time-dependent Wentzel-Kramers-Brillouin approach

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The WKB approach to quantum tunneling from a metastable well is generalized to include a weak oscillatory force. Both one-dimensional decay and multidimensional decay, appropriate for a system coupled to a dissipative environment, are considered. In both cases, general expressions are derived for the enhancement of the decay rate linear in the applied force. The enhancement is expressed exclusively in terms of the instanton trajectories of the unperturbed system. For a cubic potential with no damping, the decay rate exhibits sharp resonances when the applied frequency coincides with the energy spacing between the ground and excited states in the metastable well. In the strong-damping limit, this resonance structure is completely suppressed.

I. INTRODUCTION

The decay of a system from a long-lived metastable state arises frequently in physics. An appropriate model in many circumstances consists of a single coordinate moving in a metastable potential, as depicted in Fig. 1. At high temperatures thermal activation over the barrier is the principal decay mode. Upon cooling, however, quantum tunneling through the classically forbidden region eventually dominates. Provided the coordinate is essentially decoupled from its environment, a standard Wentzel-Kramers-Brillouin (WKB) approach[1] can then be used to calculate the decay rate.

Over the past several years there has been considerable interest in understanding the quantum tunneling of a *macroscopic* coordinate,[2–6] motivated in particular by experiments on Josephson junction systems.[7] An important feature in macroscopic quantum tunneling is the coupling of the coordinate to the microscopic or environmental degrees of freedom,[2] which may strongly modify the decay rate. In a recent experiment,[8] quantum tunneling of the phase in a current-biased Josephson junction was studied in the presence of a weak microwave perturbation. The decay rate out of the zero-voltage state was found to be resonantly enhanced when the microwave frequency matched the energy-level spacing between two states in the metastable well. A

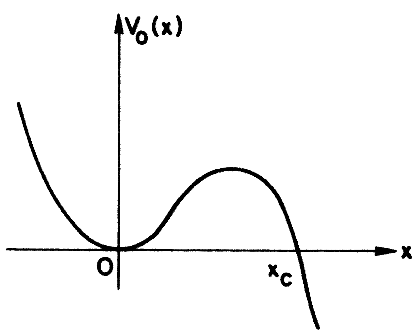

FIG. 1. Metastable potential $V_0(x)$.

theoretical understanding[9–12] of this experiment requires incorporating both dissipation and the ac perturbation into a quantum tunneling calculation. Reasonable agreement with the existing data has recently been obtained[13] by employing a master equation for the discrete states in the well, with dissipation, introduced perturbatively, mediating transitions between the states.

Motivated by this experiment, the purpose of this work is to reformulate the standard WKB approach to quantum decay in order to include a weak time-dependent perturbation. In contrast to the usual WKB treatment, the presence of an ac force necessitates studying the time-dependent Schrödinger equation. Progress can be made,[14] nevertheless, by solving the Schrödinger equation via a *double* expansion in powers of $\hbar$ and the strength of the ac force, f. In this way the contribution to the decay exponent linear in f can be calculated explicitly.

In Sec. II a time-dependent WKB calculation is developed for the decay of a single coordinate in the absence of a dissipative environment. A general expression for the resonantly modified decay exponent (linear in f) is derived for arbitrary metastable potential wells. For the cubic well, the decay rate is found to exhibit resonances at frequencies corresponding to integer multiples of ω_0, the small oscillation frequency in the well. These resonances can be interpreted as arising from a single photon absorption which excites the particle from the ground to an excited state in the well.

The additional effects of a dissipative environment are also considered. As usual the coordinate is coupled to a bath of harmonic-oscillator degrees of freedom.[2] As shown originally by Caldeira and Leggett,[2] provided no ac force is acting, the decay rate in the presence of a dissipative environment can be extracted by using instanton paths to evaluate a functional-integral representation of the partition function. More recently, Schmid[15] has demonstrated that the decay rate can alternatively be derived by employing a multidimensional WKB approach. The advantage of this latter treatment is that it can be generalized to incorporate a time-dependent term in the Hamiltonian.

In Sec. III the multidimensional WKB method is modified to include a weak ac force. A general expression for the leading enhancement to the decay rate is obtained. The final result, Eqs. (3.16)–(3.18), is expressed exclusively in terms of the instanton (or bounce) trajectory which enters into the Caldeira-Leggett theory.[2] Using previous results[2,5,16,17] for the instanton trajectories, the resonant enhancement is evaluated explicitly in several special cases in Sec. III B. For the cubic well in the strong-damping limit, the resonant structure, present at zero damping, is found to be completely suppressed.

II. UNDAMPED RESONANT ENHANCEMENT

A. General formulation

For a single quantum degree of freedom, the decay rate from a metastable well can be computed in the semiclassical limit using the WKB approximation. In this section I study the modification of the decay exponent due to the presence of a weak oscillatory force. The Hamiltonian of interest takes the form

$$H=\frac{1}{2m}p^2+V_0(x)+V_1(x,t)\ , \tag{2.1}$$

where $V_1(x,t)$ is a small oscillatory force given by

$$V_1(x,t)=fx\cos(\omega t)\ , \tag{2.2}$$

and $V_0(x)$ is a static metastable well as shown in Fig. 1. The potential $V_0(x)$ will be taken to vary quadratically about its metastable minimum at $x=0$, $V_0(x=0)=0$, and possess a unique "exit" point x_c such that $V_0(x)$ is negative for $x>x_c$. The canonical example is the simple-cubic potential, $V_0(x)=\frac{1}{2}m\omega_0^2x^2(1-x/x_c)$. In the absence of $V_1(t)$ the decay rate may be extracted[15] from quasistationary states ψ of the time-independent Schrödinger equation, $H\psi=E\psi$. In particular, the imaginary part of the energy E, for a state with an outgoing probability current, can be interpreted as the decay rate. When a time-dependent force is acting, however, it appears necessary to study rather the time-dependent Schrödinger equation, $i\hbar\partial_t\psi=H\psi$. In the quasiclassical limit it is convenient to recast the Schrödinger equation by introducing the ansatz

$$\psi(x,t)=\exp[-W(x,t)/\hbar]\ , \tag{2.3}$$

giving the time-dependent Hamilton-Jacobi equation

$$-i\partial_t W=-\frac{1}{2m}(\partial_x W)^2+V_0+V_1+\frac{\hbar}{2m}(\partial_x^2 W)\ . \tag{2.4}$$

Not surprisingly, for a harmonic-oscillator potential where no tunneling is present, (2.4) can be solved exactly (see below). For a nonlinear metastable well, the idea is to find the solution of (2.4) in the classical inaccessible region, $0<x<x_c$, which matches onto the harmonic-oscillator solution at $x=0$. Up to a normalization factor, such a procedure gives the probability of finding the particle at the exit point. When $V_1=0$ and to leading order in $\hbar\to 0$, this probability is in fact simply proportional to the decay rate, $\Gamma\sim|\psi(x_c)|^2$. When V_1 is present, however, $|\psi(x_c,t)|^2$ depends on time oscillating with the external frequency ω. The central assumption made in this paper is that, in this case, the decay rate follows by simply averaging $|\psi(x_c,t)|^2$ over one cycle of motion,

$$\Gamma\equiv\langle\Gamma(t)\rangle\ , \tag{2.5a}$$

$$\Gamma(t)\sim|\psi(x_c,t)|^2=\exp\left[-\frac{2}{\hbar}\mathrm{Re}[W(x_c,t)]\right]\ . \tag{2.5b}$$

In (2.5a) the angular brackets denote the time average over one cycle.

Under the assumption (2.5) an evaluation of Γ has been reduced to solving the Hamilton-Jacobi equation in the classical inaccessible region. The boundary condition at $x=0$ is specified by requiring that $W(x,t)$ match onto a solution of (2.4) for a harmonic-oscillator potential, namely,

$$\mathrm{Re}W=\frac{m\omega_0}{2}\left[x+\frac{f}{m}\frac{\cos\omega t}{(\omega_0^2-\omega^2)}\right]^2\ , \tag{2.6a}$$

$$\mathrm{Im}W=\frac{\hbar\omega_0 t}{2}-f\frac{\omega\sin(\omega t)x}{(\omega_0^2-\omega^2)}-\frac{f^2}{4m(\omega_0^2-\omega^2)}\left[t+\frac{(\omega_0^2+\omega^2)\sin(2\omega t)}{(\omega_0^2-\omega^2)2\omega}\right]\ . \tag{2.6b}$$

Notice that this solution corresponds to a wave packet following the motion of a classical forced harmonic oscillator and with a dispersion equal to that of the *ground state* of an unforced *quantum* oscillator.

A solution of (2.4) for a general metastable well can be obtained by expanding simultaneously in $\hbar$ and f. Specifically, inserting the expansion

$$W(x,t)=W_0(x)+fW_1(x,t)+O(f^2,\hbar) \tag{2.7}$$

in (2.4) and comparing powers of $\hbar$ and f gives the zero-order eikonal equation

$$\frac{1}{2m}(\partial_x W_0)^2=V_0(x)\ , \tag{2.8}$$

and an equation first order in f,

$$-i\partial_t W_1=-\frac{1}{m}(\partial_x W_0)(\partial_x W_1)+x\cos(\omega t)\ . \tag{2.9}$$

Notice that W_0 has been assumed to depend only on x, consistent with the harmonic-oscillator solution (2.6) which is time independent at zeroth order in $\hbar,f$. The eikonal equation is standard from WKB analysis and can be solved by straightforward integration. The boundary condition on $W_0(x)$ at $x=0$ follows by matching onto (2.6), $W_0(x)=\frac{1}{2}m\omega_0^2x^2$, giving

$$W_0(0)=0,\quad W_0(x)\geq 0\ . \tag{2.10}$$

Upon integration $W_0(x)$ is then given at the exit point by the familiar form

$$W_0(x_c)=\int_0^{x_c}dx[2mV_0(x)]^{1/2}\ . \tag{2.11}$$

The decay rate for the unforced problem (to leading order in $\hbar$) follows from (2.5) as $\Gamma_0 \sim \exp[-2W_0(x_c)/\hbar]$.

The resonant enhancement to the decay rate is obtained by solving (2.9) since

$$\Gamma(t)/\Gamma_0 \sim \exp\left[-\frac{2f}{\hbar}\mathrm{Re}[W_1(x_c,t)]\right] . \tag{2.12}$$

The function $W_1(x,t)$ is required to match onto the harmonic-oscillator solution (2.6) which takes the form

$$W_1(x,t) = \tfrac{1}{2}y(x)e^{-i\omega t} + (\omega \rightarrow -\omega) , \tag{2.13}$$

with $y_h(x) = x(\omega_0 - \omega)^{-1}$. This matching can be achieved most simply by inserting the general form (2.13) into (2.9) to obtain a differential equation for the unknown function $y(x)$,

$$\left[\frac{2}{m}V_0(x)\right]^{1/2} y'(x) - \omega y(x) = x . \tag{2.14}$$

Then $\Gamma(t)$ follows by integrating (2.14) through the classically inaccessible region, $0 < x < x_c$, with the boundary condition

$$y(x=0)=0 , \tag{2.15}$$

as in the harmonic-oscillator case, *and* the requirement that $y(x)$ be analytic at the origin. This additional requirement of analyticity is physically reasonable since $y(x)$ is directly related to the wave function [via (2.13)], which should be analytic. In practice it serves to eliminate the *unphysical* homogeneous solution of (2.14) which varies as x^{ω/ω_0} for $x \rightarrow 0$.

The general expressions (2.12)–(2.15) constitute the central results of this section. The contribution to the decay exponent, linear in the ac force, is completely specified by these expressions, for arbitrary metastable potentials. In Sec. II B $\Gamma(t)/\Gamma_0$ is evaluated explicitly for a specific class of potentials.

B. Results for specific potential wells

Below we obtain the resonantly enhanced decay rate for a class of metastable potentials of the form

$$V_0(x) = \tfrac{1}{2}m\omega_0^2 x^2[1-(x/x_c)^k], \quad x \geq 0 \tag{2.16}$$

for arbitrary (positive) integer k. This form interpolates between the cubic potential, $k=1$, and a truncated harmonic-oscillator potential, $k \rightarrow \infty$. The calculation consists of integrating (2.14) from the origin out to x_c, with the boundary condition (2.15) and the requirement of analyticity imposed at the origin. The most general solution can be expressed as a sum of a homogeneous and a particular piece. Since the homogeneous solution is, in general, nonanalytic at the origin, varying as x^{ω/ω_0} for $x \rightarrow 0$, it must be omitted from $y(x)$. A particular solution can be obtained in closed form as

$$y_p(x) = \int_c^x dx'\, x' g(x') \exp\left[\int_{x'}^x dx''\, \omega g(x'')\right] \tag{2.17}$$

with $g(x) = [m/2V_0(x)]^{1/2}$ and c an arbitrary constant. A simple analysis reveals that in the limit $x,c \rightarrow 0$, $y_p(x)$ consists of a sum of two terms, one linear in x and of the form $x(\omega_0-\omega)^{-1}$, and the other varying as $c(x/c)^{\omega/\omega_0}$. Thus, *provided* $\omega < \omega_0$, setting $c=0$ eliminates the nonanalytic piece, giving an appropriate solution for $y(x)$ satisfying the boundary condition (2.15).

For $\omega > \omega_0$, an analytic solution can be constructed by introducing $y(x) = x(\omega_0-\omega)^{-1} + y_1(x)$ into (2.14) and integrating the resulting differential equation for y_1. For the cubic potential, a particular solution for y_1 of the form (2.17) will also contribute a nonanalytic piece which, however, vanishes in the limit $c \rightarrow 0$ *provided* $\omega < 2\omega_0$. The appropriate solution for $\omega > 2\omega_0$ can, in turn, be constructed by iterating this procedure obtaining, upon the nth iteration, the solution for ω in the range $n < (\omega/\omega_0) < n+1$. Below I adopt a simpler approach which relies on an analytic continuation in the complex ω plane to avoid the singularities at the resonance frequencies, $\omega = n\omega_0$.

As discussed above, provided $\omega < \omega_0$, an expression for $y(x)$ at the exit point can be obtained from (2.17) with $c=0$ and $x=x_c$. Specializing to the cubic potential one finds

$$y(x_c;\omega) = x_c \int_0^1 \frac{dx}{(1-x)^{1/2}} \left[\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right]^{\omega/\omega_0} , \tag{2.18}$$

which is indeed convergent for $\omega < \omega_0$. The sum $y(x_c;\omega) + y(x_c;-\omega)$, which is required to evaluate the resonant enhancement (2.12), can then be obtained for $|\omega| < \omega_0$ by direct integration on (2.18), giving

$$y(x_c;\omega) + y(x_c;-\omega) = \frac{4\pi x_c(\omega/\omega_0)}{\sin(\pi\omega/\omega_0)} . \tag{2.19}$$

While only derived for $|\omega| < \omega_0$, this closed-form expression can be correctly interpreted as the appropriate analytic continuation over the entire complex ω plane. Analytic continuation has enabled us to "step around" the poles at $\omega = n\omega_0$, evident in (2.19), without having to solve separately the differential equation in *each* of the intervals, $n < (\omega/\omega_0) < n+1$, along the real axis.

The final expression for the resonantly modified decay exponent out of the cubic metastable well follows upon combining (2.19) with (2.12) and (2.13), namely,

$$-\ln[\Gamma(t)/\Gamma_0] \sim \frac{4fx_c}{\hbar\omega_0}\left[\frac{\pi\omega/\omega_0}{\sin(\pi\omega/\omega_0)}\right]\cos(\omega t) . \tag{2.20}$$

Notice that the decay rate, obtained upon averaging $\Gamma(t)$ over one cycle as in (2.5a), is resonantly enhanced at frequencies $\omega = n\omega_0$ corresponding to integer multiples of the small oscillation frequency in the quadratic part of the well. These resonances can be interpreted as arising from a single-photon absorption which excites the particle from the ground to the nth excited state in the metastable well. Since the excited states have less far to tunnel, the decay rate is in turn enhanced. Multiphoton processes, absent above, could in principle be incorporated by working to successively higher order in the external force f. In general, at mth order one would expect subharmonic resonances at frequencies $\omega = (n/m)\omega_0$.

Apparently the semiclassical approximation used above does not account fully for the anharmonicities in the metastable well which will shift the excitation energies from the harmonic-oscillator results $n\hbar\omega_0$. This fact can be understood by noting that in the WKB limit, $\hbar\omega_0 \ll m\omega_0^2 x_c^2$, the (relative) anharmonicities in (2.16), which are of order x/x_c, are vanishingly small, $O(\hbar)$, when x is comparable to the oscillator dispersion $(\hbar/m\omega_0)^{1/2}$. One therefore expects the (relative) energy shifts due to the anharmonicities to be down by order $\hbar$, and consequently absent from the *leading* $\hbar$ analysis above [see (2.7)].

It should be emphasized that the result (2.20) is applicable only for frequencies far enough off-resonance that the dimensionless exponent linear in f is small compared to the zeroth-order tunneling exponent, $2W_0(x_c)/\hbar$ given in (2.11). For the resonance at $\omega=\omega_0$, this restriction is equivalent to the condition that the harmonic-oscillator wave packet described by (2.6), oscillate with an amplitude much smaller than the exit point x_c.

For $(fx_c/\hbar\omega_0) \gg 1$ the decay rate Γ, obtained by averaging $\Gamma(t)$ over one cycle, is dominated by times when $\Gamma(t)$ is maximum, so that $\ln(\Gamma/\Gamma_0) \sim \omega$ away from the resonance. In the opposite limit (2.20) indicates that $\ln(\Gamma/\Gamma_0) \sim \omega^2$. This should be contrasted with Ref. 9 where an enhancement exponential in ω for large frequencies was obtained. The low-frequency limit, $\omega \ll \omega_0$, provides a convenient check on the above calculations. In this limit, $\Gamma(t)$ reduces to the WKB decay rate from the *static* metastable well $\tilde{V}(x)=V_0(x)+V_1(x,t)$ (fixed t), as expected.

The calculation for the cubic potential can be easily generalized to potentials of the form (2.16). One finds for arbitrary positive integer k

$$-\ln[\Gamma(t)/\Gamma_0] \sim \frac{fx_c}{\hbar\omega_0} A_k(\omega/\omega_0)\cos(\omega t) , \tag{2.21a}$$

$$A_k(x)=\frac{4^{1/k}}{k\gamma(2/k)}\gamma\left[\frac{1-x}{k}\right]\gamma\left[\frac{1+x}{k}\right] , \tag{2.21b}$$

with γ the standard γ function. Note that, in contrast to the cubic case ($k=1$), the resonances occur at frequencies $\omega/\omega_0=1+nk$, for positive integers n. In general, the resonance structure is determined by the power k of the leading anharmonicity in the potential $V_0(x)$, see (2.16). In particular, *any* metastable potential with a *leading* anharmonicity varying as x^3 will exhibit resonances at $n\omega_0$ for *all* positive integers n. It is amusing to see how the result for general k reduces to the truncated harmonic-oscillator result ($k=\infty$). For $k\to\infty$ all of the resonances, save one, are pushed off to infinite frequencies, leaving a *single* resonance at ω_0 as expected. At large frequencies the resonance strengths vary as $\omega^{(2-k)/k}$.

In the presence of dissipation one expects the (divergent) resonances found above to be rounded. Indeed, as will be shown in Sec. III, with sufficient damping the resonance structure can in fact be washed out entirely. Before considering the effects of dissipation, it is convenient to digress briefly to recast the above approach in terms of instanton or bounce trajectories.

C. Reformulation in terms of bounce trajectories

Rather than solve the eikonal equation (2.8) by direct integration, as in (2.11), it is convenient to introduce the method of characteristics,[15] which can be more easily generalized to a multidimensional decay problem (Sec. III) appropriate for a system with dissipation. If the eikonal equation is recast in the form $H_0=0$, where H_0 is a fictitious Hamiltonian given by

$$H_0=\frac{1}{2m}p^2-V_0(x), \quad p=\partial_x W_0 \tag{2.22}$$

it is natural to seek a solution by introducing fictitious trajectories $\{x(\tau),p(\tau)\}$, moving in imaginary time τ, which satisfy Hamilton's equations

$$\dot{x}(\tau)=\frac{\partial H_0}{\partial p}, \quad \dot{p}=\frac{-\partial H_0}{\partial x} . \tag{2.23}$$

Solving the eikonal equation reduces, then, to solving for $x(\tau)$ satisfying

$$\tfrac{1}{2}m\dot{x}^2(\tau)-V_0(x)=0 , \tag{2.24}$$

an equation of motion for a "particle" moving in the inverted potential. The trajectory satisfying (2.24) which starts at $x=0$ and $\tau=-\infty$, reaches x_c at $\tau=0$ and returns to the origin at $\tau=+\infty$, is referred to as a bounce (or instanton) trajectory. Since $\partial_x W_0=m\dot{x}$ one can relate the tunneling exponent (for $f=0$) to the action along the bounce trajectory, namely

$$2W_0(x_c)/\hbar=\frac{1}{\hbar}\int_{-\infty}^{\infty} d\tau\, m\dot{x}^2 . \tag{2.25}$$

The resonant enhancement follows by solving the differential equation (2.14), which can also be recast as a differential equation in imaginary time involving the bounce trajectory. Specifically, replacing $(2V_0/m)^{1/2}$ with $\dot{x}(\tau)$ and applying the chain rule enables (2.14) to be reexpressed as

$$\dot{y}(\tau)-\omega y(\tau)=x(\tau) \tag{2.26}$$

with $x(\tau)$ the bounce trajectory. Then $y(x=x_c)$, being equivalent to $y(\tau=0)$, is obtained via a temporal integration of (2.26) from $-\infty$ to 0, with boundary condition $y(\tau=-\infty)=0$. Since $x(\tau)\sim\exp(\omega_0\tau)$ for $\tau\to-\infty$, the requirement of analyticity on $y(x)$ at the origin implies that $y(\tau)$ must be analytic in $\exp(\omega_0\tau)$ at $\tau=-\infty$. Thus, the homogeneous solution of (2.26), varying as $\exp(\omega\tau)$, should be absent from the desired solution.

It is instructive to verify that integration of (2.26) agrees with the results of Sec. II B. For $\omega<\omega_0$ a solution satisfying the boundary condition and required analyticity at $\tau=-\infty$ can be written

$$y(\tau=0)=\int_{-\infty}^{0} d\tau\, x(\tau)e^{-\omega\tau} , \tag{2.27}$$

so that

$$y(x_c;\omega)+y(x_c;-\omega)=\int_{-\infty}^{\infty} d\tau\, x(\tau)\cosh(\omega\tau) . \tag{2.28}$$

Upon combining (2.28) with (2.12) and (2.13) the resonantly enhanced decay exponent is expressed directly as a simple integral over the bounce trajectory. The above expression is in fact reminiscent of (although different than) the one derived by Ivlev and Mel'nikov[9] by deforming real-time physical trajectories of the particle $x(t)$ into the complex time plane.

Despite the apparent simplicity of the result (2.28) it should be emphasized that since $x(\tau)\sim\exp(-\omega_0|\tau|)$ as $\tau\to\pm\infty$, the integral is convergent (on the real axis) only for $|\omega|<\omega_0$. As in Sec. II B, however, this solution can be extended to arbitrary ω by analytic continuation into the complex ω plane to avoid the resonance poles at $n\omega_0$. For example, for the cubic potential where the bounce trajectory is $x(\tau)=x_c\cosh^{-2}(\omega_0\tau/2)$, the integral can be evaluated in closed form giving the result (2.19) which is defined over the entire complex ω plane.

III. DISSIPATIVE RESONANT ENHANCEMENT

A. Many-dimensional WKB approach

The effects of dissipation on the resonantly enhanced decay rate can be conveniently studied within a system-plus-bath framework. To be concrete, I consider the Caldeira-Leggett model,[2] which consists of coupling the tunneling coordinate bilinearly to an infinite set of harmonic-bath degrees of freedom, although the techniques discussed below apply more generally. The Hamiltonian of interest can be expressed as

$$H=\frac{-\hbar^2}{2m}\sum_{j=1}^{N}\left[\frac{\partial}{\partial x_j}\right]^2+V(\mathbf{x}),$$
$$V(\mathbf{x})=V_0(x_1)+\sum_{j=2}^{N}\tfrac{1}{2}m\omega_j^2(x_j-x_1)^2, \tag{3.1}$$

where x_1 denotes the particle moving in the metastable well V_0 and $\{x_j\}$, $j=2,\ldots,N$, are the bath degrees of freedom. The coupling to the environment is characterized by the spectral density[2]

$$J(\omega)=\frac{\pi}{2}m\sum_j\omega_j^3\delta(\omega-\omega_j). \tag{3.2}$$

The special case of Ohmic dissipation $J(\omega)=\eta\omega$ corresponds to frictional damping with friction coefficient η. The Hamiltonian (3.1) with Ohmic dissipation is generally believed to offer a reasonable description of the dissipative dynamics of small Josephson junctions. Under the influence of a microwave perturbation, the coordinate will experience an additional oscillatory force. As in Sec. II, this is modeled by adding a term to H of the form

$$H_1=fx_1\cos(\omega t). \tag{3.3}$$

In the absence of H_1, the decay rate can be calculated by using instanton techniques to obtain the imaginary part of the ground-state energy.[2] It is unclear whether this approach can be generalized to incorporate the microwave perturbation, however, since the Hamiltonian will then be time dependent. Alternatively, the decay rate can be extracted using a multidimensional WKB approach.[15] Below, this method will be generalized to incorporate the ac perturbation. A general expression for the contribution to the decay exponent linear in f will be obtained.

As in Sec. II, the time-dependent Schrödinger equation is rewritten by inserting the semiclassical ansatz for the wave function,

$$\psi(\mathbf{x},t)=\exp[-W(\mathbf{x},t)/\hbar], \tag{3.4}$$

where the action W is now a function of all N degrees of freedom. Upon introducing an expansion for W in powers of $\hbar$ and f, as in (2.7), and comparing powers, one obtains the many-dimensional generalization of the zeroth-order eikonal equation (2.8),

$$\frac{1}{2m}\sum_j\left[\frac{\partial W_0}{\partial x_j}\right]^2-V(\mathbf{x})=0, \tag{3.5}$$

and an equation first order in f,

$$-i\partial_t W_1=-\frac{1}{m}\sum_j\left[\frac{\partial W_0}{\partial x_j}\right]\left[\frac{\partial W_1}{\partial x_j}\right]+e^{\epsilon t}x_1\cos(\omega t). \tag{3.6}$$

Terms of order $\hbar$ and f^2 have been ignored. As in the one-dimensional case, W_0 has been assumed time independent. A convergence factor $\exp(\epsilon t)$ has been introduced in (3.6) to switch on the microwave perturbation adiabatically. The limit $\epsilon\to 0$ will be taken at the end of the calculation.

Consider first the decay in the absence of the ac perturbation. This case was studied in detail in Ref. 15 and will be only briefly reviewed here. It is necessary to solve the eikonal equation in the classically forbidden region subject to the boundary condition (2.10). As in Sec. II C, it is convenient to introduce fictitious trajectories $\{x_j(\tau)\}$ which satisfy

$$m\dot{x}_j(\tau)=\frac{\partial W_0}{\partial x_j}, \tag{3.7}$$

enabling the eikonal equation to be rewritten as

$$\tfrac{1}{2}m\sum_j\dot{x}_j^2-V(\mathbf{x})=0. \tag{3.8}$$

Notice that these trajectories correspond to constant "energy" motion in the inverted potential $-V(\mathbf{x})$. Moreover, from (3.7) the change of W_0 along a trajectory is simply $\dot{W}_0=m\sum_j\dot{x}_j^2$. Therefore, $W_0(\mathbf{x})$ can be evaluated, in the classically inaccessible region, by integration in time along a trajectory which starts at $\mathbf{x}=0$ at $\tau=-\infty$ and passes through $\mathbf{x}$ at time τ.

A set of such paths is shown schematically for two dimensions in Fig. 2. The "surface" Σ denotes the boundary between classically accessible and inaccessible regions, determined from the condition $V(\mathbf{x})=0$. A typical path starting from the origin will veer away from this boundary before reaching it. Attention should be focused[15] on the single path which approaches Σ perpendicularly and reaches the boundary at the exit point $\mathbf{x}_c$

with zero velocity. This singular path is clearly the multidimensional generalization of the bounce trajectory discussed in Sec. II C. At long times the bounce will retrace its steps returning eventually to the origin; for convenience the origin of (imaginary) time is set so that the trajectory arrives at the exit point at $\tau=0$.

The decay rate to leading order in $\hbar$ is given by the probability of finding the particle at the exit point, $\Gamma_0 \sim \exp[-2W_0(\mathbf{x}_c)/\hbar]$, in complete analogy with the one-dimensional case. As indicated above, this can be related to an integral along the bounce trajectory, namely,

$$2W_0(\mathbf{x}_c)=\int_{-\infty}^{\infty} d\tau\, m \sum_j \dot{x}_j^2(\tau) \ . \tag{3.9}$$

The resonant enhancement to the decay rate can, in turn, be obtained by solving for W_1 at the exit point; see (2.12). Introducing

$$W_1(\mathbf{x},t)=\tfrac{1}{2}y(\mathbf{x};\omega)e^{-i(\omega+i\epsilon)t}+(\omega\to-\omega) \tag{3.10}$$

into (3.6) suffices to eliminate the time dependence, giving

$$\sum_j \frac{\partial y}{\partial x_j}\left[\frac{1}{m}\frac{\partial W_0}{\partial x_j}\right]-(\omega+i\epsilon)y=x_1 \ , \tag{3.11}$$

which must be supplemented by the boundary condition at the origin, (2.15), namely $y(\mathbf{x}=0)=0$. It is convenient to reexpress (3.11) in terms of the trajectories $\{x_j(\tau)\}$. Inserting (3.7) and applying the chain rule enables (3.11) to be rewritten as an equation in imaginary time,

$$\dot{y}(\tau)-(\omega+i\epsilon)y(\tau)=x_1(\tau) \ , \tag{3.12}$$

the multidimensional generalization of (2.26).

The function $y(\mathbf{x})$ can then be evaluated by integrating (3.12) in time with $x_1(\tau)$ corresponding to the appropriate trajectory passing through the point $\mathbf{x}$ of interest. Since only $y(\mathbf{x}_c)$ is required to evaluate the decay enhancement, $x_1(\tau)$ can be taken as the bounce trajectory for our purposes. With this choice

$$y(\mathbf{x}_c)=y(\tau=0) \ , \tag{3.13}$$

where $y(\tau=0)$ is obtained by integrating (3.12) from $-\infty$ to 0 with the boundary condition $y(\tau=-\infty)=0$. This boundary condition, however, is insufficient to specify a unique solution of (3.12) since the homogeneous solution, $y_h(\tau)=\exp[(\omega+i\epsilon)\tau]$, vanishes at $\tau=-\infty$. This difficulty was also encountered in Sec. II. It was pointed out there that the homogeneous solution corresponds, in general, to a nonanalytic function in $\mathbf{x}$ and must on physical grounds be eliminated from the required solution. This can be achieved here by expressing $y(\tau)$ as a sum of functions $\exp(\lambda\tau)$ in the form

$$y(\tau)=\int_0^{\infty} d\lambda\, e^{\lambda\tau}y(\lambda), \quad \tau<0 \tag{3.14}$$

with $y(\lambda)$ to be determined. For real λ, (3.14) does not contain any of the homogeneous solution which oscillates in time for nonzero ϵ.

To determine $y(\lambda)$ it is convenient to introduce a Laplace transform representation of the bounce trajectory

$$x_1(\tau)=\int_0^{\infty} d\lambda\, e^{\lambda\tau}x_1(\lambda), \quad \tau<0 \tag{3.15}$$

with $x_1(\lambda)$ defined by the Laplace inversion formula

$$x_1(\lambda)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\tau\, e^{\lambda\tau}x_1(\tau), \quad \lambda>0 \ . \tag{3.16}$$

Inserting (3.14) and (3.15) in (3.12) then relates $y(\lambda)$ directly to the bounce trajectory,

$$y(\lambda)=(\lambda-\omega-i\epsilon)^{-1}x_1(\lambda) \ .$$

The resonant enhancement can now be expressed directly as an integral over $x_1(\lambda)$. Indeed, using the definition (2.12) and combining (3.10) with (3.13) and (3.14) gives the final result

$$-\ln[\Gamma(t)/\Gamma_0]\sim \frac{2f}{\hbar}\mathrm{Re}[W_1(\mathbf{x}_c,t)]=\frac{f}{\hbar}\mathrm{Re}(Ae^{-i\omega t}) \ , \tag{3.17}$$

with the complex amplitude A given by

$$A=\int_0^{\infty} d\lambda\, x_1(\lambda)\frac{2\lambda}{\lambda^2-\omega^2-i\epsilon} \tag{3.18}$$

and the $\epsilon\to 0$ limit understood.

The general expressions (3.16)–(3.18) are the central results of this paper. The contribution to the decay exponent, linear in the microwave perturbation, is expressed simply in terms of the bounce (or instanton) trajectory. Specific details, such as the form of the metastable well and the nature of the dissipation, enter only via this bounce trajectory. It should be emphasized that the above expressions apply generally to multidimensional decay problems and are not restricted to the Caldeira-Leggett form (3.1). Moreover, as a special case, they may be applied to the undamped decay problem, which

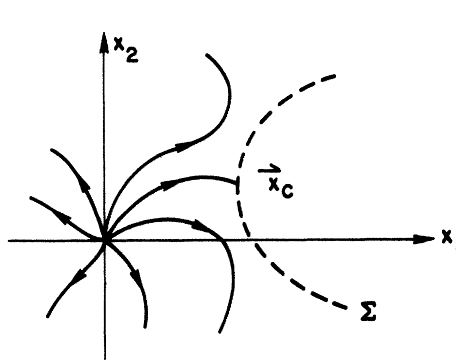

FIG. 2. Solid lines depict solutions of the classical equation of motion (3.8) which start from the origin. The dotted line Σ denotes the boundary between classically accessible and inaccessible regions, determined from the condition $V(\mathbf{x})=0$. The bounce trajectory connects the origin with the exit point $\mathbf{x}_c$ on Σ.

was treated slightly differentially in Sec. II. In Sec. III B specific results will be obtained for the Caldeira-Leggett model (with Ohmic dissipation) using previously calculated bounce trajectories.

B. Results for Ohmic dissipation

To obtain the resonant enhancement an analytic solution for the bounce trajectory is required. The equation of motion for the bounce trajectory follows from (3.1) and (3.8). Assuming a continuous spectrum of oscillators $J(\omega)$ with Ohmic form, one finds

$$-m\ddot{x}_1(\tau)+\int d\tau' K(\tau-\tau')x_1(\tau')+V_0'(x_1)=0 \qquad (3.19)$$

with $K(\omega)=\eta\,|\,\omega\,|$. The oscillator bounce trajectories are related to $x_1(\tau)$ via

$$-\ddot{x}_j+\omega_j^2 x_j=\omega_j^2 x_1,\quad j=2,\ldots,N\ . \qquad (3.20)$$

Solutions of (3.19) have previously been obtained both for the truncated harmonic-oscillator potential[16] with arbitrary η and for the cubic potential at special values of the damping.[2,17]

Consider first the truncated oscillator potential, which corresponds to the $k\to\infty$ limit of the potential (2.16). In this case the bounce trajectory can be expressed as[16]

$$x_1(\tau)=x_c I(\tau)/I(\tau=0)\ ,$$
$$I(\tau)=\int\frac{d\omega}{2\pi}e^{-i\omega\tau}(\omega^2+\omega_0^2+\gamma\,|\,\omega\,|\,)^{-1} \qquad (3.21)$$

with $\gamma=\eta/m$. Notice that $I(\tau=0)$ is equal to the variance of a damped quantum harmonic oscillator, $\langle x^2\rangle_h$. The decay rate in the absence of the microwave perturbation is computed by inserting the bounce trajectory into (3.9), giving

$$\Gamma_0\sim\exp[-2W_0(\mathbf{x}_c)/\hbar]=\exp\left[-\frac{x_c^2}{2\langle x^2\rangle_h}\right]\ , \qquad (3.22)$$

a result obtained previously by Grabert, Weiss, and Hanggi.[5] Notice that Γ_0 is proportional to the probability that a damped quantum oscillator is at the exit point x_c.

To evaluate the resonant enhancement, the Laplace inverse of the bounce $x_1(\lambda)$, defined in (3.16), is required. This can be most readily obtained by rotating the frequency integration in (3.21) to run along the imaginary axis, which gives

$$x_1(\lambda)=\frac{x_c}{\ln(\omega_+/\omega_-)}\lambda[(\lambda^2+\omega_-^2)^{-1}-(\lambda^2+\omega_+^2)^{-1}]\ , \qquad (3.23)$$

with the definition

$$\omega_\pm=\frac{\gamma}{2}[1\pm(1-4\omega_0^2/\gamma^2)^{1/2}]\ . \qquad (3.24)$$

Finally, inserting (3.23) into the general formula (3.18) and performing a contour integration gives the enhanced decay rate for the damped, truncated oscillator

$$\Gamma(t)\sim\exp\left[-\frac{[x_c-x_{c1}(t)]^2}{2\langle x^2\rangle_h}+O(f^2)\right]\ , \qquad (3.25)$$

with

$$x_{c1}(t)=\frac{f}{m}\mathrm{Re}[e^{-i\omega t}(\omega^2+i\omega\gamma-\omega_0^2)^{-1}]\ . \qquad (3.26)$$

Here $x_{c1}(t)$ is the solution for a classical forced, damped harmonic oscillator. Notice that $\Gamma(t)$ is proportional to the (instantaneous) probability of finding a forced, damped quantum oscillator at the exit point x_c. The quantum oscillator has a wave packet with dispersion $\langle x^2\rangle_h$ and a center of mass which follows the classical trajectory. For weak damping, $\gamma\ll\omega_0$, the decay rate will have a resonance for $\omega\simeq\omega_0$ with strength proportional to γ^{-1}. The divergent resonance, present with no damping, has been rounded by the dissipation.

Consider next the cubic potential, where several exact results for the bounce trajectory are known. As a check on the approach in this section consider first the undamped limit. Inserting the undamped bounce trajectory, $x_1(\tau)=x_c/\cosh^2(\omega_0\tau/2)$, into (3.16)–(3.18), gives complete agreement with the results of Sec. II, (2.20), showing once again the divergent resonances at $\omega=n\omega_0$. In the overdamped limit, $\gamma\gg\omega_0$, if the first term in (3.19) is ignored the bounce trajectory can be obtained exactly,[2]

$$x_1(\tau)=(4x_c/3)\frac{\tau_0^2}{\tau^2+\tau_0^2},\quad \tau_0=\gamma/\omega_0^2\ . \qquad (3.27)$$

Performing the Laplace inversion (3.16) and inserting into (3.18) gives, then, a contribution to the tunneling exponent for $\gamma\gg\omega_0$ of the form

$$-\ln[\Gamma(t)/\Gamma_0]\sim\frac{4\pi}{3}\left[\frac{fx_c}{\hbar\omega_0}\right](\gamma/\omega_0)\cos(\omega t-\omega\gamma/\omega_0^2)\ . \qquad (3.28)$$

Notice that the resonance structure, present at zero damping, has been destroyed. The only remnant of any sort of resonance is through the phase shift $\omega\gamma/\omega_0^2$ which cycles through 2π when ω changes by $2\pi\omega_0^2/\gamma$, a characteristic frequency in the overdamped limit. This phase shift, though, is unobservable in the decay rate Γ, which is defined as the average of $\Gamma(t)$ over one period of oscillation.

The bounce trajectory for a particular intermediate value of damping was obtained by Riseborough *et al.*[17] Specifically, for damping γ satisfying

$$\frac{\gamma\tau_B}{5}=\frac{6(\omega_0\tau_B)^2-15}{(\omega_0\tau_B)^2+30} \qquad (3.29)$$

with $\omega_0\tau_B$ the unique solution of the equation

$$(\omega_0\tau_B)^6-15(\omega_0\tau_B)^4-90(\omega_0\tau_B)^2-225=0\ , \qquad (3.30)$$

the bounce trajectory was found to be

$$x_1(\tau)=\frac{8x_c}{\omega_0^2}\left[\frac{a-1}{\tau^2+\tau_B^2}+\frac{2\tau_B^2}{(\tau^2+\tau_B^2)^2}\right]\ , \qquad (3.31)$$

with $a=\gamma\tau_B/5$. From (3.29) and (3.30) the dimensional damping is found to be $\alpha\equiv\gamma/(2\omega_0)=1.175\ldots$. Since

$\alpha > \alpha_c = 1$, the motion in the quadratic minima of the well is moderately overdamped. Inserting this trajectory into (3.16)–(3.18) gives an enhancement of the form

$$-\ln[\Gamma(t)/\Gamma_0] \sim 8\pi \left[\frac{f x_c}{\hbar\omega_0}\right](\omega_0\tau_B)^{-1}[a^2+(\omega\tau_B)^2]^{1/2} \times \cos(\omega t - \omega\tau_B + \theta) \quad (3.32)$$

with $\theta = \tan^{-1}(\omega\tau_B/a)$. As in the overdamped limit the resonant structure is absent. For $\omega \gg \omega_0$ the strength of the enhancement grows linearly in ω, similar to the undamped case (2.20).

Presumably, in the underdamped regime, $\alpha < 1$, the resonant structure will reappear and cross over smoothly to the divergent resonances as $\gamma \to 0$. A detailed analysis of the resonance shapes in this regime requires an analytic expression for the bounce trajectory at weak but nonzero damping. At the present time no such solution is known. This is unfortunate since the existing experiments[8] have focused on studying the microwave enhancement in the weak-damping limit.

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