

Edge states and tunneling of non-Abelian quasiparticles in the $\nu = 5/2$ quantum Hall state and $p + ip$ superconductors

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(Dated: July 14, 2006)

We study quasiparticle tunneling between the edges of a non-Abelian topological state. The simplest examples are a $p + ip$ superconductor and the Moore-Read Pfaffian non-Abelian fractional quantum Hall state; the latter state may have been observed at Landau level filling fraction $\nu = 5/2$. Formulating the problem is conceptually and technically non-trivial: edge quasiparticle correlation functions are elements of a vector space, and transform into each other as the quasiparticle coordinates are braided. We show in general how to resolve this difficulty and uniquely define the quasiparticle tunneling Hamiltonian. The tunneling operators in the simplest examples can then be rewritten in terms of a free boson. One key consequence of this bosonization is an emergent spin-1/2 degree of freedom. We show that vortex tunneling across a $p + ip$ superconductor is equivalent to the single-channel Kondo problem, while quasiparticle tunneling across the Moore-Read state is analogous to the two-channel Kondo effect. Temperature and voltage dependences of the tunneling conductivity are given in the low- and high-temperature limits.

I. INTRODUCTION

The possible existence of non-Abelian quantum Hall states has caused great excitement recently^{1,2}. A non-Abelian quantum Hall state would not only be a new class of quantum matter – a truly remarkable discovery in itself – but could also be a platform for fault-tolerant quantum computation³. The leading candidate is the observed^{4,5,6} quantized Hall plateau with $\sigma_{xy} = \frac{5}{2} \frac{e^2}{h}$. There is numerical evidence^{7,8} that the ground state at this filling fraction is given by a filled lowest Landau level of both spins and $\nu = 1/2$ filling of the first excited Landau level in the Moore-Read Pfaffian state⁹. The excitations of this state are charge- $e/4$ quasiparticles which exhibit non-Abelian braiding statistics^{11,12,13,14}.

The Moore-Read Pfaffian state is the quantum Hall incarnation of a $p + ip$ superconductor¹⁰, whose vortices have the same non-Abelian braiding statistics^{15,16,17,18}. Such vortices have Majorana (real) fermion zero modes in their cores. A pair of vortices, if kept far apart, therefore shares a complex fermion zero mode which can be either occupied or unoccupied. As vortices are braided, the occupancies of these zero modes are altered and phases are acquired. Such transformations do not commute, so the braiding statistics of vortices is non-Abelian. There are at least two candidate systems in which a $p + ip$ superconducting state may exist: (1) the seemingly unconventional superconductor Sr_2RuO_4 ¹⁹ and (2) ultra-cold fermions with a p -wave Feshbach resonance in an atomic trap²⁰. These systems may be fertile alternatives to the $\sigma_{xy} = \frac{5}{2} \frac{e^2}{h}$ quantum Hall state for exploring non-Abelian braiding statistics and topological quantum computation^{21,22}.

Multi-quasiparticle states of non-Abelian anyons are topologically degenerate. These states can be used as decoherence-free qubits which can be manipulated and measured using quasiparticle braiding and interferometry. This observation is the basis not only of proposals for quantum computation^{23,24,25} but also for experiments to determine

whether the $\sigma_{xy} = \frac{5}{2} \frac{e^2}{h}$ state is non-Abelian^{13,23,26,27,28}. All of these proposed experiments involve interference between trajectories in which non-Abelian quasiparticles tunnel at point contacts between the edges of a Hall droplet.

A better understanding of the behavior of a point contact would illuminate the analysis of these proposed experiments. It would also open a new avenue for exploring the physics of the $\sigma_{xy} = \frac{5}{2} \frac{e^2}{h}$ state – and, perhaps, Sr_2RuO_4 and ultra-cold fermions with a p -wave Feshbach resonance – because transport through a point contact is an important probe of quantum Hall states. Chiral topological states have gapless edge excitations whose behavior is largely determined by the topological properties of the bulk²⁹. These gapless excitations determine low-temperature transport properties. At a point contact, fractionally-charged quasiparticles tunnel from one edge of the system to the other. Consequently, the temperature and voltage dependences for transport through a point contact reflect the topological structure of the state. In the case of the Laughlin states, a small bare tunneling rate between the two sides of a Hall bar at any finite temperature increases as the temperature is decreased until the bar is effectively broken in two at zero temperature. The conductance versus temperature power laws in both the high-temperature and low-temperature limits³⁰ (and even the full crossover function between these two limits³¹) show the effects of the fractional statistics of quasiparticles in these states. Shot noise and other measurements evince the fractional charge of quasiparticles³². In the case of the hierarchy states, the topological structure is richer, but has proven more elusive experimentally³³. One might also expect even more interesting physics in a single point contact in a non-Abelian quantum Hall state³⁴, reflecting its topological properties.

In ref. 35, we analyzed the behavior of a single point contact in the Moore-Read Pfaffian quantum Hall state and also in the slightly simpler case of a $p + ip$ superconductor and found that it is highly non-trivial. A point contact between two edges of a $\nu = 5/2$ Moore-Read Hall droplet leads to a

leading correction to the vanishing of the longitudinal resistivity $R_{xx} \sim T^{-3/2}$ (at temperatures which are sufficiently high that this is a small correction). At zero temperature, the filled lowest Landau level is unaffected but the $\nu = 1/2$ first excited Landau level is broken in two so that $R_{xx} = \frac{1}{10} \frac{h}{e^2}$ (see appendix A for the definition of the four-terminal resistance). At small non-zero temperature, $R_{xx} - \frac{1}{10} \frac{h}{e^2} = -T^4$. We showed that the crossover between these two limits is a variant of the two-channel Kondo problem³⁶ and resonant tunneling in Luttinger liquids³⁰. In this paper, we explain in greater detail how to properly define tunneling at a point contact in a non-Abelian state. We expand upon our construction of a bosonized representation for the tunneling Hamiltonian, thereby clarifying the relation to the Kondo problem. As in the Kondo problem, the crossover from high to low-temperatures is accompanied by entropy loss. In a companion paper³⁷, we show that there is a very natural $2 + 1$ -dimensional interpretation for this entropy loss.

The edge excitations of the Laughlin and hierarchy states correspond to free chiral bosons. Although these are free field theories, the perturbation corresponding to a point contact, at which quasiparticles tunnel between edges, is built up from exponentials of the chiral boson. Such operators are non-trivial and capture the fractional charge and Abelian fractional statistics of quasiparticles. The Moore-Read Pfaffian state has a further wrinkle: in addition to a chiral boson there is a Majorana fermion edge mode³⁸. Although this, too, is a free field theory it is a more peculiar one than a chiral boson. The Majorana fermion is the chiral part of the critical theory of the $2D$ Ising model, which has non-trivial spin-spin correlation functions. This is related, as we will see below, to the non-Abelian statistics of the quasiparticles. The edge of a $p + ip$ superconductor has only this Majorana mode – it lacks a charge-carrying chiral boson mode – but the same issues arise.

The non-Abelian statistics of the state plays a crucial role in describing quasiparticle tunneling at a point contact. The operator which creates a charge $e/4$ quasiparticle at the edge of the Moore-Read Pfaffian state at $\nu = 5/2$ or a vortex at the edge of a $p + ip$ superconductor is the chiral part of the Ising spin field $\sigma(\mathbf{r})$, which creates a branch cut for the fermions, terminating at \mathbf{r} . Correlation functions of the tunneling operator therefore involve the chiral parts of spin-spin correlation functions. The chiral parts of correlation functions in a conformal field theory, such as the Ising model, are called *conformal blocks*³⁹. In general, they are not defined uniquely if only the positions of the fields are specified. As these coordinates are taken around each other (i.e. *braided*), the conformal blocks transform linearly. In other words, the conformal blocks form a vector space, on which the braid group is represented. In the case of exponentials of a chiral boson, braiding simply results in a phase. In the fractional quantum Hall context, this is the Abelian braiding statistics of quasiparticles in the Laughlin and hierarchy states. However, in a generic rational conformal field theory⁴⁰, there is a multi-dimensional space of such chiral parts of correlation functions, i.e. of conformal blocks. The full non-chiral correlation function is a sum of products of left- and right-conformal blocks, and must be single-valued. The consequent constraints on the conformal blocks result in

a great deal of structure, which is discussed in depth in Refs. 40.

In order to define the tunneling Hamiltonian for non-Abelian quasiparticles and compute the effects of a point contact perturbatively, we must compute multi-point chiral correlation functions such as $\langle \sigma \sigma \dots \sigma \rangle$. In conformal field theory language, the ambiguity in defining such a quantity stems from the two possible fusion channels for a pair of spin fields, written schematically as $\sigma \cdot \sigma \sim 1 + \psi$. Each of these possibilities (or any linear combination thereof) corresponds to a different possible chiral correlation function. Consequently, the vector space of chiral correlation functions with $2n$ spins is 2^{n-1} -dimensional. This can be restated in different terms by observing that a pair of quasiparticles can be in either of two topologically distinct states^{11,12,15}, the two states of the qubit which they form²³. In the language of a $p + ip$ superconductor, the complex fermion zero mode associated with a pair of vortices can be either occupied or unoccupied – the two states $|0\rangle$ and $|1\rangle$ of the qubit. When a pair of quasiparticles is in the state $|0\rangle$, they fuse to the 1; when they are in the state $|1\rangle$, they fuse to ψ . Thus, in order to properly define these correlation functions, we must also specify the state of the qubit associated with each pair of quasiparticles. How we pair them is arbitrary; changing the pairing is just a change of basis.

The resolution of this problem can be stated quite simply in physical terms: when a quasiparticle tunnels, a neutral fermion cannot be created by the tunneling process alone. Therefore, the two quasiparticle operators (corresponding to the annihilation of a quasiparticle on one edge and its subsequent creation on the other edge) must fuse to the identity. However, this is not the most convenient basis for computations. We would rather know how quasiparticle operators on the same edge fuse so that different edges can be treated perturbatively as being independent. This is simply a basis change in the space of conformal blocks. We can switch into such a basis using the braiding rules of the chiral Ising conformal field theory⁴⁰ or, equivalently, the corresponding topological field theory⁴¹.

The basic strategy outlined above can be applied to any two-dimensional gapped system with gapless edge modes described by conformal field theory. However, in the case of the Moore-Read Pfaffian state, we can massage this result into an even simpler form. When considering tunneling between two different edges, we combine the Majorana fermion modes at the two edges into a single Dirac fermion. We then bosonize this Dirac fermion. This allows us to directly compute the conformal blocks of the critical $2D$ Ising model. In order to bosonize the quasiparticle tunneling operator, we need to introduce a spin-1/2 degree of freedom. Although we introduce this degree of freedom almost as a bookkeeping device, we then find that the bosonized tunneling Hamiltonian takes a form similar to the anisotropic Kondo Hamiltonian. Armed with this knowledge, we analyze the crossover to the low-temperature limit in which tunneling becomes strong.

Sections II, III, and IV are reviews in which we explain why the edge excitations of the Moore-Read Pfaffian state and of a $p + ip$ superconductor contain a Majorana fermion

mode which is the chiral part of the critical 2D Ising model. In section V, we discuss the form of the tunneling Hamiltonian. We explain in detail the subtlety associated with defining this Hamiltonian and show how to resolve this difficulty in sections VI and VII. We then show in section VIII how this Hamiltonian can be bosonized. In section IX, we map the bosonized Hamiltonian to the Kondo and Luttinger liquid resonant tunneling Hamiltonians. In section X, we analyze the infrared behavior of this bosonic Hamiltonian using these mappings and an instanton gas expansion.

II. VORTICES AND EDGE STATES IN A $p_x + ip_y$ SUPERCONDUCTOR

The physics of a $p_x + ip_y$ superconductor is essentially identical to the neutral sector of the Moore-Read Pfaffian state^{15,16,17,18}. The vortices in the superconductor correspond to the non-Abelian quasiparticles in the Moore-Read Pfaffian state, and the superconductor has a gapless chiral Majorana fermion edge mode which is identical to the neutral sector of the Moore-Read Pfaffian edge theory, which we will discuss in section IV. Moreover, one can consider the process of passing a bulk vortex off the edge of the superconductor, which leaves behind a “twist” field or “spin” field operator σ (a terminology which we explain in section III) acting on the chiral edge state. This spin field changes the boundary conditions on the chiral edge state from periodic to anti-periodic, or vice versa. In this section we study the $p + ip$ superconductor at the level of the Bogoliubov-De Gennes equations, to gain insight into the physics of edge states and quasiparticles in this non-Abelian topological state.

The Bogoliubov Hamiltonian for the $p + ip$ superconductor is expressed in terms of the (spinless) fermion creation and destruction operators, $\hat{c}(\mathbf{x})$, $\hat{c}^\dagger(\mathbf{x})$, with \mathbf{x} denoting a two-dimensional spatial coordinate. We define the Pauli matrices $\vec{\sigma}$ to act on the two-component spinor

$$\hat{\Psi}(\mathbf{x}) = \begin{pmatrix} \hat{c}^\dagger(\mathbf{x}) \\ \hat{c}(\mathbf{x}) \end{pmatrix}.$$

The appropriate Bogoliubov-deGennes Hamiltonian is

$$\hat{\mathcal{H}} = \int d\mathbf{x} \hat{\Psi}^\dagger H \hat{\Psi}, \quad (1)$$

with single-particle Hamiltonian,

$$H = (-\nabla^2/2m + V(\mathbf{x}) - \mu)\sigma^z + i\Delta(\sigma^x\partial_x + \sigma^y\partial_y).$$

This Hamiltonian has the symmetry, $\sigma^x H^* \sigma^x = -H$, which implies that all non-zero-energy eigenstates come in $\pm E$ pairs. To wit, with $H\phi_E = E\phi_E$ for a two-component wave function ϕ_E , one has $H\phi_{-E} = -E\phi_{-E}$ for $\phi_{-E} = \sigma^x\phi_E^*$. One can then expand the spinor field operator as

$$\hat{\Psi} = \frac{1}{\sqrt{2}} \sum_{E>0} [\hat{\eta}_E\phi_E + \hat{\eta}_{-E}\phi_{-E}], \quad (2)$$

with fermion operators $\hat{\eta}_E$ which satisfy $\hat{\eta}_{-E}^\dagger = \hat{\eta}_E$. The Hamiltonian can then be written in a diagonal form,

$$\hat{\mathcal{H}} = \sum_{E>0} E \hat{\eta}_E^\dagger \hat{\eta}_E. \quad (3)$$

The ground state consists of filling up all of the negative energy states, $|\text{Ground}\rangle = \prod_{E>0} \hat{\eta}_{-E}^\dagger |\text{vac}\rangle$, and is annihilated by $\hat{\eta}_E$ and $\hat{\eta}_{-E}^\dagger$ for all $E > 0$. In the bulk of the superconductor all states will be gapped, obeying $|E| > p_F\Delta$, with p_F the Fermi momentum.

In order to establish the presence of chiral edge modes, it is convenient to consider an infinite system in which the potential $V(\mathbf{x})$ varies spatially,

$$V(x, y) - \mu = \Delta V_0(y), \quad (4)$$

with $V_0(y > 0)$ positive and increasing to large values for large y , and $V_0(y < 0)$ negative. Since the electron density will fall to zero for large positive y , we have, in effect, created a straight edge at $y = 0$. At low energy, we can ignore the first term in the single-particle Hamiltonian, H , because its second derivative makes it smaller than the zero and single-derivative terms. For this potential, there are exact eigenstates which are spatially localized near $y = 0$:

$$\phi_E^{edge}(\mathbf{x}) = e^{ikx} e^{-\int_0^y V_0(y') dy'} \phi_0, \quad (5)$$

with $\phi_0 = \binom{1}{1}$ an eigenstate of σ^x . This wavefunction describes a chiral wave propagating in the x -direction localized on the edge, with wave vector $k = E/\Delta$. One can then construct a second quantized description of these edge modes by expanding the spinor field operator in terms of both bulk states above the gap and an edge sector:

$$\hat{\Psi}(\mathbf{x}) = \hat{\Psi}_{bulk}(\mathbf{x}) + \hat{\Psi}_{edge}(\mathbf{x}), \quad (6)$$

with $\hat{\Psi}_{bulk}$ given in Eq. 2, and

$$\hat{\Psi}_{edge}(\mathbf{x}) = e^{-\int_0^y V_0(y') dy'} \sum_{k>0} [\hat{\psi}_k e^{ikx} \phi_0 + \hat{\psi}_{-k} e^{-ikx} \sigma^x \phi_0^*].$$

Here the fermion operators, $\hat{\psi}_{\pm k}$, satisfy $\hat{\psi}_{-k} = \hat{\psi}_k^\dagger$. One sees that $\hat{\psi}(x) = \sum_k \hat{\psi}_k e^{ikx}$ is a real Majorana field, $\hat{\psi}(x) = \hat{\psi}^\dagger(x)$ satisfying anticommutation relations,

$$\{\hat{\psi}(x), \hat{\psi}(x')\} = 2\delta(x - x'). \quad (7)$$

The edge Hamiltonian can be simply written in terms of a real-space Hamiltonian density:

$$\hat{\mathcal{H}}_{edge} = \sum_{k>0} v_n k \hat{\psi}_k^\dagger \hat{\psi}_k = \int dx \hat{\psi}(x) (-iv_n \partial_x) \hat{\psi}(x), \quad (8)$$

where the edge velocity $v_n \equiv \Delta$. The Lagrangian density describes chiral Majorana edge modes propagating at velocity v_n ,

$$\mathcal{L} = i\psi(x)(\partial_t + v_n \partial_x)\psi(x). \quad (9)$$

This is identical to Eq. (24) which will arise in section IV in our description of the neutral edge sector of the Moore-Read state.

Next consider the edge of a large circular sample, with circumference L . Locally, each part of the edge looks “flat”, and one expects that the edge wavefunction should take the same form as in Eq. 5. The main difference is that the spinor, ϕ_0 in Eq.5, which describes a spin one-half pointing in the x -direction, will have a direction in spin space which is parallel to the local edge tangent vector. Then, upon fully encircling the drop, the spinor will rotate by 2π around the z -axis, and will change sign. This implies that in the edge Hamiltonian of the form Eq. 8 with the spatial coordinate in the interval $0 < x < L$, must be supplemented by antiperiodic boundary conditions on the Majorana field: $\hat{\psi}(x=0) = -\hat{\psi}(x=L)$. With this boundary condition, the lowest energy edge mode will have energy $2\pi v/L$. Of course, in the thermodynamic limit this edge energy spacing vanishes, and one has truly gapless edge excitations.

Next consider introducing a single $hc/2e$ vortex which is assumed to be located at the center of the sample. In the “normal core” of the vortex the order parameter vanishes, but the effects of the vortex are “felt” by the Bogoliubov quasiparticles well outside of this region. Indeed, upon exciting a fermionic Bogoliubov quasiparticle above the bulk gap, and adiabatically transporting it around the vortex, the fermionic quasiparticle will acquire a Berry’s phase of π . This Berry’s phase is equivalent to a sign change in the boundary condition for the Bogoliubov quasiparticle upon encircling a vortex. The edge Majorana field encircling the outer edge of the sample, which in the absence of the vortex has antiperiodic boundary conditions (due to the 2π spinor rotation), will have periodic boundary conditions in the presence of a single bulk vortex. If N_v denotes the number of bulk vortices, the boundary condition on the edge Majorana fermion is

$$\hat{\psi}(x=0) = (-1)^{N_v+1} \hat{\psi}(x=L). \quad (10)$$

For a single vortex, the periodic boundary condition on the Majorana field implies that there will be one exact zero-energy eigenstate on the edge, the zero-momentum state $\hat{\psi}_{k=0} \equiv \hat{\psi}_{edge}$ independent of the spatial coordinate. Since all of the non-zero-energy states come in $\pm E$ pairs, it is natural to anticipate the existence of a second zero-energy Majorana mode associated with the vortex, and this is indeed the case. To illustrate this, consider modeling the vortex as a circular core region of radius r_{core} , within which $V(\mathbf{x}) - \mu < 0$, for $|\mathbf{x}| < r_{core}$. This depletes the fermion density within the core region, in effect making a hole in the sample, and creating an internal edge running around the circumference of the hole. Then, just as for the outer sample edge, one expects an inner Majorana chiral mode, described by a spinor with direction tangential to the edge. Moreover, the Berry’s phase of π experienced by the Bogoliubov fermions upon encircling the vortex, will be “felt” by the chiral edge fermion encircling the core. This leads to periodic boundary conditions for this inner edge Majorana fermion, which will have an exact zero-energy Majorana state, which we denote by $\hat{\psi}_{vort}$. The Majorana

zero mode associated with the vortex can be combined with the zero-energy Majorana mode on the sample outer edge to define a zero-energy complex fermion,

$$\hat{a} \equiv \frac{1}{2}(\hat{\psi}_{vort} + i\hat{\psi}_{edge}), \quad (11)$$

with \hat{a} and \hat{a}^\dagger satisfying canonical fermion anticommutation relations, $\{\hat{a}, \hat{a}^\dagger\} = 1$. Together, the vortex and edge zero modes thus constitute a two-state system, corresponding to the two eigenvalues of $\hat{a}^\dagger \hat{a} = 0, 1$. (For a finite system the parity of the total electron number precludes one of the two states, implying a unique ground state for the system with one vortex present.) The vortex “quasiparticle” is thus entangled with the edge of the system, despite the large spatial separation.

When multiple bulk vortices are present and spatially well separated from one another, each will have an associated Majorana zero-energy mode within its core. For $N_v = 2N$ such vortices, the Majorana zero modes can be combined to form N complex fermions. The dimension of the zero-energy Hilbert space will thus be, $\Omega 2^{N-1}$, and for large N_v scales as, $\Omega(N_v) \sim d^{N_v}$ with $d \equiv \sqrt{2}$. Here d is called the “quantum” dimension of the vortex quasiparticles. (We discuss this in greater depth in a companion paper³⁷.) These vortex quasiparticles have non-Abelian braiding statistics.

It is instructive to consider the process of bringing a vortex into the system by passing it through the edge into the bulk of the sample. Even when the vortex is inside the sample and well away from the edge, it leaves an imprint on the edge. Specifically, this process dynamically changes the boundary condition on the edge Majorana fermion field, $\hat{\psi}(x)$ from periodic to antiperiodic (or vice versa) upon encircling the sample. But if the vortex is brought through the edge at a particular spatial location, say x , the change in boundary conditions must, in some sense, occur locally. This can be made precise by introducing an edge vortex field operator, denoted $\hat{\sigma}(x)$, which satisfies

$$\hat{\sigma}(x)\hat{\psi}(x')\hat{\sigma}(x) = i \operatorname{sgn}(x-x')\hat{\psi}(x'). \quad (12)$$

This vortex quasiparticle field is Hermitian, $\hat{\sigma}^\dagger = \hat{\sigma}$, and squares to unity, $\hat{\sigma}(x)\hat{\sigma}(x) = 1$. Thus Eq. (12) can be re-expressed as, $\hat{\sigma}(x)\hat{\psi}(x') = i \operatorname{sgn}(x-x')\hat{\psi}(x')\hat{\sigma}(x)$.

We have employed the notation $\hat{\sigma}$ to denote this boundary-condition-changing field, because of an intimate connection between the edge theory of the $p_x + ip_y$ superconductor and a one-dimensional quantum transverse Ising model tuned to criticality. The σ field is closely related to the Ising spin operator. We will review the connection between Majorana fermions and the Ising model in the next section.

III. MAJORANA FERMIONS AND THE ISING MODEL

In the previous section, we saw how the edge modes for a $p + ip$ superconductor are described by a free chiral Majorana fermion. Near the end of the section, we saw that when a vortex passes through an edge, it changes the Majorana fermion

boundary conditions from periodic to antiperiodic and vice-versa. Such a process could be handled by introducing an edge vortex operator which effects this change of boundary condition. As we describe in this section, such an operator is closely related to the spin field of the 2D critical Ising model. We will review some key properties of chiral Majorana fermions and their relation to the Ising model⁴². Excellent reviews of the two-dimensional Ising lattice model and the Ising conformal field theory can be found respectively in Ref. 43 and Ref. 44, so we will be brief. In the next section, we will show how the edge modes for the Moore-Read Pfaffian state are also described by such a fermion (plus a free chiral boson), drawing on the notation and terminology introduced here.

The degrees of freedom in the Ising model are classical “spins” taking values $+$ or $-$ on the sites of some lattice. However, as shown in Ref. 42, the 2D Ising model can be reformulated as a theory of free fermions on the lattice; they become massless at the critical point. In the continuum limit, the model is described by a free massless Majorana fermion. The correlations of this fermion are, therefore, simple. However, the map from the Ising spins to the fermionic variables is *non-local*. The spin field introduces a branch cut for the fermions terminating at the point at which it acts. Correlators of spins are therefore highly non-trivial.

To proceed further, it is useful to rotate Euclidean time to real time, and obtain a Lorentz-invariant 1+1 dimensional field theory. At the critical point, the modes of the fermion are either right-moving or left-moving, which means the corresponding fields ψ and $\bar{\psi}$ depend only on $(vt-x)$ or $(vt+x)$ respectively. Since we are interested in describing the edge modes for a disc, we take space to be periodic, identifying $x = x + 2\pi R$. Spacetime is thus a cylinder.

It is often more convenient to study conformal field theory on the punctured plane instead of the cylinder. This can easily be done by taking advantage of the conformal invariance of critical points and performing a conformal transformation to the complex coordinates $z = e^{(vt-ix)/R}$ and $\bar{z} = e^{(vt+ix)/R}$. It is usually easiest to compute a given correlator on the plane, and then do a conformal transformation to the cylinder. One thing to note is that the transformation between the cylinder and the plane changes the boundary conditions on the fermion from periodic (antiperiodic) around the cylinder to antiperiodic (periodic) around the puncture at the origin of the plane. In complex coordinates, the action of the 1+1 dimensional critical Ising field theory is then

$$S_{\text{Ising}} = -i \int dz d\bar{z} [\psi(x) \partial_{\bar{z}} \psi(z) - \bar{\psi}(\bar{z}) \partial_z \bar{\psi}(\bar{z})]. \quad (13)$$

Going off the critical point corresponds to adding a mass term $\propto \psi\bar{\psi}$ to this field theory. Since the theory is quadratic in ψ both on and off the critical point, any correlators involving the fermions can easily be computed.

The spin field $\sigma(z, \bar{z})$, on the other hand, is a *twist field* for the fermions. A twist field at a given spacetime location puts a puncture there, so that the fermion boundary conditions around the puncture are changed from periodic to antiperiodic. We thus demand that the operator product of the twist field

with the fermions be of the form

$$\psi(z)\sigma(w, \bar{w}) \sim \frac{1}{(z-w)^{1/2}} \mu(w, \bar{w}) \quad (14)$$

$$\bar{\psi}(\bar{z})\sigma(w, \bar{w}) \sim \frac{1}{(\bar{z}-\bar{w})^{1/2}} \mu(w, \bar{w}) \quad (15)$$

where μ is another field with the same dimension as σ . μ turns out to be the continuum limit of the Kramers-Wannier dual of the spin field, which is known as the *disorder field*. From (15) we see if we rotate z around w by an angle of 2π , we pick up a factor $(e^{\pm 2\pi i})^{-1/2} = -1$. In other words, the twist field creates a square-root branch cut in fermion correlators. Thus the twist field is non-local with respect to the fermions.

To change the boundary conditions for all the fermions, one merely places twist fields $\sigma(0, 0)$ and $\sigma(\infty, \infty)$ at the origin and spacetime infinity of the punctured plane. (These points correspond respectively to $t \rightarrow -\infty$ and $t \rightarrow +\infty$ on our original spacetime cylinder; including these two points makes spacetime topologically a sphere.) This creates a branch cut from the origin to infinity on the plane, so that the fermions pick up a minus-sign change in their boundary conditions. We find, for instance, that for periodic (P) and anti-periodic (AP) boundary conditions around the origin:

$$\begin{aligned} \langle \psi(z)\psi(w) \rangle_{\text{P}} &= \frac{1}{z-w} \\ \langle \psi(z)\psi(w) \rangle_{\text{AP}} &= \langle \sigma(\infty, \infty)\psi(z)\psi(w)\sigma(0, 0) \rangle_{\text{P}} \\ &= \frac{1}{z-w} \cdot \frac{1}{2} \left(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}} \right) \end{aligned} \quad (16)$$

Using the latter correlation function, we can compute the operator product expansion of $\sigma(z, \bar{z})$ with the energy-momentum tensor, $T = \frac{1}{2} : \psi\partial\psi :$. From this operator product, we can deduce that the right and left scaling dimensions of $\sigma(z, \bar{z})$ are $(\frac{1}{16}, \frac{1}{16})$, for a total scaling dimension of $1/8$ (see, e.g. Ref 44 for details). By scaling, this gives Onsager’s famous result that $\eta = 1/4$ in the 2D Ising model.

To obtain the correlation function of an arbitrary number of spin fields, $\langle \sigma\sigma \dots \sigma \rangle$, we need to compute the ratio of the fermion partition function in the presence of the corresponding branch cuts with the partition function without any branch cuts. However, this is a very difficult calculation in general. Instead, we can use the powerful constraints which follow from two-dimensional conformal invariance, which holds at the critical point.

In two dimensions, conformal transformations take the form $z \rightarrow f(z)$, $\bar{z} \rightarrow \bar{f}(\bar{z})$, where f and \bar{f} are arbitrary analytic functions. Not only do these transformations decompose into independent right and left transformations f and \bar{f} , but the algebra of infinitesimal transformations of this form is infinite-dimensional – two copies of the Virasoro algebra, one for z and one for \bar{z} (see refs. 39,44 for details). Consequently, operators and states can be organized in representations of these two independent algebras.

The independence of these two algebras leads to separate constraints for the z and \bar{z} dependence of correlation functions. This naturally leads one to consider the two chiralities separately. In general, there is no local action for the chiral

part of a conformal field theory by itself, so the chiral theory must be considered purely algebraically. However, in the case of the Ising model, there *is* a local action for the right-moving part of the Ising model, which only has z dependence:

$$S_{\text{chiral Ising}} = -i \int dz d\bar{z} \psi(x) \partial_{\bar{z}} \psi(z) \quad (17)$$

(and there is a similar action for the left-moving part alone). Of course, in the context of edge excitations of a $p + ip$ superconductor (as we saw in the last section) or of a $\nu = \frac{5}{2}$ quantum Hall state (as we will see in the next section), the chiral theory (17) itself actually interests us. The fields in this theory can be organized in representations of a single copy of the Virasoro algebra, corresponding only to the transformations $z \rightarrow f(z)$ (since there is no \bar{z} dependence at all). The chiral spin field $\sigma(z)$, which does not appear at all in the action (17), is best understood in just such a way.

In a ‘‘rational’’ conformal field theory, like the Ising conformal field theory, all states in the theory can be found by acting with symmetry generators on a finite number of states. The fields which create these special states are known as ‘‘primary’’ fields. In other words, for each primary field, there is a corresponding irreducible representation of the Virasoro algebra (or a larger enveloping algebra), whose states are obtained by acting with all elements of the algebra on the state created by the primary field. In the context of edge states, these primary fields correspond to the different possible topological charges which can be at the edge (they must, of course, be accompanied by compensating topological charges in the bulk). By acting with symmetry generators, we produce all possible generalized oscillator excitations (such as edge magnetoplasmons) ‘on top of’ these topological charges.

For the chiral Ising model, there are just three primary fields, which are the identity field I , the twist field σ , and the fermion ψ . These three primary fields correspond to three irreducible representations of the Virasoro algebra. Hence, the product of any two such representations can be decomposed into the sum of irreducibles. In the Ising model, the corresponding *fusion rules* are

$$\begin{aligned} \sigma \cdot \sigma &= I + \psi \\ \sigma \cdot \psi &= \sigma \\ \psi \cdot \psi &= I \end{aligned} \quad (18)$$

Of course, the product of any representation with the identity is the representation itself. In terms of operators or fields, the fusion rules amount to the statement that the primary fields on the right-hand-side appear in the operator product expansion of the two fields on the left.

These fusion rules for representations correspond precisely to the rules for combining topological charges in the bulk. Two nearby Majorana fermions in the bulk of a $p + ip$ superconductor are topologically equivalent to the ground state (i.e. the absence of a topologically non-trivial excitation) as far as a distant quasiparticle is concerned. On the other hand, two nearby vortices can either be topologically equivalent to the ground state or to a single neutral fermion. (These are the two states of the topological qubit which the two vortices form.)

The chiral $\sigma(z)$ field, with scaling dimension $1/16$, is largely the subject of this paper. In the Ising model context, it would only be considered at an intermediate step of a calculation. The non-chiral field $\sigma(z, \bar{z})$ is the field which is really of interest in the Ising model. It is a primary field under both the right- and left-handed Virasoro algebras, but it is *not* simply the product of the right-handed chiral field $\sigma(z)$ with its left-handed partner. One can deduce this, for instance, from the operator product expansion:

$$\sigma(z, \bar{z}) \sigma(w, \bar{w}) = \frac{1}{|z - w|^{1/4}} + \frac{i}{2} |z - w|^{3/4} \psi(w) \bar{\psi}(\bar{w}) + \dots$$

This expansion does not factor into the product of right- and left-handed copies of (18). Therefore, the correlation functions of the chiral field $\sigma(z)$ cannot be simply obtained by factoring the z and \bar{z} dependence of the correlators of $\sigma(z, \bar{z})$. Further subtleties must be dealt with, as we discuss in section VI.

IV. EDGE EXCITATIONS OF THE $\nu = 5/2$ STATE

In this section, we will derive the form of the theory of edge excitations of a $\nu = 5/2$ droplet, assuming that the lowest Landau level (of both spins) is filled and the first excited Landau level is in the universality class of the Moore-Read Pfaffian quantum Hall state. To do this, we give the explicit form of wavefunctions for the edge excitations. Let us follow Milovanovic and Read³⁸ and take the Hamiltonian to be the three-body interaction for which the Moore-Read state is the exact ground state¹⁰ together with a confining potential which simply gives an energy proportional to the increase in angular momentum, $E \propto \Delta M$. Neither of these is realistic, but they make the counting of edge states easy, and the universal properties will not depend on these details.

The Moore-Read wavefunction^{9,10} for filling fraction $\nu = 1/m$ (m even for fermions; odd for bosons) is

$$\Psi_0 = \prod_{j < k} (z_j - z_k)^m \prod_j e^{-|z_j|^2/4} \cdot \text{Pf} \left(\frac{1}{z_j - z_k} \right). \quad (19)$$

As opposed to the last section, z here is a complex coordinate for two-dimensional space, not 1+1-dimensional spacetime. The Pfaffian is the square root of the determinant of an anti-symmetric matrix or, equivalently, the antisymmetrized product over pairs

$$\text{Pf} \left(\frac{1}{z_j - z_k} \right) = \mathcal{A} \left(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \dots \right) \quad (20)$$

We will assume for now that there is an even number of electrons in the system and consider the odd electron number later. The form (20) is strongly reminiscent of the real-space form of the BCS wavefunction. Indeed, the Moore-Read state may be viewed as a quantum Hall state of p -wave paired fermions^{15,16,17}. At $\nu = 5/2$, we take $m = 2$ for the electrons in the $N = 1$ Landau level. Other even-denominator quantum

Hall states of electrons could be described by m even. Quantum Hall states of bosons (e.g. cold atoms in rotating traps) would correspond to m odd.

There are $3m$ topologically-distinct quasiparticle types in this state which we will enumerate below. On a closed surface, the total topological charge must be trivial. In a system with boundaries, the total topological charge in the bulk is equal to the topological charge at the boundaries. Therefore, the Moore-Read state on a disk has $3m$ different sectors of edge excitations, corresponding to the different possible topological charges at the edge.

There are sectors corresponding to different numbers of Laughlinesque charge e/m quasiparticles in the bulk:

$$\Psi = \prod_i z_i^n \prod_{j < k} (z_j - z_k)^m \prod_j e^{-|z_j|^2/4} \text{Pf} \left(\frac{1}{z_j - z_k} \right).$$

These different charge sectors correspond to the different sectors (or primary fields) of a chiral boson $\phi \equiv \phi + 2\pi\sqrt{m}$: $e^{in\phi/\sqrt{m}}$, $n = 0, 1, \dots, m-1$. As in the case of the Laughlin states, in each of these sectors there are edge excitations which correspond to the multiplication of (19) by a symmetric polynomial $S(z_1, z_2, \dots, z_N)$:

$$\Psi = S(z_1, z_2, \dots, z_N) \prod_i z_i^n \Psi_0$$

The low-degree symmetric polynomials are in one-to-one correspondence with the oscillator modes of a free chiral boson²⁹:

$$\mathcal{L}^{\text{charge}} = \frac{1}{4\pi} \partial_x \phi (\partial_t + v_c \partial_x) \phi. \quad (21)$$

These are the only edge excitations for the Laughlin states, but the Moore-Read state has fermionic edge excitations as well. Consider the following states for F even:

$$\Psi = \prod_i z_i^r \prod_{j < k} (z_j - z_k)^m \prod_j e^{-|z_j|^2/4} \times \mathcal{A} \left(z_1^{p_1} z_2^{p_2} \dots z_F^{p_F} \frac{1}{z_{F+1} - z_{F+2}} \frac{1}{z_{F+3} - z_{F+4}} \dots \right). \quad (22)$$

The antisymmetrization requires that we take $0 \leq p_1 < p_2 < \dots < p_F$. Therefore, there is an exclusion principle for these excitations: we are populating fermionic edge modes with neutral fermions obtained by breaking pairs (they are neutral because the charge density is unchanged). The angular momentum increase is:

$$\Delta M = \sum_i \left(p_i + \frac{1}{2} \right) \quad (23)$$

These excitations are in one-to-one correspondence with the basis states of a Majorana fermion:

$$\psi_{-p_F - \frac{1}{2}} \dots \psi_{-p_2 - \frac{1}{2}} \psi_{-p_1 - \frac{1}{2}} |0\rangle$$

with Lagrangian:

$$\mathcal{L}^{\text{neutral}} = i\psi(\partial_t + v_n \partial_x)\psi \quad (24)$$

From (23), we see that ψ has angular momentum quantized in half-integers in the sectors $e^{in\phi/\sqrt{m}}$, $n = 0, 1, \dots, m-1$.

Breaking pairs isn't the only way to populate these modes. We could also add an electron, so that the electron number is now odd. The ground state wavefunction of lowest angular momentum is

$$\Psi = \prod_i z_i^r \prod_{j < k} (z_j - z_k)^m \prod_j e^{-|z_j|^2/4} \times \mathcal{A} \left(z_1^0 \frac{1}{z_2 - z_3} \frac{1}{z_4 - z_5} \dots \right). \quad (25)$$

We have now added a neutral fermion to the system, giving us the odd fermion number sectors $\psi e^{ir\phi/\sqrt{m}}$, with $r = 0, 1, \dots, m-1$. We can, of course, multiply by symmetric polynomials to obtain bosonic oscillator excitations in these sectors as well. We can also break pairs as in (22) – but with F now odd – in order to populate an arbitrary odd number of fermionic modes.

The paired nature of the Moore-Read state allows for quasiparticles carrying half of a flux quantum and, therefore, charge $1/2m$. A wavefunction for a two-quasihole state may be written by exploiting the Pfaffian factor in (19) to split a Laughlin quasihole into two half-flux-quantum quasiholes at η_1 and η_2 :

$$\Psi = \prod_{j < k} (z_j - z_k)^m \prod_j e^{-|z_j|^2/4} \times \text{Pf} \left(\frac{(z_j - \eta_1)(z_k - \eta_2) + z_j \leftrightarrow z_k}{z_j - z_k} \right).$$

If we take η_1 to infinity and η_2 to the origin, we have a wavefunction for a state with one half-flux quantum quasihole:

$$\Psi = \prod_{j < k} (z_j - z_k)^m \prod_j e^{-|z_j|^2/4} \text{Pf} \left(\frac{z_j + z_k}{z_j - z_k} \right).$$

The extra factor of $z_j + z_k$ in the numerator gives each pair an additional unit of angular momentum. Majorana fermion edge excitations in this sector have wavefunction

$$\Psi = \prod_i z_i^s \prod_{j < k} (z_j - z_k)^m \prod_j e^{-|z_j|^2/4} \times \mathcal{A} \left(z_1^{p_1} z_2^{p_2} \dots z_F^{p_F} \frac{z_{F+1} + z_{F+2}}{z_{F+1} - z_{F+2}} \frac{z_{F+1} + z_{F+2}}{z_{F+3} - z_{F+4}} \dots \right).$$

As a result of the extra angular momentum of each pair, the angular momenta of these excitations takes integral values:

$$\Delta M = \sum_i p_i \quad (26)$$

Therefore, a half-flux quantum quasiparticle has the effect of changing the quantization condition on fermion momenta

from integer to half-integer values, in addition to the electrical charge it carries. Therefore, this is the $\sigma e^{i\phi/2\sqrt{m}}$ sector. The Ising spin field σ introduces a branch cut for fermions ψ , thereby shifting their angular momenta by half a unit. We discuss correlation functions of σ in detail in section VI.

Of course, we can also have an additional s quasi-holes at the origin, corresponding to topological charges $\sigma e^{(2s+1)i\phi/2\sqrt{m}}$:

$$\Psi = \prod_i z_i^s \prod_{j<k} (z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \text{Pf} \left(\frac{z_j + z_k}{z_j - z_k} \right).$$

These $3m$ sectors can essentially be divided into m different charge sectors and 3 neutral sectors, where the non-Abelian structure lies. The one subtlety is that the space of states is not simply a tensor product of charged and neutral sectors but only includes those invariant under the combined transformation $\sigma \rightarrow -\sigma$, $\phi \rightarrow \phi + 2\pi\sqrt{m}$.

To summarize, the edge excitations of a droplet of $\nu = 1/m$ Moore-Read liquid obey the Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{edge}}(\psi, \phi) &= \mathcal{L}_{\text{fermion}}(\psi) + \mathcal{L}_{\text{boson}}(\phi) \\ &= i\psi(\partial_t + v_n \partial_x)\psi + \frac{1}{2\pi} \partial_x \phi (\partial_t + v_c \partial_x \phi) \end{aligned} \quad (27)$$

with $\phi \equiv \phi + 2\pi\sqrt{m}$. The neutral and charge velocities v_n, v_c are, in general, different, and one expects that $v_n < v_c$. The normalization above is such that the operator $e^{ia\phi}$ has scaling dimension $a^2/2$, or equivalently that the two point function, $\langle e^{ia\phi(\tau)} e^{-ia\phi(\tau')} \rangle \sim |\tau - \tau'|^{-2d_a}$, evaluated for the Gaussian Lagrangian $\mathcal{L}_{\text{boson}}(\phi)$ decays as a power law with $d_a = a^2/2$. The different primary fields, i.e. topologically distinct quasiparticles, are:

$$\begin{aligned} \Phi_{q/m} &= e^{iq\phi/\sqrt{m}} \\ \Phi_{\psi, q/m} &= \psi e^{iq\phi/\sqrt{m}} \\ \Phi_{(2r+1)/2m} &= \sigma e^{i(2r+1)\phi/2\sqrt{m}} \end{aligned} \quad (28)$$

with $q, r = 0, 1, \dots, m-1$. There are also quasiparticles with $q \geq m$ or $r \geq m$, but these do not correspond to primary fields (they are, instead, descendant fields) because a quasiparticle with $q \geq m$ has the same topological properties as the quasiparticle with $q \rightarrow q(\text{mod } m)$ above, and similarly for $r \geq m$. $\Phi_0 = 1$ is the identity operator, which is topologically trivial and has the same quantum numbers as the vacuum. Other topologically trivial operators are descendants of the identity. There is one such descendant of the identity which is particularly important physically, namely the electron (fermionic for m even):

$$\Phi_{\text{el}} = \psi e^{i\phi\sqrt{m}} \quad (29)$$

Two other topologically trivial operators which will interest us later are an operator which annihilates a charge-2 boson, which we will interpret as a Cooper pair,

$$\Phi_{\text{pair}} = e^{2i\phi\sqrt{m}} \quad (30)$$

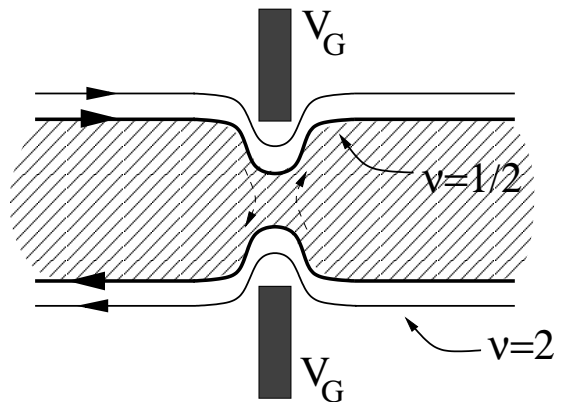


FIG. 1: A voltage V_G applied to gates on either side of a Hall bar forms a constriction, causing tunneling between the edges. For a weak constriction in a $\nu = 5/2$ state, quasiparticles can tunnel between the edges of the half-filled first excited Landau level. Tunneling between the integer quantum Hall edges of the filled lowest Landau level can be neglected because these edges are further apart.

and the fermion kinetic energy operator

$$\Phi_{\psi, \text{kin}} = \psi \partial_x \psi \quad (31)$$

which we can interpret as the creation/annihilation operator for a p -wave pair of neutral fermions. For the $\sigma_{xy} = (2 + \frac{1}{2}) \frac{e^2}{h}$ quantum Hall state, we take $m = 2$ in the above formulas.

V. THE POINT CONTACT

A voltage V_G applied to the gates on either side of a Hall droplet effectively pinches the droplet, as illustrated in figure 1. Quasiparticle tunneling between the edges, which is negligible when they are far apart, will now become important in the vicinity of the constriction. If the gate voltages are large, then the Hall bar will be cut in two by the gates so that there are two Hall droplets, as depicted in figure 2. On the other hand, if the droplet is not pinched too strongly, we might naively expect that the rate at which quasiparticles tunnel between the top and bottom edges will be small. However, as we will discuss in detail, a weak pinch will always become effectively stronger as the temperature is decreased, until it reaches the limit of a Hall droplet which is broken in two at zero temperature.

In the strong constriction limit, there is vacuum separating the two droplets, so only electrons and excitations which are made up of several electrons – i.e. topologically trivial excitations – can tunnel between the left and right droplets. (In a $\nu = 2 + \frac{1}{2}$ Moore-Read Pfaffian state, we want to consider the case in which there is $\nu = 2$ integer quantum Hall liquid, which also does not support fractional excitations, between the two $\nu = 5/2$ droplets.) Therefore, for example, a charge-1 boson cannot tunnel between the two droplets, for m even. Even though its charge is integral, it braids non-trivially with a charge $1/2m$ excitation.

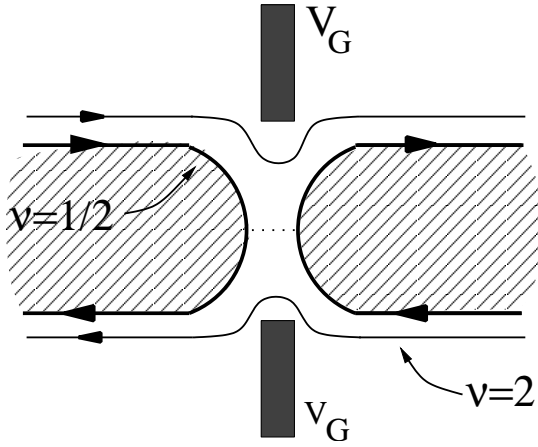


FIG. 2: When a large gate voltage V_G is applied, the quantum Hall droplet is broken in two. Electrons can tunnel between the two droplets. At $\nu = 5/2$, it is the $\nu = 1/2$ droplet in the first excited Landau level which is broken in two. The $\nu = 2$ integer quantum Hall droplet remains unbroken.

Let us first consider the action describing a strong constriction. There are two edges, a counterclockwise edge on the left, which we will denote by L ; and a clockwise edge on the right, which we will denote by R . We mark distance along both of these edges with a spatial coordinate x . The point contact is at $x = 0$. The most important operators coupling the two edges tunnel an electron, a pair of electrons, or a pair of neutral fermions from the point $x = 0$ on one edge to $x = 0$ on the other:

$$S_{\text{strong}} = \int d\tau \left[\int dx (\mathcal{L}_{\text{edge}}(\psi_L, \phi_L) + \mathcal{L}_{\text{edge}}(\psi_R, \phi_R)) + t_{\text{el}} \Phi_{\text{el},R}^\dagger(0) \Phi_{\text{el},L}(0) + t_{\text{pair}} \Phi_{\text{pair},R}^\dagger(0) \Phi_{\text{pair},L}(0) + t_{\psi,\text{kin}} \Phi_{\psi,\text{kin},R}^\dagger(0) \Phi_{\psi,\text{kin},L}(0) + \text{h.c.} \right] \quad (32)$$

A term coupling the kinetic energies of the charged bosons could also appear in this action, but it is less important than $t_{\psi,\text{kin}}$ at low temperatures and is not particularly important in the analysis which follows.

As for tunneling between Abelian quantum Hall edges, one can readily define a renormalization group (RG) transformation which leaves the edge Lagrangians invariant³⁰. The edge Lagrangian is an RG “fixed point”. To read off the scaling dimensions of the operators, it is convenient to write the action (32) in terms of the boson introduced in section IV:

$$S_{\text{strong}} = \int d\tau \left[\int dx (\mathcal{L}_{\text{edge}}(\psi_L, \phi_L) + \mathcal{L}_{\text{edge}}(\psi_R, \phi_R)) + t_{\text{el}} i\psi_R(0)\psi_L(0) \cos((\phi_R(0) - \phi_L(0))\sqrt{m}) + t_{\text{pair}} \cos(2(\phi_R(0) - \phi_L(0))\sqrt{m}) + t_{\psi,\text{kin}} (\psi_R(0)\partial_x\psi_R(0)) (\psi_L(0)\partial_x\psi_L(0)) \right] \quad (33)$$

Under this RG transformation, the leading terms for the

“flows” of the tunneling operators are:

$$\begin{aligned} \frac{d}{d\ell} t_{\text{el}} &= -m t_{\text{el}} \\ \frac{d}{d\ell} t_{\text{pair}} &= (1 - 4m) t_{\text{pair}} \\ \frac{d}{d\ell} t_{\psi,\text{kin}} &= -3 t_{\psi,\text{kin}} \end{aligned} \quad (34)$$

Since these operators are all irrelevant at the fixed point, the limit of two decoupled droplets is stable to weak inter-droplet tunneling.

The case $m = 2$ is relevant to the half-filled first excited Landau level in a Moore-Read Pfaffian state at $\nu = 2 + \frac{1}{2}$. Here t_{el} has RG eigenvalue -2 and is the least irrelevant coupling for both charge and energy transport. Consequently, the 4-terminal longitudinal resistance (see appendix A) scales as:

$$R_{xx} - \frac{1}{10} \frac{h}{e^2} \sim -t_{\text{el}}^2 T^4 \quad (35)$$

$t_{\psi,\text{kin}}$ is the first sub-leading irrelevant operator coupling the two droplets, but it does not contribute to charge transport. The leading sub-dominant contribution to the charge transport between the two droplets is thus from the pair tunneling term, t_{pair} , which is strongly irrelevant. If t_{el} is tuned to zero, then

$$R_{xx} - \frac{1}{10} \frac{h}{e^2} \sim -t_{\text{pair}}^2 T^{14} \quad (36)$$

In the case of a $p + ip$ superconductor, there is no charged mode, so $t_{\psi,\text{kin}}$ is the only one of these three couplings which can occur. It is the most relevant coupling between the edge modes of two such superconductors.

Now we turn our attention to the case of a weak constriction. In this case, quasiparticles can tunnel across the bulk of a Hall droplet, as in figure 1. In the $\nu = 2 + \frac{1}{2}$ case, we assume that tunneling only occurs between the $\nu = \frac{1}{2}$ edges and ignore tunneling between the $\nu = 2$ integer quantum Hall edges, which are further apart. It is convenient to treat the top and bottom edges of the bar as independent, but we must keep in mind that, ultimately, they are the two edges of the same bar. We will use the subscripts a and b for the top and bottom edges so that ψ_a, ϕ_a are the fermion and boson operators at edge a and ψ_b, ϕ_b are the corresponding operators at edge b . As drawn in the figure, the a modes are right-moving and the b modes are left-moving.

At the point contact, which we will assume is at $x = 0$, the two edges are coupled by quasiparticle tunneling. There are no restrictions on what kind of quasiparticles can tunnel at the point contact. In general, we must consider not only primary fields but also all of their descendants. However, descendant fields have higher scaling dimensions than primaries; as a consequence, the tunneling of descendants is strongly irrelevant. For instance, the tunneling of electrons, electron pairs, and neutral fermion pairs – all due to descendant fields – can occur not only between droplets but also across a droplet. As we saw above, they are all irrelevant.

Hence, if we retain only the tunneling of primary fields, the action will take the form

$$\begin{aligned}
S_{\text{weak}} = & \int d\tau \left[\int dx (\mathcal{L}_{\text{edge}}(\psi_a, \phi_a) + \mathcal{L}_{\text{edge}}(\psi_b, \phi_b)) \right. \\
& + \sum_{r=0}^{m-1} \left(\lambda_{(2r+1)/2m} \Phi_{(2r+1)/2m}^{a,\dagger}(0) \Phi_{(2r+1)/2m}^b(0) + \text{h.c.} \right) \\
& + \sum_{q=0}^{m-1} \left(\lambda_{\psi,q/m} i \Phi_{\psi,q/m}^{a,\dagger}(0) \Phi_{\psi,q/m}^b(0) + \text{h.c.} \right) \\
& \left. \sum_{q=1}^{m-1} \left(\lambda_{q/m} \Phi_{q/m}^{a,\dagger}(0) \Phi_{q/m}^b(0) + \text{h.c.} \right) \right] \quad (37)
\end{aligned}$$

($q = 0$ is omitted from the last sum because it is simply the identity operator.) The leading terms in the RG equations for the couplings above are:

$$\begin{aligned}
\frac{d}{d\ell} \lambda_{(2r+1)/2m} &= \left(\frac{7}{8} - \frac{(2r+1)^2}{4m} \right) \lambda_{(2r+1)/2m} \\
\frac{d}{d\ell} \lambda_{\psi,q/m} &= -\frac{q^2}{m} \lambda_{\psi,q/m} \\
\frac{d}{d\ell} \lambda_{q/m} &= \left(1 - \frac{q^2}{m} \right) \lambda_{q/m} \quad (38)
\end{aligned}$$

In this equation, we have used the known scaling dimension of the chiral part of the Ising spin field, $1/16$.

We now specialize to the two cases of greatest experimental interest, a possible Moore-Read Pfaffian state at $\nu = 2 + \frac{1}{2}$ and a $p + ip$ superconductor. A Moore-Read Pfaffian state at $\nu = 2 + \frac{1}{2}$ corresponds to $m = 2$. Keeping only the terms in the action which are not irrelevant, we have:

$$\begin{aligned}
S = & \int d\tau \left[\int dx (\mathcal{L}_{\text{edge}}(\psi_a, \phi_a) + \mathcal{L}_{\text{edge}}(\psi_b, \phi_b)) \right. \\
& + \lambda_{1/2} \cos((\phi_a(0) - \phi_b(0))/\sqrt{2}) + \lambda_{\psi,0} i \psi_a \psi_b \\
& \left. + \lambda_{1/4} \sigma_a \sigma_b \cos((\phi_a(0) - \phi_b(0))/2\sqrt{2}) \right] \quad (39)
\end{aligned}$$

In this action we have labeled the σ field in the same fashion as the other fields: by an index a and b indicating which edge it is on. However, these sigma fields can be entangled with each other as well as with other fields, and so defining the action precisely requires more information than just the a, b labels. We discuss this issue in depth in section VI. The leading weak-tunneling corrections derived in this section are not affected by this (important) subtlety.

The RG equations to lowest order for the three couplings in (39) are:

$$\begin{aligned}
\frac{d}{d\ell} \lambda_{1/2} &= \frac{1}{2} \lambda_{1/2} \\
\frac{d}{d\ell} \lambda_{\psi,0} &= 0 \\
\frac{d}{d\ell} \lambda_{1/4} &= \frac{3}{4} \lambda_{1/4} \quad (40)
\end{aligned}$$

Since $\lambda_{1/4}$ and $\lambda_{1/2}$ are relevant couplings, the weak tunneling limit is unstable. The longitudinal resistivity increases

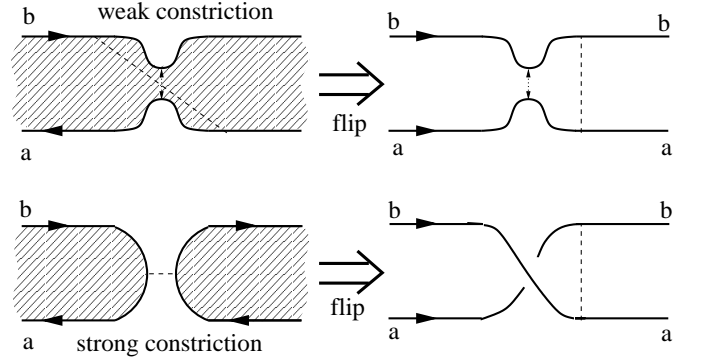


FIG. 3: We redraw the point contact with the bottom edge flipped so that the description is completely chiral. In the strong constriction limit, an incoming a mode becomes an outgoing b mode and vice versa. The dotted line with arrows on both ends in the top two pictures represents the tunneling path between the top and bottom edges in the weak constriction limit. The dashed line in the bottom two pictures represents a tunneling path from the left droplet to the right one. This dashed line is transposed to the top two pictures to illustrate how these tunneling paths cross in real space but are parallel in the flipped representation.

with decreasing temperature:

$$R_{xx} \sim \lambda_{1/4}^2 T^{-3/2} \quad (41)$$

If $\lambda_{1/4}$ were tuned to zero, we would instead have $R_{xx} \sim \lambda_{1/2}^2/T$. Since both $\lambda_{1/4}$ and $\lambda_{1/2}$ grow as the temperature is decreased, we expect that the constriction will become strong and the droplet will be effectively broken into two droplets. In the subsequent sections we will show that this is the case³⁵ and describe the crossover between these two limits.

In the preceding discussion, we have represented edge excitations at the top and bottom edges of the droplet as opposite chirality $(1+1)$ -D theories. However, we are not required to do so since tunneling occurs at only a single point. We can exploit a trick used in the analysis of the related (but distinct) problem of the Ising model with a defect line^{45,46}. If we flip the bottom edge, as shown in figure 3, then both edge modes are right-moving. We now have the problem of a point defect in a purely chiral theory. In the strong constriction limit, the incoming chiral modes are exchanged as they pass through the contact at $x = 0$, as depicted in figure 3. The (irrelevant) tunneling processes which transfer electrons between the two droplets can be implemented with chiral operators acting ‘downstream’ from the point contact. This is depicted by the dashed lines in figure 3. Note that when such an operator is transposed back to the unflipped picture for the weak constriction limit (the upper left of figure 3), the dashed electron tunneling path crosses the dotted quasiparticle tunneling path. Since quasiparticle and electron tunneling operators commute, the corresponding chiral operators in the upper right of figure 3 also commute.

This ‘flipped’ representation will prove to be more convenient, as we will see in section VIII. Even before we get the real payoff in that section, however, we can benefit

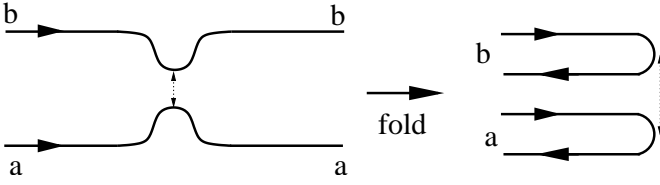


FIG. 4: We can fold the $x > 0$ half-plane onto the $x < 0$ half-plane so that right-moving modes in the $x > 0$ half-plane now become left-moving modes in the $x < 0$ half-plane. The resulting non-chiral modes are coupled only at the origin.

from a minor simplification. ϕ_a and ϕ_b are now both right-moving chiral bosons. We can form their sum and difference: $\phi_c = (\phi_a + \phi_b)/\sqrt{2}$, $\phi_\rho = (\phi_a - \phi_b)/\sqrt{2}$. Only ϕ_ρ is affected by the point contact. ϕ_c decouples completely, so we drop it from the action:

$$S = \int d\tau dx (\mathcal{L}_{\text{fermion}}(\psi_a) + \mathcal{L}_{\text{fermion}}(\psi_b) + \mathcal{L}_{\text{boson}}(\phi_\rho)) \\ + \int d\tau \lambda_{1/2} \cos \phi_\rho + \int d\tau \lambda_{\psi,0} i\psi_a \psi_b \\ + \int d\tau \lambda_{1/4} \sigma_a \sigma_b \cos(\phi_\rho/2) \quad (42)$$

We can now recast this problem as a boundary conformal field theory problem by folding the $x > 0$ half-plane onto the $x < 0$ half-plane, as shown in figure 4. The $x > 0$ part of right-moving modes now become left-moving modes in the $x < 0$ half-plane: $\phi_{\rho R}(x) \equiv \phi_\rho(x)$, $\phi_{\rho L}(x) \equiv \phi_\rho(-x)$, for $x < 0$ and similarly for $\psi_{a,b}$.

In the case of a $p + ip$ superconductor, there is no charged mode. Dropping irrelevant terms, the action in the ‘flipped’ representation is simply

$$S = \int d\tau dx (\mathcal{L}_{\text{fermion}}(\psi_a) + \mathcal{L}_{\text{fermion}}(\psi_b)) \\ + \int d\tau \lambda_\psi i\psi_a \psi_b + \int d\tau \lambda_\sigma \sigma_a \sigma_b \quad (43)$$

with

$$\frac{d}{d\ell} \lambda_\psi = 0 \\ \frac{d}{d\ell} \lambda_\sigma = \frac{7}{8} \lambda_\sigma \quad (44)$$

VI. CONFORMAL BLOCKS AND TUNNELING OPERATORS

In order to follow the crossover from the unstable weak tunneling limit to the limit of two droplets, we need to go beyond lowest-order perturbation theory in the relevant couplings described in the previous section. In so doing, we see that the preceding discussion requires refinement in one important respect. Terms of the form $\sigma_a \sigma_b$ in the above actions (37) and (43) are not well defined without additional information. The

technical reason is that the correlation functions of the Ising spin field (as defined by taking the continuum limit of the lattice model at its critical point, or by using conformal field theory) do not factor into a product of a right-moving and a left-moving part. For instance, the four-point function of the non-chiral spin field $\sigma(z, \bar{z})$ on the plane is of the form

$$\langle \sigma(z_1, \bar{z}_1) \sigma(z_2, \bar{z}_2) \sigma(z_3, \bar{z}_3) \sigma(z_4, \bar{z}_4) \rangle = |\mathcal{F}_I(z)|^2 + |\mathcal{F}_\psi(z)|^2 \quad (45)$$

where the *conformal blocks* \mathcal{F}_I and \mathcal{F}_ψ depend only on the z coordinates with no \bar{z} dependence. The sum in (45) means however that one *cannot* simply decompose $\sigma(z, \bar{z})$ into a product $\sigma(z)\sigma(\bar{z})$. Therefore, the correlation functions of, say, σ_a are not well defined until more information is provided. In the four-point case, the correlation function could be \mathcal{F}_I , \mathcal{F}_ψ , or even any linear combination of the two.

The physical reason underlying this ambiguity is the reason why the problem is so interesting: the non-abelian braiding. Quasiparticles on the edge can be entangled with others, even if they are far away. Moreover, they also have non-trivial braiding with bulk quasiparticles: when a fermion goes around the disk, the sign the wavefunction picks up depends on the number of σ quasiparticles in the bulk. (Thus the boundary conditions on the edge theory depend on whether the number of bulk quasiparticles is even or odd.)

These ambiguities are resolved by specifying the *fusion channels* of the fields in addition to their positions in space-time. As we saw in (18, two chiral spin fields can fuse either to the identity field I or the fermion ψ . What this means is that two spin fields form a two-state quantum system, with I and ψ as the basis elements. Consequently, conformal blocks of four chiral spin fields form a two-dimensional vector space. The conformal blocks in (45) \mathcal{F}_I and \mathcal{F}_ψ are those which the chiral fields at z_1 and z_2 fuse to I and ψ respectively.

In this section, we define unambiguously the tunneling operators \mathcal{T}_σ and \mathcal{T}_ψ , which correspond respectively to tunneling a σ and ψ quasiparticle across a $p + ip$ superconductor. This amounts to defining their conformal blocks uniquely. In the subsequent sections, we exploit the work of ref. 40 to express these blocks in terms of correlators of bosonic fields. This will enable us to derive the surprising result that \mathcal{T}_σ for a simple point contact is equivalent to the interaction in the single-channel anisotropic Kondo problem.

The effects of the non-Abelian statistics are independent of the charged mode, so in this section, for simplicity, we study only the $p + ip$ superconductor, where there are σ and ψ quasiparticles. In the edge conformal field theory, all descendant fields are irrelevant, so the only fields we need to worry about are the σ and ψ primary fields. Since there is only one fusion channel for the edge fermion ψ , its tunneling operator is easy to define. Since the edge theory is conformally invariant, the correlations for any shape droplet can be obtained from those on a circular disk. We can then label the position on the edge by an angular coordinate $0 \leq \theta < 2\pi$ going around the circle. For simplicity, we put the contact in the middle of the circular disc, so that it tunnels particles between $\theta = \pi/2$ and $\theta = 3\pi/2$. Tunneling a fermion at Euclidean time τ then

corresponds to the operator

$$\mathcal{T}_\psi(\tau) = \psi(\tau + i\pi/2)\psi(\tau + i3\pi/2) \quad (46)$$

The fermion is free, so correlators of \mathcal{T}_ψ are trivial to compute. Since ψ is of dimension $1/2$, \mathcal{T}_ψ is of dimension 1, and is marginal. The operator \mathcal{T}_σ is relevant, but we will show how the presence of \mathcal{T}_ψ still plays a crucial role in understanding the ‘‘infrared fixed point’’, which describes the low-temperature limit of two decoupled droplets. (We refer to the opposite limit of a very weak pinch across the Hall bar as the ‘‘ultraviolet fixed point’’, since it describes the behavior at temperatures much higher than the scale where the system crosses over to two decoupled drops.)

The interesting complications occur for tunneling σ quasiparticles. In order to properly define the tunneling operator \mathcal{T}_σ , we need to account for the fact that our two edges are, in fact, different sections of the same edge, bounding a single Hall droplet. In figures 1-4, the edges going off to the left are connected to each other, and likewise for those on the right.

The fact that the both point contacts are a single edge means that the fields denoted in the last section by σ_a and σ_b are part of the same edge conformal field theory. When defining their conformal blocks, we must therefore specify the appropriate fusion channels. As shown in the fusion rules (18), two σ fields can fuse to either the identity or the fermion. Tunneling a quasiparticle at Euclidean time τ involves two quasiparticle fields: one to annihilate a quasiparticle, and the other to create one on the other side. To define this operator, we must therefore specify which fusion channel these two fields are in. *For a simple point contact, this must be the identity channel.* The reason is that simply transferring a quasiparticle from one point to another nearby point cannot create a neutral fermion. One can of course imagine more complicated physical situations, where it is possible for the tunneling to change fusion channels. For example, if there were an antidot with the point contact, the tunneling could be in the ψ channel, with a compensating ψ particle created at the anti-dot. Consequently, a tunneling event in the ψ channel corresponds to changing the topological charge on the antidot by fusing with ψ – if the antidot were originally in the I channel, the tunnelling event would leave it in the ψ channel, while if it were originally in the ψ channel it would be left in the I channel. Such point contacts can be analyzed using the formalism we develop here, but the physics becomes considerably more involved. We therefore focus in this paper on a simple point contact.

To explain in more detail what it means for the tunneling operator to be in the identity channel, we discuss its conformal blocks. A convenient pictorial representation of a chiral σ field at z_1 is given by the trivalent vertices

$$\begin{array}{c} 1 \\ | \\ c \end{array} \quad \text{or} \quad \begin{array}{c} 1 \\ | \\ c \end{array}$$

where $c=I$ or ψ . (In this figure, the 1 refers to the spacetime point z_1 and must not be confused with the identity label, I .) Such operators are generally known in the mathematical literature as ‘‘chiral vertex operators’’. In general, one labels all the

legs of the vertex; here we adopt the convention that unlabeled legs correspond to the σ channel.

A non-vanishing conformal block of $2n$ σ fields located at $z_1 \dots z_{2n}$ is then pictorially represented as

$$\begin{array}{ccccccccccc} & 1 & 2 & & 3 & 4 & & & & 2n-1 & 2n \\ I & | & | & c_1 & | & | & c_2 & \dots & c_{n-1} & | & | & I \end{array}$$

where c_j represents the fusion channel for the first $2j$ particles; we must have $c_0 = c_{2n} = I$ for the conformal block to be non-vanishing. This means that for a conformal block to be non-vanishing, the fusion channels for all the operators combined must be the identity. Each choice of $c_j = I, \psi$, with $j = 1, \dots, n-1$, corresponds to a basis element for the vector space of $2n$ -point conformal blocks. In ref. 35, we introduced the notation $[m_1, \dots, m_n]$ with $m_i = 0, 1$ for such a conformal block. The relation with the pictures above is $c_j = I$ if $\sum_{i=1}^j m_i \equiv 0 \pmod{2}$ and $c_j = \psi$ if $\sum_{i=1}^j m_i \equiv 1 \pmod{2}$.

An arbitrary conformal block of $2n$ σ fields is therefore a linear combination of these 2^{n-1} basis elements. Again, we see that the quantum dimension of σ is $\sqrt{2}$. Conformal blocks with four σ quasiparticles form a two-dimensional vector space. Thus they effectively form a two-state quantum system, which can be used as a qubit in a topological quantum computer.

In this pictorial notation, we then have for the tunneling operator of a simple point contact

$$\mathcal{T}_\sigma \equiv \begin{array}{ccc} & 1 & 1' \\ c & | & | \\ & c & c \end{array} \quad (47)$$

where 1 represents the spacetime location $\theta = \pi/2, \tau = \tau_1$, while $1'$ represents the spacetime location $\theta = 3\pi/2, \tau = \tau_1$. The index c can be either I or ψ ; the fact that a is the same on both sides here is the precise meaning of \mathcal{T}_σ being in the identity channel.

The tunneling Lagrangian for a point contact in a $p + ip$ superconductor is therefore

$$\mathcal{L}_{\text{tun}} = \lambda_\psi \mathcal{T}_\psi(\tau) + \lambda_\sigma \mathcal{T}_\sigma(\tau)$$

with $\mathcal{T}_\sigma(\tau)$ defined as in (47). The partition function is then defined perturbatively by expanding in powers of λ_σ and λ_ψ . The coefficients are the conformal blocks defined by the requirement that \mathcal{T}_σ be in the identity channel. Such conformal blocks of \mathcal{T}_σ then have all $c_j = I$, i.e. $\langle \mathcal{T}_\sigma(\tau_1) \mathcal{T}_\sigma(\tau_2) \dots \mathcal{T}_\sigma(\tau_j) \rangle$ is pictorially represented as

$$\begin{array}{ccccccccccc} & 1 & 1' & & 2 & 2' & & & & j & j' \\ I & | & | & I & | & | & I & \dots & I & | & | & I \end{array}$$

To make this more concrete, we give the simplest cases explicitly. There is only one vanishing two-point function of two sigma operators, which on the plane is

$$\begin{array}{ccc} & 1 & 2 \\ I & | & | \\ & I & I \end{array} = \frac{1}{(z_1 - z_2)^{1/8}}$$

The four-point conformal blocks \mathcal{F}_c discussed above were defined by demanding that the fields at z_1 and z_2 fuse to the c channel, so

$$\mathcal{F}_c \equiv \begin{array}{cccc} & 1 & 2 & & 3 & 4 & & \\ I & | & | & c & | & | & I & . \end{array}$$

Explicitly, one finds³⁹

$$\begin{aligned} \mathcal{F}_I &= \left(\frac{1}{z_{12}z_{34}x(1-x)} \right)^{1/8} (1 + \sqrt{1-x})^{1/2}, \\ \mathcal{F}_\psi &= \left(\frac{1}{z_{12}z_{34}x(1-x)} \right)^{1/8} (1 - \sqrt{1-x})^{1/2}, \end{aligned} \quad (48)$$

where $z_{ij} = z_i - z_j$ and $x = z_{12}z_{34}/z_{13}z_{24}$.

More complicated conformal blocks can be computed explicitly by using the algebraic structure of the conformal field theory to derive differential equations for them; specifying the fusion channels for a given conformal block then amounts to choosing the particular solution of the differential equation. Computing conformal blocks in this fashion gets very tedious beyond low orders. In section VIII we therefore explain how to compute them using a much simpler procedure: bosonization.

VII. CLUSTER DECOMPOSITION OF CONFORMAL BLOCKS

In the previous section, we showed how to properly define the quasiparticle tunneling operator. One notable feature of the definition is that a quasiparticle on one side of the point contact will be entangled with the quasihole which was left behind on the other side when it tunneled across. This is necessitated by the physics of non-Abelian statistics. However, this kind of non-local correlation makes it difficult to treat the two edges as independent. Nonetheless, with a little work we can disentangle the fields on the two edges, enabling us in the next section to bosonize the quasiparticle tunneling operator.

It should be possible to treat the two σ fields in \mathcal{T}_σ as independent when the circumference of the disc is “long”, meaning that the radius R of the disk is much larger than the inverse temperature β . In this limit, conformal blocks of tunneling operators obey cluster decomposition: correlations between fields on the opposite sides fall off as some power of β/R relative to correlations of fields at the same spatial point but at different times. The tension here is that we would like to treat $\sigma(\pi/2)$ and $\sigma(3\pi/2)$ as two independent fields, σ_a and σ_b , since the points $\theta = \pi/2$ and $\theta = 3\pi/2$ are far apart so their correlations satisfy cluster decomposition. On the other hand, the tunneling operator is defined by the fusion channel of $\sigma(\pi/2)$ and $\sigma(3\pi/2)$. Although $\theta = \pi/2$ and $\theta = 3\pi/2$ are far apart, the choice of fusion channel is topological and is insensitive to the distance between these points. Since $\sigma(\pi/2)$ and $\sigma(3\pi/2)$ are entangled in this way, we cannot cluster decompose their correlation functions even though $\theta = \pi/2$ and $\theta = 3\pi/2$ are far apart.

The resolution is to switch into a basis in which we specify the fusion of σ fields at the same spatial point. Consider the four-point conformal blocks. We can define an alternate basis \mathcal{G}_c , for this two-dimensional vector space:

$$\mathcal{G}_c \equiv \begin{array}{cccc} & 1 & 2 & & 1' & 2' & & \\ I & | & | & c & | & | & I & . \end{array}$$

with $c = I, \psi$. In this diagram, the number j represents the spacetime location ($\theta = \pi/2, \tau = \tau_j$), while j' represents the spacetime location ($\theta = 3\pi/2, \tau = \tau_j$). Both bases, \mathcal{F}_c and \mathcal{G}_c , are perfectly valid. The advantage of the \mathcal{F}_c basis is that it arises naturally in perturbation theory. The advantage of the \mathcal{G}_c basis is that $\sigma(\pi/2)$ and $\sigma(3\pi/2)$ are not entangled with each other. Cluster decomposition then gives \mathcal{G}_I and \mathcal{G}_ψ each to be the product of two-point functions for each edge (to leading order in β/R). This means that $\mathcal{G}_\psi = 0$ to this order, because it doesn't have I on both ends. Neglecting terms suppressed by powers of β/R , we have

$$\begin{aligned} \mathcal{G}_I &= \begin{array}{cccc} & 1 & 2 & & 1' & 2' & & \\ I & | & | & I & | & | & I & . \end{array} \\ &= \left(\frac{1}{\sin(\tau_1 - \tau_2)} \right)^{1/4} \end{aligned} \quad (49)$$

To obtain the latter we conformally map the punctured plane to a infinitely-long cylinder by using $w = \exp(\theta R/\beta + i\tau)$. These results for \mathcal{G}_I and \mathcal{G}_ψ can easily be checked by taking $R \gg \beta$ limit of the full four-point conformal blocks, which were originally derived in ref. 39.

When the $2n$ -point correlation function is decomposed into the product of the n -point correlation function of σ_a s multiplied by the n -point correlation function of σ_b s, we must then consider the conformal blocks for each of these. In ref. 35, we introduced the notation $(m_1, \dots, m_{n/2})_{a(b)}$ with $m_i = 0, 1$ to specify the conformal blocks on the a(b) edges, respectively. In terms of the “vertex” notation described above,

$$\begin{aligned} (m_1, \dots, m_{n/2})_{a(b)} &= \\ & \begin{array}{cccc} & 1 & 2 & & 3 & 4 & & & n-1 & n & & \\ I & | & | & c_1 & | & | & c_2 & \dots & c_{\frac{n}{2}-1} & | & | & I \end{array} \end{aligned} \quad (50)$$

with $c_j = I$ or ψ if $\sum_{i=1}^j m_i$ is $0 \pmod{2}$ or $1 \pmod{2}$ respectively. In this notation, $\mathcal{G}_I = (0)_a \cdot (0)_b$, and $\mathcal{G}_\psi = (1)_a \cdot (1)_b = 0$.

In order to make use of the \mathcal{G}_c basis, we must transform to it from the \mathcal{F}_c basis which arises in perturbation theory for the tunneling operator. Consider

$$\langle \mathcal{T}_\sigma(\tau_1) \mathcal{T}_\sigma(\tau_2) \rangle = \begin{array}{cccc} & 1 & 1' & & 2 & 2' & & \\ I & | & | & I & | & | & I & . \end{array}$$

which is simply \mathcal{F}_I . As Moore and Seiberg⁴⁰ explained in an in-depth analysis of the topological properties of conformal

blocks, there is a linear transformation which connects this to the \mathcal{G}_c basis:

$$\mathcal{F}_c = \sum_d \mathcal{B}_{cd} \mathcal{G}_d \quad (51)$$

Moore and Seiberg⁴⁰ showed that the requirement that one obtains the same non-chiral correlators from any basis for conformal blocks results in a huge number of constraints on the matrix \mathcal{B}_{cd} . This matrix is called the ‘‘braid matrix’’, for fairly obvious reasons: the only difference between \mathcal{F} and \mathcal{G} is the exchange or braid of 2 with 1'. For the Ising model, the braid matrix for two σ fields with I or ψ (such as, for instance, 1' and 2) connecting them is⁴⁰

$$\mathcal{B} = \frac{e^{-i\pi/8}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad (52)$$

This braid matrix can be applied not only to the conformal blocks (48), but to any conformal block (tensored by the identity acting on all other indices).

Here, we are using a braiding operation in a ‘passive’ sense: not as an actual physical braid, but as a change of basis. Indeed, any element of the braid group can be viewed either in an ‘active’ sense (moving particles around) or in a ‘passive’ sense of a change of basis. The braid matrices associated with this chiral conformal field theory are the same as those associated with the Chern-Simons topological field theory for the bulk state⁴¹. In the latter context, it is more natural to view the braid in the ‘active’ sense: as quasiparticles are exchanged in the bulk, different topologically degenerate states are transformed into each other. The elements of the braid matrices can be viewed as the amplitudes for various quasiparticle histories, which can be computed from the Jones-Kauffman invariants of knot theory^{47,48}. For any topological class of quasiparticle trajectories, we can thereby associate a braid matrix.

Therein lies a quandary: what braiding operation should we use? If we simply use \mathcal{B} , then we obtain

$$\langle \mathcal{T}(\tau_1) \mathcal{T}(\tau_2) \rangle = \mathcal{F}_I = e^{i\pi/8} (\mathcal{G}_I + i\mathcal{G}_\psi). \quad (53)$$

However, we could just as well use \mathcal{B}^{-1} , which exchanges 1' and 2 in a clockwise (rather than a counterclockwise) manner. Indeed, we could then follow this with any braid which is diagonal in the \mathcal{G}_a basis, such as winding 1' around 2' any number of times. So which braid (interpreted in a passive sense) should we use? Said differently, there are many possible bases \mathcal{G}_c which are diagonal in the fusion 1, 2 and 1', 2' fusion channels. Which of these should we use to decompose the σ multi-point correlation functions?

To answer this question, let us take a closer look at the tunneling process. We can view it as the creation of a quasiparticle-quasihole pair at the middle of the junction and the subsequent motion of the quasiparticle to one edge at 1 and the quasihole to the other edge at 1', as depicted in figure 5. Since the quasiparticle-quasihole pair is created out of the ground state, it fuses to the identity, as we argued in the previous section. The next tunneling process occurs in the same way: a quasiparticle-quasihole pair (which fuses to the identity) is created from the ground state, and the quasiparticle and

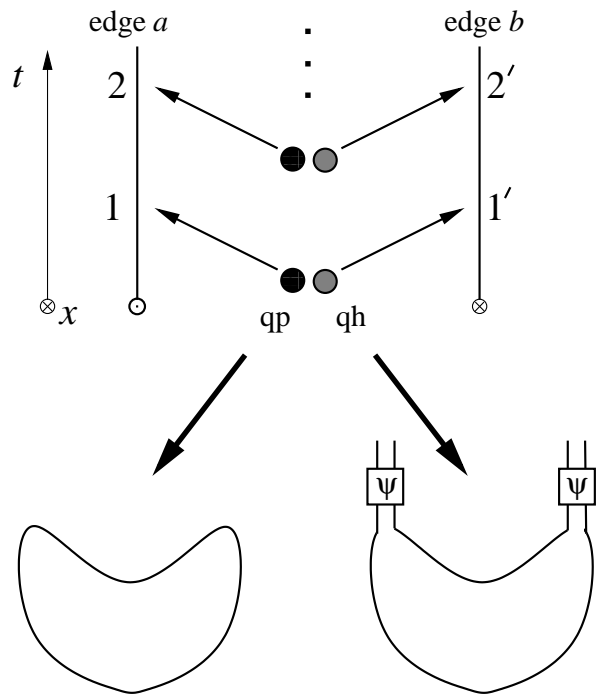


FIG. 5: Spacetime history of quasiparticle-quasihole pair creation processes contributing to transport across the point contact. A pair is created at the middle of the point contact; the quasiparticle moves to one edge and the quasihole moves to the other. When two such processes happen in succession, we can ask how the two quasiparticles which end up on the same edge fuse. The amplitude for such fusion into the 1 and ψ channels is given by the Jones-Kauffman bracket evaluation of the knot diagrams at the bottom left and right of the figure.

quasihole move to the two edges at 2 and 2'. We would now like to know how 1 and 2 fuse and how 1' and 2' fuse. At the bottom of figure 5, we have redrawn the spacetime history of the four quasiparticles in the form of knot diagrams which specify the desired basis change.

However, there is still an ambiguity in the change of basis. Because the quasiparticles are created by chiral fields, they are not invariant under a rotation by 2π : they pick up phases, just like a fermion picks up a minus sign. To keep track of these phases, we must give a framing to these histories. This is done pictorially by thickening these lines into ribbons. This is physically natural since quasiparticles have a finite size. It is also mathematically essential; otherwise, the distinction between different braids will be lost. For instance, without the framing, the two pictures in figure 6b will be topologically equivalent, even though they are associated with different braids in 6c. The framing is determined by the physics of the situation. In this case, the geometry of the tunnel junction prefers the ‘blackboard framing’, in which the curve in figure 6a is thickened into a ribbon which is contained entirely in the plane of the page (or the proverbial blackboard on which it is drawn). In any other history, one of the quasiparticles would have to spin around, as in the picture on the right side of figure 6a, which would be energetically costly.

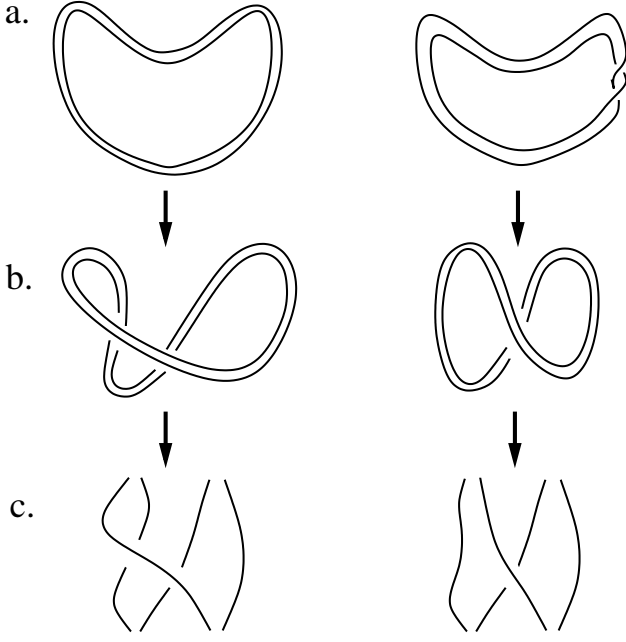


FIG. 6: (a) Two different histories in which two quasiparticle quasipoles are created and the two quasipoles are fused and the two quasipoles are fused. In the history on the right, one of the quasipoles twists by 2π before fusing. (b) A topologically equivalent set of histories in which the pair creations and annihilations occur at the same times. (c) The braid group elements corresponding to these histories.

In general, when we have four operators in a row, i.e.

$$\dots \quad \begin{array}{c} 2j-1 \quad (2j-1)' \quad 2j \quad (2j)' \\ \hline I \quad | \quad | \quad I \quad | \quad | \quad I \end{array} \quad \dots$$

we braid the operators at $(2j-1)'$ and $2j$ and then braid $(2j-1)'$ with $(2j)'$. We then obtain the ordering

$$\dots \quad \begin{array}{c} 2j-1 \quad 2j \quad (2j)' \quad (2j-1)' \\ \hline I \quad | \quad | \quad c \quad | \quad | \quad I \end{array} \quad \dots \quad (54)$$

To determine c , we need the relation⁴⁰

$$\begin{array}{c} a \\ \hline c \quad | \quad b \end{array} = e^{i\pi(h_a+h_b-h_c)} \begin{array}{c} b \\ \hline c \quad | \quad a \end{array} \quad (55)$$

where h_a is the chiral scaling dimension of the operator labelled by a . (For more complicated theories than the Ising model, there can be extra signs in (55), but they do not appear here.) We have already seen that $h_\sigma = 1/16$ and $h_\psi = 1/2$. Putting this all together yields $c = I + \psi$, i.e.⁴⁰

$$\begin{aligned} & \langle \dots \mathcal{T}_\sigma(\tau_{2j-1}) \mathcal{T}_\sigma(\tau_{2j}) \dots \rangle \\ &= \dots \quad \begin{array}{c} 2j-1 \quad 2j \quad (2j)' \quad (2j-1)' \\ \hline I \quad | \quad | \quad I + \psi \quad | \quad | \quad I \end{array} \quad \dots \quad (56) \end{aligned}$$

Recall that conformal blocks are elements of a vector space, so $I + \psi$ just means the sum of the blocks with a I and a ψ in the middle. Using this braiding, we can now unambiguously decompose any of our conformal blocks into the product of conformal blocks on the two edges.

VIII. BOSONIZING CONFORMAL BLOCKS

We now describe a method which not only allows us to compute conformal blocks explicitly, but also to write \mathcal{T}_σ in a form in which the physics is much clearer. This is to use *bosonization* to write \mathcal{T}_σ in terms of a free boson ϕ_σ . (This is not to be confused with the charge boson ϕ_ρ associated with the Moore-Read state, but rather a new boson introduced to make computation of the conformal blocks possible.)

It has long been known how to write the correlation functions of the non-chiral Ising model in terms of the correlation functions of a free boson⁵¹. The trick is to note that one can combine two independent Majorana fermions into a single Dirac fermion, which can be bosonized. The current is bosonized as:

$$i\psi_a\psi_b = \frac{1}{2\pi} \partial_x \phi_\sigma. \quad (57)$$

Meanwhile, the Dirac fermion

$$\psi_a + i\psi_b = e^{i\phi_\sigma}. \quad (58)$$

In order to use this method, we need (1) to treat the two edges as independent and (2) to use the ‘flipped’ setup; since both of the Majorana modes then have the same chirality, they can be combined into a Dirac fermion.

However, representing the chiral spin field, $\sigma(z)$, is trickier. The full non-chiral spin fields $\sigma_a(z, \bar{z})$ and $\sigma_b(z, \bar{z})$ can be bosonized according to⁵¹

$$\sigma_a(z, \bar{z})\sigma_b(z, \bar{z}) = \sqrt{2} \cos((\phi_\sigma(z) + \bar{\phi}_\sigma(\bar{z}))/2) \quad (59)$$

where we have now introduced $\bar{\phi}_\sigma(\bar{z})$, which is a left-moving counterpart to $\phi_\sigma(z)$. The square of any non-chiral Ising correlator can be written as a product of a correlator of fields with label a with a correlator with b labels, since the a and b fields are independent. Since correlators of exponentials of a boson are easily evaluated, the correlators of $\cos((\phi_\sigma + \bar{\phi}_\sigma)/2)$ are also easy to find, yielding the Ising correlator as their square root.

We now show how to extend (59) to write the conformal blocks of the tunneling operator \mathcal{T}_σ in terms of bosons. The formula (59) does not decompose into a product of chiralities, so it takes more work to find bosonic expressions for chiral sigma fields. One method is to use the fact that the product of *disorder* fields has a bosonic expression

$$\mu_a(z, \bar{z})\mu_b(z, \bar{z}) = \sqrt{2} \sin((\phi_\sigma(z) + \bar{\phi}_\sigma(\bar{z}))/2) \quad (60)$$

By using the operator product expansion, one can derive bosonized expressions for $\sigma_a\mu_b$ and $\mu_a\sigma_b$ as well. Since both σ_a and μ_a can be expressed in terms of chiral and antichiral

vertex operators, one can then find linear combinations which split into the product of chiral and antichiral vertex operators. This is conceptually straightforward but in practice is tricky, because one must keep track of a variety of phases. It is much simpler to deal with the chiral conformal blocks directly. The bosonization of the non-chiral fields does give us valuable information, namely that bosonization of conformal blocks is possible. Consider the product $\mathcal{T}_\sigma(z_1)\mathcal{T}_\sigma(z_2)$, as in (54). We have a four-dimensional space of conformal blocks, corresponding to having either I or ψ in the middle and at the right end. All correlators of the non-chiral spin and disorder fields can be built up by products of these blocks. As discussed earlier, we then take the limit of a “long” disc, so that the fields on opposite sides can be treated independently. In the bosonized formulation, there are four linearly-independent chiral fields describing these conformal blocks, namely

$$e^{\pm i\phi_\sigma(z_1)/2}e^{\pm i\phi_\sigma(z_2)/2} \quad (61)$$

for any of the four choices of \pm signs. Since both spaces are four-dimensional, and the elements of each are linearly independent, we must be able to write one in terms of the other.

Since we know that products of tunneling operators in (56) can be bosonized, consistency under braiding allows us to find exactly what they are. One key fact is that braiding operations leave the overall fusion channel for $\mathcal{T}_\sigma(\tau_{2j-1})\mathcal{T}_\sigma(\tau_{2j})$ unchanged, i.e. the ends of (54) and (56) remain I no matter what braiding we do inside. The bosonized version must do likewise, so only two of the four fields in (61) can contribute to (54) or (56), namely

$$e^{i\phi_\sigma(z_1)/2}e^{-i\phi_\sigma(z_2)/2} \quad \text{and} \quad e^{-i\phi_\sigma(z_1)/2}e^{i\phi_\sigma(z_2)/2}.$$

Hence, we find that

$$\begin{aligned} & \dots \quad \begin{array}{c} 2j-1 \quad 2j \quad \quad \quad (2j)' \quad (2j-1)' \\ \hline I \quad | \quad | \quad \quad \quad c \quad | \quad | \quad I \end{array} \quad \dots \\ & \propto \dots \left(e^{i\phi_\sigma(z_1)/2}e^{-i\phi_\sigma(z_2)/2} \pm e^{-i\phi_\sigma(z_1)/2}e^{i\phi_\sigma(z_2)/2} \right) \dots \end{aligned} \quad (62)$$

where the relative $+$ sign is for $c = I$, and the minus sign for $c = \psi$. The simplest way of seeing this is to note that in the limit of a long disk, non-vanishing conformal blocks must contain an even number of pieces like (62) with $c = \psi$. For example, we saw above that the four-point function $\mathcal{G}_\psi = 0$ in this limit. Since correlators of free bosons are invariant under sending ϕ_σ to $-\phi_\sigma$, having $\mathcal{G}_\psi = 0$ requires the $-$ sign for $c = \psi$, which then requires the $+$ sign for $c = I$. We can restate this result in the notation of (50):

$$\begin{aligned} & (m_1, m_2, \dots, m_{n/2})^2 \left\langle \prod_{j=1}^{n/2} \left(e^{i(\phi_\sigma(\tau_{2j-1}) - \phi_\sigma(\tau_{2j}))/2} \right. \right. \\ & \quad \left. \left. + (-1)^{m_j} e^{-i(\phi_\sigma(\tau_{2j-1}) - \phi_\sigma(\tau_{2j}))/2} \right) \right\rangle \quad (63) \end{aligned}$$

As a consistency check on (62), consider interchanging 1 and 2, and 2 with $1'$. (This amounts to removing the aforementioned ambiguity by changing 1 with 2 instead of $2'$ with $1'$.) This interchange results in an overall phase $e^{i\pi/4}$ for $c = I$, while $c = \psi$ gets a phase $-e^{i\pi/4}$, the extra minus sign coming from the two factors of i in (55). In the bosonic formulation, this interchange just corresponds to braiding the two $e^{\pm i\phi_\sigma}$. The operator product expansion yields

$$e^{i\phi_\sigma(z_1)/2}e^{-i\phi_\sigma(z_2)/2} = e^{i\pi/4}e^{-i\phi_\sigma(z_2)/2}e^{i\phi_\sigma(z_1)/2},$$

giving the needed factor of $e^{i\pi/4}$. The extra minus sign for $c = \psi$ then arises because the two terms on the right-hand-side of (62) have been interchanged.

The last (and trickiest) thing to get straight is the phase in front of (62). The phase for $c = I$ can be fixed by comparing to \mathcal{G}_I , but the important part is the relative phase between $c = I$ and $c = \psi$. The importance is because as shown in (56), the conformal blocks contain the sum of the two terms. To get this sign straight, consider a conformal block of the form (54) with N_ψ intermediate ψ states. This means we have N_ψ primed states in the ψ channel, and N_ψ unprimed states in the ψ channel. To cluster decompose the correlator, we need to move all the primed fields together at one end, and the unprimed fields to the other. This is done by the usual braiding rules, and does not change the block, except when a pair of fields in the ψ channel is interchanged with another pair in the ψ channel, where one picks up a minus sign. Moving all the fields to the appropriate sides results in an overall sign $(-1)^{N_\psi/2}$ (N_ψ must be even for the conformal block to not vanish in the long-disc limit.) However, this sign is cancelled by the fact that to get the same conformal blocks for the primed and unprimed cases, we need to braid all the $(2j)'$ and $(2j-1)'$ terms, so that the primed block will be ordered $1'2'3'4' \dots$ instead of $2'1'4'3' \dots$. This gives us a factor $i^{N_\psi} = (-1)^{N_\psi/2}$, cancelling the previous factor.

The upshot is that the two cases in (62) have the same overall phase, which can be absorbed in the coefficient λ_σ . We have now bosonized both basis elements for the product of tunneling operators. To obtain the tunneling operator, we now just add the bosonized expression for I and ψ channels together, as indicated in (56). Our result is therefore

$$\mathcal{T}_\sigma(\tau_{2j-1})\mathcal{T}_\sigma(\tau_{2j}) = e^{i\phi_\sigma(\tau_{2j-1})/2}e^{-i\phi_\sigma(\tau_{2j})/2} \quad (64)$$

Thus the conformal blocks in the perturbative expansion in powers of λ_σ are given by a single correlator of bosonic vertex operators. These are easily evaluated, with $2n$ tunneling operators we have

$$\left| \frac{\prod_{i < j \leq n} \sin((\tau_{2i-1} - \tau_{2j-1})/2) \sin((\tau_{2i} - \tau_{2j})/2)}{\prod_{i < j \leq n} \sin((\tau_{2i-1} - \tau_{2j})/2)} \right|^{1/4}$$

With these bosonization formulas in hand, we can compute all of the conformal blocks of the Ising model, not just the combination (64). See appendix B for details.

IX. MAPPING TO THE KONDO PROBLEM AND RESONANT TUNNELING BETWEEN LUTTINGER LIQUIDS

In the previous section, we learned that the perturbation expansion of the partition function and correlation functions can be bosonized using:

$$\begin{aligned} & \langle \dots \mathcal{T}_\sigma(\tau_1) \mathcal{T}_\sigma(\tau_2) \dots \mathcal{T}_\sigma(\tau_{2k-1}) \mathcal{T}_\sigma(\tau_{2k}) \rangle = \\ & \left\langle \dots e^{i\phi_\sigma(\tau_1)/2} e^{-i\phi_\sigma(\tau_2)/2} \dots e^{i\phi_\sigma(\tau_{2k-1})/2} e^{-i\phi_\sigma(\tau_{2k})/2} \right\rangle \end{aligned} \quad (65)$$

On the right-hand-side, the operators $e^{i\phi_\sigma}$ and $e^{-i\phi_\sigma}$ alternate. This is precisely the same as

$$\left\langle \left(S^+ e^{-i\phi_\sigma/2} + S^- e^{i\phi_\sigma/2} \right) \left(S^+ e^{-i\phi_\sigma/2} + S^- e^{i\phi_\sigma/2} \right) \dots \right\rangle$$

if \vec{S} is a spin-1/2 operator. Since $(S^+)^2 = (S^-)^2 = 0$, $e^{i\phi_\sigma}$ and $e^{-i\phi_\sigma}$ alternate in the same way in this expression, too.

Therefore, we can rewrite the Hamiltonian (43) for a point contact in a $p + ip$ superconductor in the bosonic form:

$$\begin{aligned} \mathcal{H}_{p+ip} = & \int dx \left(\frac{v_n}{2\pi} (\partial_x \phi_\sigma)^2 \right) + \frac{\lambda_\psi}{2\pi} \partial_x \phi_\sigma(0) \\ & + \lambda_\sigma \left(S^+ e^{-i\phi_\sigma(0)/2} + S^- e^{i\phi_\sigma(0)/2} \right) \end{aligned} \quad (66)$$

The Hamiltonian for a Moore-Read Pfaffian state at $\nu = 5/2$ can be bosonized similarly. The charge $e/4$ quasiparticle tunneling operator is the product of $\mathcal{T}_\sigma = (S^+ e^{-i\phi_\sigma(0)/2} + S^- e^{i\phi_\sigma(0)/2})$ with the cosine of the charged mode, $\cos(\phi_\rho(0)/2)$. The neutral fermion tunneling operator is the same as in (66). The charge $e/2$ tunneling operator is independent of the Majorana fermion and is the same as in (42). Hence, we have the Hamiltonian

$$\begin{aligned} \mathcal{H}_{5/2} = & \int dx \left(\frac{v_c}{2\pi} (\partial_x \phi_\rho)^2 + \frac{v_n}{2\pi} (\partial_x \phi_\sigma)^2 \right) \\ & + \lambda_{1/4} \left(S^+ e^{-i\phi_\sigma(0)/2} + S^- e^{i\phi_\sigma(0)/2} \right) \cos(\phi_\rho(0)/2) \\ & + \lambda_{1/2} \cos \phi_\rho(0) + \frac{\lambda_{\psi,0}}{2\pi} \partial_x \phi_\sigma(0), \end{aligned} \quad (67)$$

An advantage of the bosonic formulation is that it allows for a semiclassical analysis in the infrared limit and it reveals the relationship between our problem and resonant tunneling between Luttinger liquids⁵⁰, the Kondo problem³⁶, and dissipative quantum mechanics⁴⁹.

The bosonic Hamiltonian for a $p + ip$ superconductor (66): is literally the single-channel Kondo problem. To see this, consider the anisotropic Kondo Hamiltonian

$$\begin{aligned} \mathcal{H}_{\text{Kondo}} = & \mathcal{H}_{\text{cond}} + J_z S^z \cdot s^z(0) \\ & + J_\perp \left(S^+ \cdot s^-(0) + S^- \cdot s^+(0) \right) \end{aligned} \quad (68)$$

where \mathbf{S} is the impurity spin and $\mathbf{s}(\mathbf{x}) = \psi_\alpha^\dagger \vec{\sigma}_{\alpha\beta} \psi_\beta$ is the conduction electron spin density. Since the impurity spin only

interacts with electrons in the s -wave channel about the origin, we can focus on this channel and treat the problem as one-dimensional. If there is only a single channel of conduction electrons, then when we bosonize the s -wave electrons, we obtain:

$$\begin{aligned} \mathcal{H}_{1\text{-ch. Kondo}} = & \int dx \frac{v_F}{2\pi} \left((\partial_x \phi_c)^2 + (\partial_x \phi_\sigma)^2 \right) \\ & + J_z S^z \partial_x \phi_\sigma(0) + J_\perp \left(S^+ e^{-i\sqrt{2}\phi_\sigma(0)} + S^- e^{i\sqrt{2}\phi_\sigma(0)} \right) \end{aligned} \quad (69)$$

where $\psi_{\uparrow,\downarrow} e^{i(\phi_c \pm \phi_\sigma)/\sqrt{2}}$. The charge mode ϕ_c does not interact with the Kondo impurity. Dropping this mode, we see that this is very similar to (66) if we identify $J_\perp = \lambda_\sigma$. The main difference is that the exponential operators in (69) are $e^{\pm i\sqrt{2}\phi_\sigma(0)}$ rather than $e^{\pm i\phi_\sigma(0)/2}$, but as we will see in the next section, this difference is unimportant. Furthermore, the $\partial_x \phi_\sigma$ term in (69) has an S^z , unlike (66).

If we follow a similar bosonization procedure in the two-channel case, following ref. 36, then the electron operator is written as

$$\psi_{\uparrow,\downarrow;\epsilon} e^{i((\phi_c + \epsilon\phi_f) \pm (\phi_\sigma + \epsilon\phi_{\sigma f}))/2} \quad (70)$$

where $\epsilon = \pm 1$ signify the two channels. The Kondo Hamiltonian (68) then takes the form:

$$\begin{aligned} \mathcal{H}_{2\text{-ch. Kondo}} = & \int dx \frac{v_F}{2\pi} \left((\partial_x \phi_\sigma)^2 + (\partial_x \phi_{\sigma f})^2 \right) \\ & + J_\perp \left(S^+ e^{-i\phi_\sigma(0)} + S^- e^{i\phi_\sigma(0)} \right) \cos(\phi_{\sigma f}(0)) \\ & + J_z S^z \partial_x \phi_\sigma(0) \end{aligned} \quad (71)$$

where we have omitted two bosons which do not couple to the Kondo impurity. If we identify $\phi_{\sigma f}$ with ϕ_ρ in (67), then this is very similar to the Hamiltonian for a point contact in a $\nu = 5/2$ Moore-Read Pfaffian state. The difference between $e^{\pm i\phi_\sigma(0)}$ and $e^{\pm i\phi_\sigma(0)/2}$ is unimportant, as in the one-channel case. However, the factor of 2 difference between $\cos(\phi_{\sigma f})$ and $\cos(\phi_\rho/2)$ results from exchanging the fermion field in the two-channel Kondo problem for the spin field of our problem. This difference is important, and will be discussed in the next section. Equation 67 has an extra term, the $\lambda_{1/2}$ term, which we will also discuss in the next section.

Resonant tunneling between two semi-infinite Luttinger liquids is also described by a very similar Hamiltonian⁵⁰. To see this, consider first an infinite one-dimension spinless fermion system. In the absence of interactions, the right and left moving fermions, $\psi_{R/L} \sim e^{i\varphi_{R/L}}$, are not coupled together. With interactions present it is convenient to define new chiral boson fields,

$$\phi_{R/L} = \frac{1}{2}(\sqrt{g} \pm 1/\sqrt{g})\varphi_R + \frac{1}{2}(\sqrt{g} \mp 1/\sqrt{g})\varphi_L, \quad (72)$$

where the dimensionless conductance g gives a measure of the interaction strength, with $g < 1$ for repulsive interactions and $g > 1$ for attractive interactions. The Hamiltonian is simply,

$$\mathcal{H}_{Lutt} = \int dx \frac{v}{2\pi} \left((\partial_x \phi_R)^2 + (\partial_x \phi_L)^2 \right). \quad (73)$$

As before, the operator $e^{ia\phi_{R/L}}$ has scaling dimension $a^2/2$.

Now consider breaking the system at $x = 0$ into two semi-infinite Luttinger liquids, which we denote as a, b . The incident chiral bosons $\phi_{R/L}$ are completely reflected at $x = 0$, and so it is convenient to define two separate chiral bosons for the two semi-infinite Luttinger liquids,

$$\begin{aligned}\phi_a(x < 0) &\equiv \phi_R(x); & \phi_a(x > 0) &\equiv \phi_L(-x), \\ \phi_b(x < 0) &\equiv \phi_L(-x); & \phi_b(x > 0) &\equiv \phi_R(x),\end{aligned}$$

so that the appropriate Hamiltonian is of the same form,

$$\mathcal{H}_0 = \int dx \frac{v}{2\pi} \left((\partial_x \phi_a)^2 + (\partial_x \phi_b)^2 \right). \quad (74)$$

Now introduce a quantum dot between the two semi-infinite Luttinger liquids, which is assumed to have a single state at the Fermi energy, occupied (with $S^z = 1/2$) or unoccupied (with $S^z = -1/2$). Fermions on the ends of the two leads are allowed to hop on and off the quantum dot with tunneling amplitude t . The tunneling Hamiltonian is,

$$\mathcal{H}_{tun} = t(S^+ e^{i\phi_a(x=0)/\sqrt{g}} + h.c.) + t(S^+ e^{i\phi_b(x=0)/\sqrt{g}} + h.c.). \quad (75)$$

Finally, upon changing variables one last time,

$$\phi_{a/b} = \frac{1}{\sqrt{2}}(\phi_\sigma \pm \phi_\rho), \quad (76)$$

one can readily see that for $g = 2$ the full Hamiltonian $\mathcal{H}_0 + \mathcal{H}_{tun}$ is identical to the point contact Hamiltonian, $\mathcal{H}_{5/2}$ in Eq. (67), with $\lambda_{1/4} = t$ and $\lambda_{1/2} = \lambda_{\psi,0} = 0$.

X. STRONG COUPLING FIXED POINT AND INSTANTON EXPANSION

We now analyze the Hamiltonians (66) and (67).

A. $p + ip$ superconductor with $\lambda_\psi = 0$

When $\lambda_\psi = 0$, the Hamiltonian for the $p + ip$ superconductor, \mathcal{H}_{p+ip} in (66), has an extra symmetry, being invariant under $\phi_\sigma \rightarrow -\phi_\sigma$ together with a π rotation of the spin about the x -axis:

$$\begin{aligned}\phi_\sigma &\rightarrow -\phi_\sigma \\ S^z &\rightarrow -S^z \\ S^\pm &\rightarrow S^\mp\end{aligned} \quad (77)$$

This is a Kramers-Wannier duality symmetry for the non-chiral Ising model. This symmetry is shared by the one-channel Kondo Hamiltonian in (69). The potentially important remaining differences between the two Hamiltonians when $\lambda_\psi = 0$ are, firstly, an additional term of the form $J_z S^z \partial_x \phi_\sigma$ present in the Kondo Hamiltonian, and, secondly the exponential operators in the Kondo problem are

$e^{\pm i\sqrt{2}\phi_\sigma(0)}$ rather than $e^{\pm i\phi_\sigma(0)/2}$ for the $p + ip$ superconductor. However, under the unitary transformation generated by

$$U = \exp(iS^z \phi_\sigma/2), \quad (78)$$

these exponential factors can be readily eliminated from (66),

$$U \mathcal{H}_{p+ip} U^\dagger = \int dx \left(\frac{v_n}{2\pi} (\partial_x \phi_\sigma)^2 + \lambda_\sigma S^x - v_n S^z \partial_x \phi_\sigma(0) \right). \quad (79)$$

Similarly, under the unitary transformation, $\tilde{U} = e^{i\sqrt{2}S^z \phi_\sigma}$, we can eliminate the exponential factors from (69),

$$\tilde{U} \mathcal{H}_{1\text{-ch. Kondo}} \tilde{U}^\dagger = \int dx \left(\frac{v_F}{2\pi} (\partial_x \phi_\sigma)^2 + J_\perp S^x + (J_z - v_F) S^z \partial_x \phi_\sigma(0) \right). \quad (80)$$

This demonstrates the equivalence of the crossover from weak to strong coupling in the point contact in a $p + ip$ superconductor when $\lambda_\psi = 0$, to the analogous crossover in the single channel anisotropic Kondo problem with $J_z = 0$.

Moreover, varying the value of J_z in the anisotropic Kondo problem, while affecting the scaling dimension of various operators in the ultraviolet, does not effect the behavior in the infrared. The reason for this is that the $J_\perp S^x$ perturbation at the ultraviolet fixed point is strongly relevant, whereas the $(J_z - v_F) S^z \partial_x \phi_\sigma$ operator is a marginal perturbation. Upon scaling down in energies, crossing over to the infrared fixed point, the J_\perp term rapidly grows in importance, pinning the value of the spin to $S_x = 1/2$. Then, upon integrating out the spin degree of freedom, the $(J_z - v_F) S^z \partial_x \phi_\sigma$ term will generate irrelevant terms such as $(\partial_x \phi_\sigma(0))^2$. We mention in passing, that the soluble Toulouse limit corresponds to $J_z = v_F$, which entirely decouples the spin from the bosonic field ϕ_σ .

We can now use the well-understood behavior of the single-channel anisotropic Kondo problem to find the infrared behavior of the point contact in a $p + ip$ superconductor (with $\lambda_\psi = 0$). As discussed above, in the infrared limit the Kondo spin points along the x -direction, $S^x = 1/2$, and we can perturbatively eliminate entirely the spin degree of freedom. However, the unitary transformation (78) has made ϕ_σ discontinuous at $x = 0$. This is the $\pi/2$ phase shift which occurs at the strong-coupling fixed point of the single-channel Kondo problem. A $\pm\pi/2$ phase shift for ϕ_σ corresponds in fermionic language to

$$\psi_1(0^+) = \pm\psi_2(0^-), \quad \psi_2(0^+) = \mp\psi_1(0^-). \quad (81)$$

Thus, it translates, in our problem, to perfect backscattering of the Majorana fermions at the point contact.

Hence, the RG flow from the weak-coupling fixed point for a point contact in a $p + ip$ superconductor (the Hamiltonian (66) with $\lambda_\sigma = \lambda_\psi = 0$), for non-zero λ_σ crosses over to the strong-coupling fixed point which we discussed in section V. The leading irrelevant operator at the strong-coupling fixed point is the dimension-2 operator $(\partial_x \phi_\sigma)^2$ (which leads to a low-temperature resistivity $\sim T^2$ in the Kondo problem). (As

discussed above, this can be obtained by integrating out the gapped fluctuations of S^z about the ground state, $\langle S^z \rangle = 0$). In terms of the Majorana modes, this operator is:

$$(\partial_x \phi_\sigma)^2 \sim \psi_a \partial_x \psi_a + \psi_b \partial_x \psi_b. \quad (82)$$

Therefore, the leading irrelevant operator at the strong coupling fixed point does not couple the two edges. It merely shifts their velocities locally. The next order term obtained by integrating out the fluctuations of S^z does couple the edges:

$$(\partial_x \phi_\sigma)^4 \sim (\psi_a \partial_x \psi_a) (\psi_b \partial_x \psi_b) \quad (83)$$

This is the leading irrelevant operator coupling the two edges, whose coupling constant was called $t_{\psi, \text{kin}}$ in (32) and (34). It may be interpreted as tunneling a p -wave pair of neutral fermions from one edge to the other.

B. $p + ip$ superconductor with $\lambda_\psi \neq 0$

With non-zero λ_ψ the behavior in the infrared changes qualitatively. The reason for this is that the $\lambda_\psi \partial_x \phi_\sigma$ term does not respect the Kramers-Wannier duality symmetry, being odd under $\phi_\sigma \rightarrow -\phi_\sigma$. As a result, the Hamiltonian \mathcal{H}_{p+ip} is no longer symmetry equivalent to the one-channel Kondo model. The importance of this broken symmetry can be more readily appreciated after transforming the $p + ip$ Hamiltonian with the unitary transformation in (78), which with non-zero λ_ψ is now given by:

$$U \mathcal{H}_{p+ip} U^\dagger = \int dx \left(\frac{v_n}{2\pi} (\partial_x \phi_\sigma)^2 \right) - \frac{\lambda_\psi}{2} S^z + \lambda_\sigma S^x + \left(\frac{\lambda_\psi}{2\pi} - v_n S^z \right) \partial_x \phi_\sigma(0). \quad (84)$$

Notice the presence of the $\lambda_\psi S^z$ term, which implies that in the infrared we can no longer drop the last term above. Indeed, upon flowing towards the infrared the impurity spin will no longer point along the x -axis, but will have a non-zero value of S^z :

$$\langle S^z \rangle = \frac{\lambda_\psi/2}{\sqrt{\lambda_\sigma^2 + (\lambda_\psi/2)^2}}. \quad (85)$$

This implies the presence of the marginal perturbation, $\langle S^z \rangle \partial_x \phi_\sigma(0)$, at the $\lambda_\psi = 0$ strong coupling infrared fixed point. In terms of Majorana fermions, at the strong coupling fixed point this operator corresponds to a tunneling of a Majorana fermions across the point contact,

$$\partial_x \phi_\sigma(0) \sim i\psi_a(0)\psi_b(0). \quad (86)$$

As a result, in the presence of a small non-zero λ_ψ in the ultraviolet, the Majorana fermions in the infrared will no longer be perfectly backscattered, but will have a small amplitude for transmission.

C. $\nu = \frac{5}{2}$ QH State with $\lambda_{\psi,0} \neq 0, \lambda_{1/2} = \lambda_{1/4} = 0$.

We now turn to the more complicated case of the $\nu = 5/2$ fractional quantum Hall state, but first we consider some simpler special cases. If $\lambda_{1/2} = \lambda_{1/4} = 0$ and only $\lambda_{\psi,0}$ is non-vanishing, then the Hamiltonian is quadratic:

$$\mathcal{H}_{5/2} = \int dx \left(\frac{v_c}{2\pi} (\partial_x \phi_\rho)^2 + \frac{v_n}{2\pi} (\partial_x \phi_\sigma)^2 \right) + \frac{\lambda_{\psi,0}}{2\pi} \partial_x \phi_\sigma(0) \quad (87)$$

The $\lambda_{\psi,0}$ term can be eliminated by shifting $\phi_\sigma(x) \rightarrow \phi_\sigma(x) + \frac{1}{2v_n} \lambda_{\psi,0} \theta(x)$. Hence, this tunneling operator causes incoming Majorana fermions to be scattered from one edge to the other according to:

$$\psi_a(0^+) + i\psi_b(0^+) e^{i\lambda_{\psi,0}/2v_n} (\psi_a(0^-) + i\psi_b(0^-)) \quad (88)$$

This should affect thermal transport, but charge transport is completely unaffected, i.e. $R_{xx} = 0$, in this special case.

This free theory lies along a fixed line connecting the limit in which ψ_a and ψ_b are unaffected by the point contact and the other extreme in which they are switched. In appendix C, we analyze the inter-edge resonant tunneling of Majorana fermions via a zero mode on a localized $e/4$ quasiparticle or superconducting vortex. In this case, there is an RG flow between these two limits.

D. $\nu = \frac{5}{2}$ QH State with $\lambda_{1/2} \neq 0, \lambda_{\psi,0} = \lambda_{1/4} = 0$.

When $\lambda_{1/2} \neq 0$ is the only non-vanishing coupling, the Hamiltonian takes the form:

$$\mathcal{H}_{5/2} = \int dx \left(\frac{v_c}{2\pi} (\partial_x \phi_\rho)^2 + \frac{v_n}{2\pi} (\partial_x \phi_\sigma)^2 \right) + \lambda_{1/2} \cos(\phi_\rho(0)) \quad (89)$$

so the neutral Majorana mode is unaffected by the point contact. However, charge- $e/2$ quasiparticles can tunnel between the edges. In the infrared limit, the coupling $\lambda_{1/2}$ grows large, according to (40), so $\phi_\rho(0)$ is localized in the minima of the potential:

$$\phi_\rho(0) = (2n + 1)\pi \quad (90)$$

This translates to a Dirichlet boundary condition on the non-chiral boson in the ‘folded’ setup. An incoming charge difference between the two edges is reversed upon passing through the point contact, since (90) means that $(\phi_\rho(0^+) + \phi_\rho(0^-))/2 = (2n + 1)\pi$. Therefore, $R_{xx} = h/10e^2$ in this limit (see appendix A for the definition of the four-terminal resistance).

Hence, the flow is to a ‘mixed’ fixed point: the charged mode is in the strong constriction limit, but the neutral mode is in the weak constriction limit. The flow into this fixed point

can be understood in terms of instantons which take the system from one minimum of the cosine (90) to another. Suppose that $\phi_\rho(0, \tau) - \pi + 2\pi f(\tau)$ is a solution to the classical equations of motion of (89) which interpolates between $\phi_\rho(0, -\infty) = -\pi$ and $\phi_\rho(0, \infty) = \pi$. Then the classical action for a multi-instanton history,

$$\phi_\rho(0, \tau) = (2n + 1)\pi + 2\pi \sum_i e_i f(\tau - \tau_i), \quad (91)$$

with $e_i = \pm 1$, has a Coulomb gas form:

$$S_{\text{cl}} = 4 \sum_{i,j} e_i e_j \ln |\tau_i - \tau_j|_{\tau_c}, \quad (92)$$

where the subscript on the right signifies that this is the asymptotic form for $|\tau_i - \tau_j| \gg \tau_c$, where τ_c is a short-time cutoff. The prefactor on the right-hand-side of (92) is, more generally, given by $(\Delta\phi_\rho(0)_{\text{inst}}/\pi)^2$, where $\Delta\phi_\rho(0)_{\text{inst}}$ is the separation of the minima between which the instanton interpolates. In this case, $\Delta\phi_\rho(0)_{\text{inst}} = 2\pi$.

If we sum over the possible numbers of instantons and integrate over their (temporal) locations τ_i , we have a contribution to the partition function

$$Z_{\text{inst}} = \sum_N \sum_{e_m = \pm 1} \int \prod_{k=1}^N d\tau_k \prod_{i>j} |\tau_i - \tau_j|_{\tau_c}^{4e_i e_j}. \quad (93)$$

We observe that this instanton gas expansion (93) about the strong coupling fixed point is the same as the perturbative expansion of

$$\mathcal{H}^{\text{dual}} = \int dx \left(\frac{v_c}{2\pi} (\partial_x \phi_\rho)^2 + \frac{v_n}{2\pi} (\partial_x \phi_\sigma)^2 \right) + t_1 \cos(2\phi_\rho(0)).$$

This is equivalent, in the unfipped setup, to:

$$S = \int d\tau dx \left(\mathcal{L}_{\text{edge}}(\psi_L, \phi_L) + \mathcal{L}_{\text{edge}}(\psi_R, \phi_R) \right) + \int d\tau t_1 \cos((\phi_R(0) - \phi_L(0))\sqrt{2}) \quad (94)$$

The leading irrelevant coupling at this ‘mixed’ fixed point, t_1 , therefore tunnels a charge- e boson across the point contact (*not* an electron). However, this is peculiar to the special case in which $\lambda_{1/4}$ is tuned to zero. Various properties of this mixed fixed point are discussed in Ref. 52.

E. $\nu = \frac{5}{2}$ QH State with $\lambda_{1/4} \neq 0$, $\lambda_{1/2} = \lambda_{\psi,0} = 0$.

This case is similar to the two-channel Kondo problem, but with an important difference. After the unitary transformation, $\lambda_{1/4}$ becomes a dimension-7/8 coupling (rather than dimension-1/2 in the two-channel Kondo case):

$$U \mathcal{H}_{5/2} U^\dagger = \int dx \left(\frac{v_c}{2\pi} (\partial_x \phi_\rho)^2 + \frac{v_n}{2\pi} (\partial_x \phi_\sigma)^2 \right) + \lambda_{1/4} S^x \cos(\phi_\rho(0)/2) - v_n S^z \partial_x \phi_\sigma(0) \quad (95)$$

While this difference in dimension is quite important for the detailed behavior of the model, the qualitative features of the two problems is similar. The coupling $\lambda_{1/4}$ is extremely relevant, so in the infrared limit $\phi_\rho(0)$ is localized in the minima of the cosine while the spin points in the corresponding direction:

$$S_x = \pm 1/2, \quad \phi_\rho(0) = (4n + 1 \pm 1)\pi \quad (96)$$

When $\phi_\rho(0)$ is localized, the charge mode is completely reflected, as before. The neutral Majorana fermions are also completely reflected, according to the unitary transformation, as in the $p + ip$ case above. Hence, the strong coupling fixed point of section V is reached in the infrared limit.

We can deduce the form of the irrelevant perturbations of the infrared fixed point by considering instantons which connect the minima (96). So long as the spin points in a fixed direction (either the $+x$ or $-x$ direction), these minima are twice as far apart as those in (90). The minimum action instantons of this type which contribute to charge transport have $\Delta\phi_\rho(0)_{\text{inst}} = \pm 4\pi$, $\Delta\phi_\sigma(0)_{\text{inst}} = 0$, $\Delta\mathbf{S}_{\text{inst}} = 0$. By the arguments of the previous subsection, these instantons correspond to the irrelevant tunneling operator

$$H_{\text{pair}}^{\text{tun}} = t_{\text{pair}} \cos(4\phi_\rho(0)) \quad (97)$$

which tunnels a charge-2 boson between the two droplets.

There is a possible complication here, namely that the spin \vec{S} can also rotate as $\phi_\rho(0)$ is varying. In the special case which we are considering in this subsection, however – in particular, when $\lambda_{\psi,0} = 0$ – the Hamiltonian (67) is invariant under the Kramers-Wannier duality symmetry in Eq. (77). This symmetry constrains which irrelevant perturbations of the strong coupling fixed point can appear in the flow along this direction from the weak-coupling fixed point, just as for the case of the $p + ip$ superconductor. For instance, a single electron tunneling operator, which takes the form $\Phi_{\text{el}} \sim \partial_x \phi_\sigma e^{2i\phi_\rho}$ cannot occur because it is not invariant under this symmetry.

We can understand this in instanton language as follows by treating the spin as a charged particle on the surface of a sphere with a magnetic monopole at its center. For a single electron to tunnel, an instanton with $\Delta\phi_\rho = \pm 2\pi$, $\Delta\phi_\sigma = 0$ is needed. Such an instanton only connects two minima of the Hamiltonian if the spin S is also reversed, e.g. from $S_x = 1/2$ to $S_x = -1/2$. There are many such equally good classical paths from one point on the sphere to its antipode, but they will contribute with different Berry phases, as result of the monopole, and cancel.

Therefore, the dimension-8 operator (97) is the leading irrelevant operator in the infrared when $\lambda_{1/4}$ is the only non-zero relevant perturbation in the ultraviolet. Hence, we obtain the non-generic low-temperature transport of (36).

F. $\nu = \frac{5}{2}$ QH State, General Case: $\lambda_{1/4}, \lambda_{1/2}, \lambda_{\psi,0} \neq 0$

The approach to the limit of two decoupled droplets is so rapid in the previous special case because a single electron can’t tunnel from left to right. However, when $\lambda_{\psi,0}$ is also

non-zero, the Hamiltonian (67) is no longer invariant under the symmetry (77). Therefore, an electron tunneling term is not forbidden. Such a term arises from an instanton gas expansion because the term $i\lambda_{\psi,0}\psi_1\psi_2 = \lambda_{\psi,0}\partial_x\phi_\sigma$ leads to a term $\lambda_{\psi,0}S^z$ after application of the unitary transformation (78), just as for the $p + ip$ superconductor as seen explicitly in Eq. (84). The symmetry between the different classical paths is now broken, and there is a unique minimum action instanton in spin space connecting two classical minima such as $S_x = 1/2$, $\phi_\rho(0) = 2\pi$ and $S_x = -1/2$, $\phi_\rho(0) = 0$. This instanton transfers charge e while simultaneously flipping the sign of the neutral part of the quasiparticle tunneling operator $\sigma_a\sigma_b$. (Thereby leaving the Hamiltonian unchanged.) Therefore, it corresponds to the electron tunneling operator

$$\begin{aligned} H_{\text{el}}^{\text{tun}} &= t_{\text{el}} \partial_x \phi_\sigma(0) \cos(2\phi_\rho(0)) \\ &= t_{\text{el}} i\psi_a(0)\psi_b(0) \cos(2\phi_\rho(0)) \end{aligned} \quad (98)$$

The presence of non-zero $\lambda_{1/2}$ does not lift the degeneracy between the minima; it just makes them deeper and suppresses the maxima. Therefore, it does not change the analysis above. Hence, we now obtain the generic low-temperature resistance of (36).

XI. DISCUSSION

We have developed a framework to describe quantitatively the tunneling of edge modes in two-dimensional systems with non-Abelian statistics. Since the edge modes are both chiral and have non-trivial fusion rules, we utilized the formalism of Moore and Seiberg to first define the tunneling operator uniquely, and then compute its conformal blocks.

One immediate result of our mapping onto the Kondo problem is the entropy loss in the flow from the ultraviolet to the infrared. In Ref. 37, we discuss in depth the entanglement entropy loss in such systems. We define and compute *holographic partition functions*, which describe the entanglement entropy of both bulk and edge quasiparticles in conformal field theory language.

Unfortunately, experiments on fractional quantum Hall systems do not generally measure the entropy or the specific heat, but instead transport quantities such as the tunneling current, which we have discussed here. To compute transport quantities in the non-Abelian case, one must utilize the precise definition of the tunneling operator which we have given. Moreover, we showed that the perturbative expansion is given in terms of conformal blocks, not the usual non-chiral correlation functions. Transport noise measurements may also shed light on the properties of non-Abelian quasiparticles³⁴. The Keldysh formalism, useful for non-equilibrium transport situations, will require some refinement to be used in such chiral theories, building on the formulation of tunneling given here.

We have mainly focused on the simplest type of tunneling, which does not change the fusion channel of the edge modes—no qubit is flipped as a result of tunneling. This is the only possibility for tunneling through a simple point contact, but more complicated geometries can result in more complicated

tunneling events. To give the simplest of such possibilities, consider an antidot at the point contact. Then a tunneling event can cause a fermion to leave or join the edge of the antidot, effectively adding or removing a zero mode from the bulk of the system. Such a tunneling event would correspond to the σ quasiparticle annihilation and creation operators on the two edges to be in the ψ channel, instead of the I channel. More complicated geometries, such as an antidot with two point contacts at both ends, can result in more complicated possibilities. A single tunneling event is described by two-dimensional vector space of operators, so for a given geometry, the tunneling operator can be any element of this vector space.

Our formalism should be applicable to any gapped two-dimensional quantum system with gapless chiral edge modes. Moore and Seiberg’s results (and hence ours) apply to any rational conformal field theory; “Rational” means that there are a finite number of chiral primary fields under some extended symmetry algebra. In the fractional quantum Hall effect, the extended symmetry algebra of the edge modes arises from the requirement that all quasiparticle and quasihole creation/annihilation operators commute with the electron operator. This effectively turns the electron operator into a symmetry operator⁹, making it likely that the resulting edge conformal field theory is rational.

One obvious candidate for applying our results is the Read-Rezayi states⁵³, which extend the Moore-Read state to an entire series of fractional quantum Hall states with non-Abelian braiding. Here, the description of the edge modes in terms of rational conformal field theory is already known⁵³, so defining the appropriate conformal blocks is straightforward. Bosonizing these conformal blocks is not straightforward, but may be possible. Bosonizing the Moore-Read tunneling operator was possible because the edge modes are described by the chiral Ising model, which has central charge $c = 1/2$; since there are two edges, the combined theory has $c = 1$, the central charge of the free boson. The non-Abelian part of the k th Read-Rezayi theory is the Z_k parafermion conformal field theory, which has central charge $c = 2(k-1)/(k+2)$. To bosonize such theories, one must combine them with other theories to get c integer. This was done for the Z_3 chiral theory (or to be precise, for the equivalent 1+1 dimensional quantum impurity problem) in Ref. 54. We expect that this analysis can be adapted to our situation.

We also note that while this formalism is necessary to define precisely the tunneling for non-Abelian states, we believe it may be fruitful to utilize it even for abelian fractional quantum Hall states. For more general states than Laughlin’s, the structure of quasiparticles and the phases which result under their braiding can get quite elaborate. Whereas keeping track of phases is tricky in any formalism, exploiting the symmetry algebra may provide a useful tool for simplifying the analysis.

Acknowledgments

We would like to thank E. Fradkin, M. Freedman, E.-A. Kim, A. Kitaev, A. Ludwig, J. Preskill, N. Read, and A. Stern

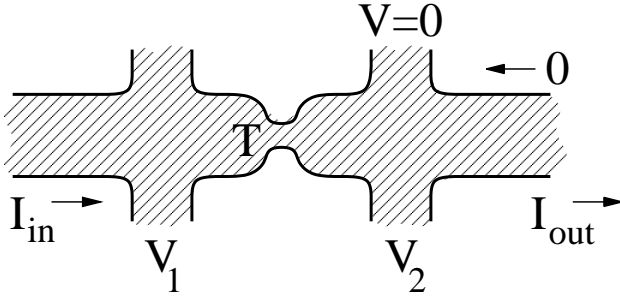


FIG. 7: Four-terminal transport setup. Voltages V_1 and V_2 are measured at the lower left and right, respectively. Current I_{in} is injected and current I_{out} flows out. When there is no inter-edge tunneling between the top and bottom edges, the transmission coefficient vanishes, $T = 0$. In the strong constriction limit, $T = 1$.

for discussions. This research has been supported by the NSF under grants DMR-0412956 (P.F.), PHY-9907949 and DMR-0529399 (M.P.A.F.) and DMR-0411800 (C.N.), and by the ARO under grant W911NF-04-1-0236 (C.N.).

APPENDIX A: FOUR-TERMINAL TRANSPORT AT $\nu = 5/2$

In this appendix, we discuss four-terminal transport in the $\nu = 5/2$ quantum Hall state in the presence of a point contact. In figure 7, an current I_{in} is injected along the lower edge at the left. The voltage to the left of the point contact is related to this current by the Hall relation:

$$I_{\text{in}} = \frac{5}{2} \frac{e^2}{h} V_1 \quad (\text{A1})$$

For notational convenience, we set the voltage of the top edge at the right of the point contact to zero. Then the current going out to the right is related to the voltages on the bottom edge to the right and left of the point contact according to:

$$I_{\text{out}} = \frac{5}{2} \frac{e^2}{h} V_2 = \left[2 + \frac{1}{2}(1 - T) \right] \frac{e^2}{h} V_1 \quad (\text{A2})$$

where T is the transmission coefficient for tunneling between the edges. Hence,

$$\begin{aligned} R_{xx} &\equiv \frac{V_1 - V_2}{I_{\text{out}}} = \frac{1}{\left[2 + \frac{1}{2}(1 - T) \right] \frac{e^2}{h}} - \frac{1}{\frac{5}{2} \frac{e^2}{h}} \\ &= \frac{\frac{h}{e^2} \frac{2}{5} \frac{(T/5)}{1 - T/5}}{e^2} \end{aligned} \quad (\text{A3})$$

In the strong coupling limit, $T = 1$, and $R_{xx} = \frac{h}{e^2} \frac{1}{10}$. At small, non-zero temperature, $1 - T \sim t_{\text{cl}}^2 T^4$, so we find:

$$R_{xx} - \frac{1}{10} \frac{h}{e^2} \sim -t_{\text{cl}}^2 T^4 \quad (\text{A4})$$

APPENDIX B: EXPLICIT COMPUTATION OF CONFORMAL BLOCKS

Ordinarily, one is not interested in the explicit forms of the conformal blocks of the $2n$ -point spin-field correlation functions in the Ising model. They are merely an intermediate step on the way to computing the quantities which are actually of interest, the non-chiral correlation functions. The latter can be computed by the methods of ref. 51. However, in the context of topological states and their edge excitations, the explicit forms of conformal blocks themselves are important quantities. Here, we show through some examples how they can be obtained using the methods of section VIII.

From (63), we see, for instance that:

$$\begin{aligned} (0, 0)^2 &= \left\langle \left(e^{i(\phi_\sigma(z_1) - \phi_\sigma(z_2))/2} + e^{-i(\phi_\sigma(z_1) - \phi_\sigma(z_2))/2} \right) \right. \\ &\quad \times \left. \left(e^{i(\phi_\sigma(z_3) - \phi_\sigma(z_4))/2} + e^{-i(\phi_\sigma(z_3) - \phi_\sigma(z_4))/2} \right) \right\rangle \\ &= 2 \left\langle e^{\frac{i}{2}\phi_\sigma(z_1)} e^{-\frac{i}{2}\phi_\sigma(z_2)} e^{\frac{i}{2}\phi_\sigma(z_3)} e^{-\frac{i}{2}\phi_\sigma(z_4)} \right\rangle \\ &\quad + 2 \left\langle e^{\frac{i}{2}\phi_\sigma(z_1)} e^{-\frac{i}{2}\phi_\sigma(z_2)} e^{-\frac{i}{2}\phi_\sigma(z_3)} e^{\frac{i}{2}\phi_\sigma(z_4)} \right\rangle \end{aligned} \quad (\text{B1})$$

In obtaining the second equality, we used the symmetry of the free boson theory under $\phi_\sigma \rightarrow -\phi_\sigma$. In the same way, we can obtain $(1, 1)^2$.

Hence, we have:

$$\begin{aligned} (0, 0) &= \left[\left(\frac{z_{13}z_{24}}{z_{12}z_{14}z_{23}z_{34}} \right)^{1/4} + \left(\frac{z_{14}z_{23}}{z_{12}z_{13}z_{24}z_{34}} \right)^{1/4} \right]^{1/2} \\ (1, 1) &= \left[\left(\frac{z_{13}z_{24}}{z_{12}z_{14}z_{23}z_{34}} \right)^{1/4} - \left(\frac{z_{14}z_{23}}{z_{12}z_{13}z_{24}z_{34}} \right)^{1/4} \right]^{1/2} \end{aligned}$$

These expressions are the same as $\mathcal{F}_I, \mathcal{F}_\psi$ in (48), which were obtained by solving a differential equation following from the existence of null vectors in ref. 39. Higher-point conformal blocks can be obtained by solving even more complicated differential equations. However, we can obtain all of these by similar bosonization formulas as above. For instance, a calculation of a conformal block of a six-point function gives:

$$\begin{aligned} (0, 0, 0) &\left[\left(\frac{z_{13}z_{15}z_{35}z_{24}z_{26}z_{46}}{z_{12}z_{14}z_{16}z_{23}z_{34}z_{36}z_{25}z_{45}z_{56}} \right)^{1/4} \right. \\ &\quad + \left(\frac{z_{13}z_{16}z_{36}z_{24}z_{25}z_{45}}{z_{12}z_{14}z_{15}z_{23}z_{34}z_{35}z_{26}z_{46}z_{56}} \right)^{1/4} \\ &\quad + \left(\frac{z_{14}z_{15}z_{45}z_{23}z_{26}z_{36}}{z_{12}z_{13}z_{16}z_{24}z_{34}z_{46}z_{25}z_{35}z_{56}} \right)^{1/4} \\ &\quad \left. + \left(\frac{z_{13}z_{16}z_{45}z_{23}z_{25}z_{35}}{z_{12}z_{13}z_{15}z_{24}z_{34}z_{45}z_{26}z_{35}z_{56}} \right)^{1/4} \right]^{1/2} \end{aligned} \quad (\text{B2})$$

Conformal blocks of correlation functions of σ 's and ψ 's can also be obtained in this way, using (57).

APPENDIX C: RESONANT TUNNELING BETWEEN EDGES

In this appendix, we discuss the situation in which a charge- $e/4$ quasiparticle is localized in the middle of a point contact in a Pfaffian quantum Hall state at $\nu = 5/2$. Such a quasiparticle has a zero-energy Majorana zero mode localized at its core, so a Majorana fermion at the edge can tunnel resonantly through this mode to the other edge. Let us suppose that all other types of tunneling are much smaller. (With a charged quasiparticle localized in the point contact, Coulomb blockade might strongly suppress the tunneling of charge $e/4$ and $e/2$ quasiparticles.) The action for this situation is:

$$S = \int dx d\tau (\mathcal{L}_{\text{fermion}}(\psi_L) + \mathcal{L}_{\text{fermion}}(\psi_R)) + \psi_{\text{loc}} \partial_\tau \psi_{\text{loc}} + it_R \psi_{\text{loc}} \psi_R(0) + it_L \psi_{\text{loc}} \psi_L(0) \quad (\text{C1})$$

where ψ_{loc} is Majorana zero mode at the localized quasiparticle and t_R, t_L are the hopping matrix elements between, respectively, the right and left edges and the localized zero mode. As we will see momentarily, a resonance occurs when $t_R = \pm t_L$ (the relative sign can be absorbed in ψ_{loc}). Since this action is quadratic, we can diagonalize it explicitly. We find:

$$\begin{pmatrix} \psi_R(0^+, \omega) \\ \psi_L(0^+, \omega) \end{pmatrix} = \frac{1}{\omega + \frac{i}{2}(t_R^2 + t_L^2)} \mathcal{M} \begin{pmatrix} \psi_R(0^-, \omega) \\ \psi_L(0^-, \omega) \end{pmatrix}$$

$$\mathcal{M} \equiv \begin{pmatrix} \omega + \frac{i}{2}(t_L^2 - t_R^2) & -it_R t_L \\ -it_R t_L & \omega + \frac{i}{2}(t_R^2 - t_L^2) \end{pmatrix}$$

When $t_R = t_L$, we find that $\psi_R(0^+, 0) = \psi_L(0^-, 0)$ and $\psi_L(0^+, 0) = \psi_R(0^-, 0)$. Thus, the ‘RG flow’ is to the strong

constriction limit.

It is instructive to re-express this result in bosonic terms, by adopting the same definition as in Eq.(58), $e^{i\phi_\sigma} \sim \psi_R + i\psi_L$. This gives,

$$\phi_\sigma(x = 0^+) = (\pi/2) - \phi_\sigma(x = 0^-). \quad (\text{C2})$$

In striking contrast to the strong coupling $p + ip$ and Kondo fixed points which have a simple $\pi/2$ phase shift, $\phi_\sigma(x = 0^+) = \phi_\sigma(x = 0^-) + (\pi/2)$, the strong coupling resonant tunneling fixed point is qualitatively different involving not only a phase shift but a sign change of the bosonic field. This demonstrates that despite the common ultraviolet fixed points in these two situations, the different form of the two inter-edge tunneling perturbations (ie. tunneling of σ particles versus Majorana tunneling through a localized σ particle, respectively) causes both the nature of the crossovers and the ‘‘destination’’ strong coupling infrared fixed points to be qualitatively distinct. Nevertheless, the leading irrelevant operator at the two infrared fixed points is the same, given in (82). To see this, we linearize for small ω in (C2) to obtain $(i\omega + t_R t_L \partial_x)(\psi_R + \psi_L)|_{x=0} = 0$. Upon Fourier transforming back to real time we obtain,

$$\partial_t(\psi_R + \psi_L) = -i[\psi_R + \psi_L, \mathcal{H}_{\text{irr}}] = \delta(x)t_R t_L \partial_x(\psi_R + \psi_L). \quad (\text{C3})$$

This is consistent with a leading irrelevant operator of the form,

$$\mathcal{H}_{\text{irr}} \sim (i\psi_R \partial_x \psi_R + i\psi_L \partial_x \psi_L)|_{x=0} \sim (\partial_x \phi_\sigma(0))^2, \quad (\text{C4})$$

the same as in (82).

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