

# Bond algebraic liquid phase in strongly correlated multiflavor cold atom systems

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When cold atoms are trapped in a square or cubic optical lattice, it should be possible to pump the atoms into excited  $p$ -level orbitals within each well. Following earlier work, we explore the metastable equilibrium that can be established before the atoms decay into the  $s$ -wave orbital ground state. We will discuss the situation with integer number of bosons on every site, and consider the strong correlation “insulating” regime. By employing a spin-wave analysis together with a new duality transformation, we establish the existence and stability of a novel gapless “critical phase”, which we refer to as a “bond algebraic liquid”. The gapless nature of this phase is stabilized due to the emergence of symmetries which lead to a quasi-one dimensional behavior. Within the algebraic liquid phase, both bond operators and particle flavor occupation number operators have correlations which decay algebraically in space and time. Upon varying parameters, the algebraic bond liquid can be unstable to either a Mott insulator phase which spontaneously breaks lattice symmetries, or a  $\mathbb{Z}_2$  phase. The possibility of detecting the algebraic liquid phase in cold atom experiments is addressed. Although the momentum distribution function is insufficient to distinguish the algebraic bond liquid from other phases, the density correlation function can in principle be used to detect this new phase of matter.

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## I. INTRODUCTION

Interacting bosonic systems hopping on two or three dimensional lattices ( $d = 2, 3$ ) typically have one of two ground states: superfluid or a Mott insulator<sup>1</sup>. Searching for bosonic phases other than these two has attracted much attention, especially gapless bosonic phases which do not break the global  $U(1)$  symmetry. In such gapless featureless bosonic phases, the correlation functions between physical operators usually fall off algebraically. Being the lattice boson analog to the algebraic spin liquids which can occur in spin models, these phases can be referred to as algebraic boson liquids. So far several algebraic liquid phases have been realized in toy models. In  $d = 3$  stable algebraic spin liquids with gapless photon excitations have been realized in dimer model on a cubic lattice<sup>2</sup>, as well as spin models on a pyrochlore lattice<sup>3,4</sup>. In two-dimensional bosonic models with strong ring exchange terms, an “excitonic Bose liquid” phase has been proposed in<sup>5,6</sup>. More recently another type of stable boson algebraic liquid phase with softened graviton excitations has been shown to be stable on an fcc lattice<sup>7</sup>. Establishing the stability of such algebraic liquid phases is oftentimes quite subtle, since the gapless excitations are not due to the breaking of a continuous symmetry - they are not Goldstone modes<sup>8</sup>.

The proliferation of topological defects in  $d > 1$  often drives the instability of algebraic spin liquids. For example, as pointed out by Polyakov<sup>9,10</sup>, the monopole (instanton) excitation in compact electrodynamics in  $d = 2$  will always proliferate and will gap out the photon excitation. On the other hand, if a bosonic model has a set of emergent symmetries at low energies which leads to a quasi-one dimensional behavior, as in the excitonic Bose

liquid studied in<sup>5,6</sup>, stability can be achieved by analogy with Luttinger liquids in  $d = 1$ . In the present paper we explore a new algebraic boson liquid phase in which the gaplessness is protected by a similar mechanism of an emergent quasi one-dimensional behavior.

The model under consideration is motivated by cold bosonic atoms systems which can be trapped in square or cubic lattices formed by laser beams. The single particle ground state wavefunction within each well of the optical lattice will have an approximate  $s$ -wave symmetry, with wave function  $\phi_0(r) \sim \exp(-\alpha r^2)$ . The first excited states are three fold degenerate  $p$ -wave states, with wave functions  $\phi_x(r) \sim x \exp(-\alpha r^2)$ ,  $\phi_y(r) \sim y \exp(-\alpha r^2)$  and  $\phi_z(r) \sim z \exp(-\alpha r^2)$ . The wave function  $\phi_x$  extends further in the  $x$  than in the  $y$  and  $z$  directions, and so will preferentially hop to adjacent wells along the  $x$ -axis. As discussed in<sup>11</sup>, although the  $p$ -wave states are not the ground states, the life time for these states can be much longer than the average tunnelling time between neighboring optical lattice sites. If all of the atoms are pumped from the ground state to the  $p$ -wave states, they will establish a metastable equilibrium which can survive for considerable time. Here, following<sup>11</sup>, we explore the properties of bosonic atoms trapped in such excited  $p$ -wave states.

In previous studies, several phases have been identified for this system, including a superfluid phase with 1 dimensional  $\mathbb{Z}_2$  gauge symmetry<sup>11</sup>, the Mott insulator phase, a novel stripe phase<sup>12</sup>. A supersolid phase has been predicted in even higher excited bands in cold atom systems in optical lattices<sup>13</sup>. In our work, the same system with strong interaction is revisited. The interaction induces a big gap to the excitations which change the total number of bosons on each site, and hence precludes

superfluidity. We establish that in addition to the Mott insulating state, a novel bond-algebraic liquid phase can be realized in both  $d = 2$  and  $d = 3$  systems. In these phases, bond-bond correlation falls off algebraically, but the correlation function is very anisotropic. We examine the instabilities of this gapless bond liquid phase towards both Mott Insulator and the  $\mathbb{Z}_2$  phase. When the average filling per site is not a multiple of the spatial dimensions  $d$ , we find that, over a range of parameters, the bond liquid phase is stable. In  $d = 2$ , this liquid phase has a nice self-dual structure, which closely resembles the exciton bose liquid phase studied earlier<sup>5</sup>. In the  $d = 3$  case, we obtain a dual representation in terms of vortex loops which can hop on the dual lattice.

This paper is organized as follows. In section II, the low energy effective Hamiltonian is derived, and the quasilocal symmetry of the low energy Hamiltonian is discussed. In section III, we implement a spin-wave analysis which enables us to access and explore the bond algebraic liquid phase. In section IV, dual representations are derived in both  $d = 2$  and  $d = 3$ . Section V is devoted to an analysis of the stability of the bond liquid phase. Instabilities towards both the  $\mathbb{Z}_2$  phase and the Mott Insulator are also discussed. Employing the dual representation we identify the crystalline symmetry breaking pattern of the Mott insulator. In section VI, we consider modifying the model by doping with  $s$ -wave particles and softening the constraint on the particle number fluctuations within each well. Several additional phases are thereby obtained. Finally, section VII is devoted to a discussion of the prospects for experimental detection of the algebraic bond liquid phase.

## II. MODELS AND SYMMETRIES

The Hamiltonian describing the cold atom system includes the kinetic term, and the  $s$ -wave scattering:

$$H = \int d^3x \psi^\dagger(x) \left( -\frac{\hbar^2}{2m} \nabla^2 - \mu + V_T(x) \right) \psi(x) + \frac{1}{2} \frac{4\pi a_s \hbar^2}{m} \int d^3x \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x), \quad (1)$$

where  $\mu$  is the chemical potential which determines the average filling of atoms  $\bar{n}$ ,  $V_T(x)$  is the optical trap potential, and  $a_s$  is the  $s$ -wave scattering length. We now assume that within each well all of the atoms are maintained in the  $p$ -wave excited states. One can then expand the atom field operators in terms of the atomic  $p$ -wave states on each site

$$\psi(x) = \sum_i \sum_a (-1)^{\sum_b i_b} d_i^{(a)} \phi_a(x - i), \quad (2)$$

with  $a, b = x, y$  in  $d = 2$  system, and  $a, b = x, y, z$  in  $d = 3$  system. Here  $d_i^{(a)}$  denotes a boson destruction operator of flavor  $a$  on site  $i$ , and  $i_x, i_y, i_z$  denote the  $x, y, z$  coordinates of the lattice site  $i$ . Inserting this expansion

into equation (1), the kinetic term generates an on-site energy term for each flavor of atoms:

$$H_o = \sum_{i,a} \epsilon_a d_i^{(a)\dagger} d_i^{(a)}, \quad \epsilon_a = \int d^3x \phi_a(x) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_T(x) \right) \phi_a(x). \quad (3)$$

as well as a (dominant) nearest neighbor hopping term of the form,

$$H_h = - \sum_i \sum_{a,b,c} t'_{a,b,c} (d_i^{(a)\dagger} d_{i+\hat{c}}^{(b)} + h.c.), \quad t'_{a,b,c} = \int d^3x \phi_a(x) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_T(x) \right) \phi_b(x - \hat{c}). \quad (4)$$

The on-site energy for each flavor of atom needs not be the same, if the lattice symmetry is broken. For instance, if the laser beams in  $x$  and  $y$  directions have different wavelengths or amplitudes,  $p_x$  particles should have a different energy from the  $p_y$  particles. The hopping in (4) is preferentially along one of the three axes, that is the integral in (4) is small unless  $a = b = c$ . The kinetic term is dominated by the very anisotropic hopping terms, for instance  $p_x$  particles can only hop easily in the  $\hat{x}$  direction. Since the other hopping terms will be small, we expect that over a large range of energies they will be unimportant. So we henceforth set these other hopping terms to zero. This is the central assumption of all that follows. Thus the hopping term is approximated as

$$H_h = -t' \sum_i \sum_a (d_i^{(a)\dagger} d_{i+\hat{a}}^{(a)} + h.c.). \quad (5)$$

Due to the form of the  $p$ -wave orbitals the remaining single hopping strength,  $t'$  should be positive.

In addition to the kinetic energy, the ( $s$ -wave scattering) interaction between the atoms will generate an on-site repulsion, as well as an intra well pair-tunnelling term between different flavors,

$$H_s = \frac{1}{2} \frac{4\pi a_s \hbar^2}{m} \int d^3x \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x) \sim \sum_i \sum_{a,b} U_{ab} n_i^{(a)} n_i^{(b)} + \sum_{a \neq b} \gamma (d_i^{(a)\dagger} d_i^{(a)\dagger} d_i^{(b)} d_i^{(b)} + h.c.) \quad (6)$$

The  $U_{ab}$  term encodes the repulsion between particles with the same flavor as well as between different flavors. The  $\gamma$  term is the flavor mixing term, where, for example, two  $p_x$  particles can be converted into two  $p_y$  particles, and vice versa. The flavor mixing is due to the fact that, despite the vanishing overlap between the wave functions  $\phi_x(r)$  and  $\phi_y(r)$ , the overlap integral  $\int d^3x (\phi_x(r)^*)^2 (\phi_y(r))^2$  does not vanish. Angular momentum conservation does not rule out this pair conversion term because, while the  $p_x$  state is an eigenstate of  $L_x$ ,  $p_y$  is not. Moreover, single particle flavor conversion is strictly forbidden being odd under parity symmetry,

$$P_x \phi_x \rightarrow -\phi_x, P_x \phi_y \rightarrow \phi_y,$$

$$P_y \phi_y \rightarrow -\phi_y, P_x \phi_x \rightarrow \phi_x. \quad (7)$$

Because the integral  $\int d^3x (\phi_x(r)^*)^2 (\phi_y(r))^2$  is positive,  $\gamma$  will be positive.

The interflavor and the intraflavor interactions will generally not be the same. For simplicity, we assume that the interaction matrix  $U_{ab}$  together with the chemical potential can be adequately approximated by a term of the form,

$$U \left( \sum_a n_i^{(a)} - \bar{n} \right)^2 + \sum_a u (n_i^{(a)} - \bar{n}_a)^2, \quad (8)$$

$$\sum_a \bar{n}_a = \bar{n}.$$

Here  $\bar{n}_a$  is the average filling of each flavor. If all flavors are symmetric,  $\bar{n}_a = \bar{n}/d$ . Also, let us not forget that since the wave functions of  $p_a$  particles have comparatively large overlap between two nearest neighbor sites in the same  $\hat{a}$  axis ( $a = x, y, z$ ), the  $s$ -wave scattering can generate a small off-site repulsion interaction between  $p_a$  particles along  $\hat{a}$  direction  $\delta H = \sum_{i,a} \delta u n_i^{(a)} n_{i+\hat{a}}^{(a)}$ , this anisotropic nearest neighbor repulsion is supposed to be much stronger than the nearest neighbor repulsion between  $s$ -wave particles, which is usually negligible.

Due to the flavor mixing  $\gamma$  term in (6), the boson number  $n_x - n_y$  is not conserved. However, as we will show below, in the phase of primary interest, the algebraic bond liquid phase, the flavor mixing term can be irrelevant, therefore in this phase the boson numbers of each flavor of bosons can be considered as independent conserved quantities, and each flavor of boson has its own separate average filling.

Let us assume  $U$  is the biggest energy scale in (8), and  $u$  is a small energy penalty for density fluctuation around average filling of each flavor, then the main effect of the term in (8) is to keep the number of atoms in each well close to the average filling  $\bar{n}$ , which we will always take as an integer. Moreover, we consider the strong correlation limit, where  $U$  is much larger than the kinetic energy term,  $t'$ . Since the average filling for each flavor is  $\bar{n}_a$ , when  $\bar{n}_a$  is not an integer, the system is effectively “fractionally” filled. Below, we will first focus on the case with  $\bar{n} \geq 2$ , deferring a discussion of the behavior in the case with  $\bar{n} = 1$ , which is special because the intrawell pair conversion term alone cannot directly operate on the low energy subspace.

The full Hubbard type Hamiltonian without  $\delta H$  then takes the form,

$$H = H_h + H_U + H_s =$$

$$-t' \sum_i \sum_a (d_i^{(a)\dagger} d_{i+\hat{a}}^{(a)} + h.c.) + U \left( \sum_a n_i^{(a)} - \bar{n} \right)^2$$

$$+ \sum_a u (n_i^{(a)} - \bar{n}_a)^2 + \sum_i \sum_{a \neq b} \gamma ((d_i^{(a)\dagger})^2 (d_i^{(b)})^2 + h.c.). \quad (9)$$

In the large  $U = \infty$  limit, there will be a hard constraint on the Hilbert space, with each site occupied by

exactly  $\bar{n}$  particles if  $\bar{n}$  is integer. For large but finite  $U$  we will project the Hubbard Hamiltonian into this constrained Hilbert space. A single particle hopping event will take one out of this low energy Hilbert space. At the second order, a “trivial” process which hops a  $p_a$  particle at site  $i$  to site  $i + \hat{a}$ , and then hops back to  $i$  can take place. This hopping process does not change the distribution of particles, but two  $p_a$  particles at two nearest neighbor sites along  $\hat{a}$  direction can benefit from this “trivial” process. Therefore this hopping generates a small effective attractive interaction between two particles  $\delta H' = -\sum_{i,a} \delta v n_i^{(a)} n_{i+\hat{a}}^{(a)}$ ,  $\delta v \sim t'^2/U$ . This off-site attractive interaction competes against  $\delta H$ , and generally drives the system into orbitally ordered phase. If  $\delta H$  is dominated by  $\delta H'$ , ferromagnetic distribution of flavors is favored; otherwise if  $\delta H'$  wins,  $p_a$  particle at site  $i$  does not want to see the same flavor of particle stay at site  $i + \hat{a}$ , in 2 dimensional case the system will be driven to the antiferromagnetic order of  $n_x - n_y$ .

From now on let us focus on the case with  $\delta u \approx \delta v$ , thus there is no obvious tendency to orbital order, the physics is controlled by higher order perturbations (our main result does not require  $\delta u$  exactly equals to  $\delta v$ , we will argue that a small residual off-site density interaction does not destabilize the phase of main interest). At fourth order perturbation, a flavor-mixing ring exchange term of the form,

$$H_{ring} =$$

$$-\tilde{t} \sum_i \sum_{a \neq b} (d_i^{(a)} d_{i+\hat{a}}^{(a)\dagger} d_{i+\hat{a}+\hat{b}}^{(a)} d_{i+\hat{b}}^{(a)\dagger} d_{i+\hat{b}}^{(b)} d_i^{(b)\dagger} d_{i+\hat{a}}^{(b)} d_{i+\hat{a}+\hat{b}}^{(b)\dagger} + h.c.) \quad (10)$$

will survive the low energy projection. Here,  $\tilde{t} \sim (t')^4/U^3$ . The ring exchange in the XY plane is depicted in Fig. 1. This term looks fairly complicated, but if we make the usual approximation, replacing the lattice boson operators with rotors  $d^{(a)} \sim e^{i\theta_a}$ , it will appear much simpler. The effective rotor Hamiltonian reads,

$$H_r = -t \sum_i \sum_{a \neq b} \cos(\partial_a \partial_b (\theta_{ai} - \theta_{bi})) + \sum_a u (n_{ai} - \bar{n}_a)^2$$

$$+ \sum_{a \neq b} \gamma \cos(2(\theta_{ai} - \theta_{bi})) \quad (11)$$

All the derivatives in (11) should be understood as lattice derivative, i.e.  $\partial_a \theta(x) = \theta(x + \hat{a}) - \theta(x)$ .

With  $\bar{n} \geq 2$ , the Hamiltonian  $H_r$  in equation (11) is the starting point of our subsequent analysis. We first consider the symmetries of this Hamiltonian, initially tuning  $\gamma$  to be zero (the legitimacy of doing this will be discussed later). The symmetry in (11) can then be summarized as follows. When  $d = 2$ , the Hamiltonian is invariant under the transformation,

$$\theta_x \rightarrow \theta_x + f_1(y) + g_1(x) + \alpha(x, y)$$

$$\theta_y \rightarrow \theta_y + f_2(x) + g_2(y) + \alpha(x, y), \quad (12)$$

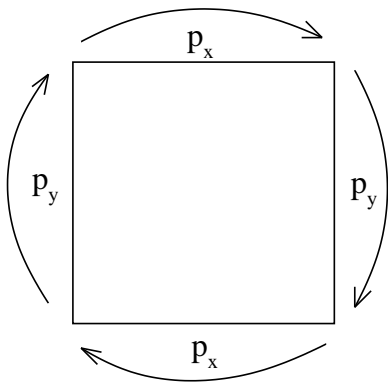


FIG. 1: A pictorial representation of the ring exchange term in (11). If a  $p_x$  particle hops to the right, another  $p_x$  particle must hop back to left, and similarly for the  $p_y$  particles.

and for  $d = 3$  an analogous form,

$$\begin{aligned} \theta_x &\rightarrow \theta_x + f_1(y, z) + g_1(x) + \alpha(x, y, z) \\ \theta_y &\rightarrow \theta_y + f_2(x, z) + g_2(y) + \alpha(x, y, z) \\ \theta_z &\rightarrow \theta_z + f_3(x, y) + g_3(z) + \alpha(x, y, z). \end{aligned} \quad (13)$$

Here  $f_i, g_i$  and  $\alpha$  are arbitrary functions of their arguments. We can understand these symmetries as follows: the  $f_i$  are functions of  $d - 1$  coordinates reflecting the fact that each flavor can only hop in one dimension. The conservation laws associated with these  $U(1)$  symmetries correspond, for example, to the conserved total number of  $x$ -flavor particles at each fixed value of  $y$  and  $z$ . By contrast, the symmetries associated with the  $g_i$  functions are additional emergent symmetries, present only after projecting into the low energy Hilbert space with a fixed number of particles on each site. Physically, in the projected Hilbert space each flavor of boson cannot hop freely: When one  $p_x$  particle hops in the  $+\hat{x}$  direction, another  $p_x$  particle must hop in the opposite direction (see Fig 1). Finally, the  $\alpha$  functions, depending on all  $d$  of the spatial coordinates, are (unphysical) gauge symmetries corresponding to the hard projection imposed on the system,  $\sum_a n_a = \bar{n}$ .

If the  $\gamma$  term is present, the continuous symmetries in (12) and (13) are broken down to  $\mathbb{Z}_2$  symmetries. All the functions  $f_i$  and  $g_i$  can only take two values, 0 or  $\pi$ .

### III. ALGEBRAIC PHASES

We first analyze the Hamiltonian in (11) in the special case with  $\gamma = 0$ . As we shall see, this enables us to access a novel algebraic bose liquid phase. We will then turn to the effects of a small but non-zero  $\gamma$ , and find that under certain conditions  $\gamma$  is an irrelevant perturbation. Under a coarse graining in space and time it will scale to zero.

To this end, we implement a “spin-wave” approximation, expanding the cosine terms in (11). The resulting action describes a Gaussian theory, with a Euclidian La-

grangian of the form,

$$\begin{aligned} L_{d=2} &= \sum_{a=x}^y \frac{K}{2} (\partial_\tau \theta_a)^2 - \frac{K}{2} (\partial_x \partial_y (\theta_x - \theta_y))^2, \\ L_{d=3} &= \sum_{a=x}^z \frac{K}{2} (\partial_\tau \theta_a)^2 - \sum_{a \neq b} \frac{K}{2} (\partial_a \partial_b (\theta_a - \theta_b))^2. \end{aligned} \quad (14)$$

Here we have rescaled space and time to set the velocity to unity. There is then just one remaining dimensionless parameter,  $K$ . Such a quadratic Lagrangian in which all terms involve derivatives of the fields, describes a scale invariant phase at long lengthscales. In a sense it can be viewed as a “fixed point” Lagrangian. The resulting phase is the desired algebraic boson liquid.

The gapless modes of this Gaussian theory can be readily calculated. In  $d = 2$  there are two modes at each momenta with frequencies given by,

$$\begin{aligned} \omega_0^2 &= 0, \\ \omega_1^2 &\sim \sin(k_x/2)^2 \sin(k_y/2)^2. \end{aligned} \quad (15)$$

The first nondispersing mode  $\omega_0$  is an unphysical gauge mode, corresponding to the function  $\alpha(x, y)$  in (12). The second mode vanishes along both coordinate axes in momentum space, as required by the symmetries encoded in the functions  $f_i$  and  $g_i$  in (12). This mode gives rise to a quasi one dimensional behavior, which is ultimately responsible for the stability of the novel bond algebraic liquid phase. If  $\omega_1$  remains gapless in the presence of all allowed perturbations, this algebraic liquid phase is stable. The double intraflavor conversion term and a set of “vertex” operators present in a dual representation (see below) can potentially destabilize the algebraic phase and gap out the mode  $\omega_1$ . The stability of the algebraic liquid phase will be discussed later.

In  $d = 3$ , in addition to the unphysical mode,  $\omega_0 = 0$ , two other modes are obtained at each momenta. At small momentum  $k$  the dispersion relations take the form,

$$\begin{aligned} \omega_1^2 &\sim 2(k_x^2 k_y^2 + k_x^2 k_z^2 + k_y^2 k_z^2), \\ \omega_2^2 &\sim \frac{3}{2} k_x^2 k_y^2 k_z^2 \frac{k^2}{k_x^2 k_y^2 + k_x^2 k_z^2 + k_y^2 k_z^2}. \end{aligned} \quad (16)$$

Notice that  $\omega_1$  vanishes along every coordinate axis in momentum space, whereas  $\omega_2$  vanishes on every coordinate plane. It is  $\omega_2$  that encodes the quasi-1d behavior, and as in  $d = 2$  will be the ultimate reason for the stability of the algebraic liquid phase.

Due to the symmetries in (12) and (13), many correlation functions vanish at this Gaussian fixed point. For instance, the correlation function between boson operators  $d^{(x)\dagger}$  and  $d^{(x)}$  vanishes, as it breaks these symmetries. For the same reason, the correlation functions between two pair conversion operators vanish. But this does not

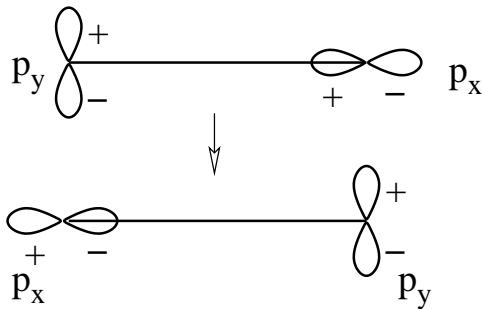


FIG. 2: A pictorial representation of the bond operator  $B_x$ . If on one site there is one  $p_x$  particle and on its adjacent site there is one  $p_y$  particle,  $B_x$  will exchange the  $p_x$  and  $p_y$  particles on the two sites.

imply that this operator is irrelevant, because two adjacent pair conversion operators can generate potentially relevant operators, which will be discussed later.

The only correlators which do not vanish are correlation functions between bond operators. In the  $d = 2$  case, the bond operators are defined as  $B_x = \cos(\partial_x(\theta_x - \theta_y))$  and  $B_y = \cos(\partial_y(\theta_x - \theta_y))$ . In the  $d = 3$  case, the bond operators are defined in a similar way, for instance,  $B_{xy,x} = \cos(\partial_x(\theta_x - \theta_y))$ . If on one site there is one  $p_x$  particle and on its adjacent site there is one  $p_y$  particle,  $B_x$  will exchange the  $p_x$  and  $p_y$  particles on the two sites (Fig. 2).

The correlation functions between bond operators are

$$d = 2 : \langle B_x(0, 0, 0) B_x(\tau, 0, y) \rangle \sim \frac{1}{|\tau^2 + y^2|^{1/(\pi^2 K)}},$$

$$d = 3 : \langle B_{xy,x}(0, 0, 0, 0) B_{xy,x}(\tau, 0, y, 0) \rangle$$

$$\sim \frac{1}{|y^2 + \tau^2|^{\eta'/K}}. \quad (17)$$

Here, the dimensionless number  $\eta'$  depends on the form of the fixed point action on the lattice scale, that is not just the small momentum form. Notice that the correlation functions between two  $B_x$  ( $B_{xy,x}$ ) operators are only nonzero when they are at the same  $(\tau, y)$  plane.

## IV. DUAL REPRESENTATION

### A. $d = 2$

In the previous section the Gaussian fixed point was obtained by expanding the cosine terms and allowing the fields to take on any real value. Here we address the legitimacy of this procedure. Specifically, we must take into account that the  $\theta$  fields are really phase angles (ie. compactified bosons) with the identification  $\theta = \theta + 2\pi\mathbb{Z}$ . This implies that particular topological defects will be allowed. The effect of such defects can be most conveniently addressed by passing to a dual representation, wherein one can identify “vertex” operators which insert

such topological defects. The dual transformation of the theory should solve the constraint  $\sum_a n_i^{(a)} = \bar{n}$ . In  $d = 2$  we define dual variables  $\phi$  and  $N$  as follows,

$$\begin{aligned} n_x - \bar{n}_x &= \partial_x \partial_y \phi, \\ n_y - \bar{n}_y &= -\partial_x \partial_y \phi, \\ \partial_x \partial_y (\theta_x - \theta_y) &= N, \end{aligned} \quad (18)$$

so that the constraint is automatically satisfied. Here,  $\phi$  and  $N$  are both defined on the dual site (plaquette centers). We can check the commutator and see that  $N$  and  $\phi$  are a pair of conjugate variables. As defined  $\phi$  takes on only discrete values, whereas  $N$  lives on the interval 0 to  $2\pi$ . But we will henceforth allow  $\phi$  (and  $N$ ) to roam over all the reals, letting  $\phi \rightarrow \varphi$  and approximately imposing the discreteness by the addition of terms in the action.

In terms of such a “coarse grained” field,  $\varphi$ , the Gaussian part of the dual action reads

$$L_{d=2} = \frac{1}{2K} (\partial_\tau \varphi)^2 + \frac{1}{2K} (\partial_x \partial_y \varphi)^2. \quad (19)$$

When  $\bar{n}$  is integer, and both  $\bar{n}_x$  and  $\bar{n}_y$  are integers, the  $\phi_i$  should all be integers. Therefore in the dual Hamiltonian there is a vertex operator  $\cos(2\pi\varphi)$  on each site. However, since the dual Gaussian action (19) has a similar symmetry as in (12),

$$\varphi \rightarrow \varphi + f(x) + g(y), \quad (20)$$

the correlation function between two such  $\cos(2\pi\varphi)$  operators on different sites is zero. The vertex operators which show up in the low energy theory are dipole operators  $V_x = \cos(2\pi\partial_x\varphi)$  and  $V_y = \cos(2\pi\partial_y\varphi)$ . At the Gaussian fixed point, the correlation function between two dipole operators is

$$V_x(0, 0, 0) V_x(\tau, 0, y) \sim \frac{1}{|\tau^2 + y^2|^{4K}}. \quad (21)$$

When  $\bar{n}_x$  and  $\bar{n}_y$  are not integers,  $\phi_i$  cannot all be integers. For simplicity, let us assume  $\bar{n}$  is even and,  $\bar{n}_x = \bar{n}/2 + 1/m$  and  $\bar{n}_y = \bar{n}/2 - 1/m$ , with  $m$  taking on all the integers. The arrangement of  $\phi$  can be taken as  $\phi = \mathbb{Z} + xy/m$ . Now the potentially relevant vertex operator in the dual action is  $\cos(2\pi\partial_x\varphi + \mathcal{B}_x)$ . Here  $\mathcal{B}_x$  is a Berry’s phase,  $\mathcal{B}_x = 2\pi y/m$ . When  $m > 1$ , this Berry’s phase gives rise to an oscillating sign on the lattice, hence the vertex operator  $\cos(2\pi\partial_x\varphi + \mathcal{B}_x)$  does not appear in the low energy coarse grained theory. However, vertex operator  $V_{qp,x} = \cos(2q\pi\partial_x\varphi) \equiv \cos(2q\pi\varphi(p, 0) - 2q\pi\varphi(0, 0))$  (with integer  $p, q$ ), as well as “parallel jump” ( $PL$ ) term  $V_{pl} = \cos(2\pi\partial_x\varphi(0, 0) - 2\pi\partial_x\varphi(1, 0))$  do not contain any oscillating sign on the lattice as long as  $qp$  is a multiple of  $m$ , and hence should appear in the low energy theory.

The leading (most relevant) operators are  $V_{1m,x}$  and  $V_{m1,x}$ , as well as  $V_{pl}$  operators. The correlation functions between  $V_{qp,x}$  and  $V_{pl}$  are

$$\langle V_{qp,x}(0, 0, 0) V_{qp,x}(\tau, 0, y) \rangle \sim \frac{1}{|\tau^2 + y^2|^{q^2 K \delta(p)}}$$

$$\langle V_{pl}(0,0,0)V_{pl}(\tau,0,y) \rangle \sim \frac{1}{|\tau^2 + y^2|^{32K/3}}. \quad (22)$$

Here  $\delta(p)$  is an integral which depends on  $p$ ,

$$\delta(p) = \frac{1}{2} \int_0^\pi dk_x \frac{(2 \sin(pk_x/2))^2}{\sin(k_x/2)}. \quad (23)$$

When  $p$  is large,  $\delta(p)$  roughly scales as  $\delta(p) \sim \ln p$ , therefore, in the case of large  $p$ ,  $V_{pl}$  is the most relevant operator in the dual formalism.

### B. $d = 3$

In the  $d = 3$  case, the compactification of the phase angles can also be dealt with in a dual representation. In the case with  $d = 3$ , we only focus on the situation with  $\bar{n}_x = \bar{n}_y = \bar{n}_z = \bar{n}/3$ , as in this case, so long as  $\bar{n} \neq 3\mathbb{Z}$ , it is enough to guarantee a stable liquid phase. Let us define dual variables  $N_i$  and  $\phi_i$  as

$$\begin{aligned} N_z &= \partial_x \partial_y (\theta_x - \theta_y), \\ N_y &= \partial_z \partial_x (\theta_z - \theta_x), \\ N_x &= \partial_y \partial_z (\theta_y - \theta_z), \\ \partial_z \partial_x \phi_y - \partial_z \partial_y \phi_x &= \bar{n}/3 - n_z, \\ \partial_x \partial_y \phi_z - \partial_x \partial_z \phi_y &= \bar{n}/3 - n_x, \\ \partial_y \partial_z \phi_x - \partial_y \partial_x \phi_z &= \bar{n}/3 - n_y, \\ [\phi_i, N_j] &= i\delta_{ij}. \end{aligned} \quad (24)$$

Here  $N_x$ ,  $N_y$  and  $N_z$  can be interpreted as elements of “vortex” lines in the  $\theta_a$  configurations, along the  $\hat{x}$ ,  $\hat{y}$  and  $\hat{z}$  directions, respectively. Moreover,  $\cos(\partial_z \partial_x \phi_y - \partial_z \partial_y \phi_x)$  is a term which hops a unit size vortex loop in the  $xy$ -plane along the  $\hat{z}$  direction (Fig. 3). Therefore the dual picture describes a system of vortex-like loops which can hop in one dimension. This is analogous to the roton hopping model studied in  $d = 2$ <sup>14</sup>.

Various phases can be understood in terms of vortex loops in boson systems. The normal three dimensional boson theory is dual to a theory of vortex loops minimally coupled to a rank-2 antisymmetric gauge field. The superfluid phase is a phase in which the vortex loops tend to shrink, and the gapless rank-2 antisymmetric gauge field corresponds to the Goldstone mode in the original theory. In the 3 dimensional crystal (or Mott insulator) phase of the bosons, the vortex loops proliferate and “condense”, gapping out the rank-2 antisymmetric gauge field through the Higgs mechanism. If there are two flavors of vortex loops, a gapless boson liquid phase can be realized, where the fluctuation of vortex loops become the photon excitations in the photon boson liquid<sup>15</sup>. Here we show that when the vortex loops hop in a perpendicular direction only, we can recover a dual representation of the algebraic boson liquid of the original  $p$ -wave boson system.

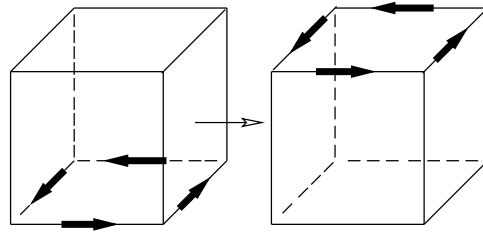


FIG. 3: The dual theory defined in (24) describes vortex loops which can only hop across a cube, as shown above.

The Gaussian part of the dual vortex loop hopping action reads:

$$L_{loop} = \sum_a \frac{1}{2K} (\partial_\tau \varphi_a)^2 + \sum_{a \neq b, b \neq c, c \neq a} \frac{1}{2K} (\partial_a (\partial_b \varphi_c - \partial_c \varphi_b))^2. \quad (25)$$

Again,  $\varphi_a$  is a coarse grained field of  $\phi_a$ . The symmetries of this dual Gaussian action are

$$\begin{aligned} \varphi_x &\rightarrow \varphi_x + f_1(x, y) + f_2(x, z) + \partial_x \alpha(x, y, z), \\ \varphi_y &\rightarrow \varphi_y + g_1(x, y) + g_2(y, z) + \partial_y \alpha(x, y, z), \\ \varphi_z &\rightarrow \varphi_z + h_1(x, z) + h_2(y, z) + \partial_z \alpha(x, y, z). \end{aligned} \quad (26)$$

Due to its definition in (24),  $\phi_a$  only takes on discrete values. Thus, as in the  $d = 2$  case, vertex operators are allowed in the dual theory. Most of these vertex operators are irrelevant at the Gaussian fixed point since the correlation functions between them break the symmetries listed in (26). One important vertex operator with nonzero correlations is a loop growing operator  $G_{ab} = \cos(2\pi(\partial_a \varphi_b - \partial_b \varphi_a) + \mathcal{B})$ , where  $\mathcal{B}$  is again a Berry’s phase. This loop growing term can enlarge, shrink, annihilate and create a vortex loop (Fig. 4). A Berry’s phase term will generally be present in the dual vertex operators, except when  $\bar{n} = 3\mathbb{Z}$  where it vanishes. Since the filling of the boson is  $\bar{n}/3$  for each flavor, when  $\bar{n}$  is not an multiple of 3, the vertex operators which survive in the low energy field theory should be  $G_{31,ab} = \cos(6\pi(\partial_a \varphi_b - \partial_b \varphi_a))$  and  $G_{13,ab} = \prod_{i=0}^2 \cos(2\pi(\partial_a \varphi_b - \partial_b \varphi_a)_{(a,b+i)})$ .  $G_{13,ab}$  is an operator that creates a “large” vortex loop which covers three unit squares (Fig. 6). Another potentially relevant vertex operator is the parallel jumping term, denoted as  $V_{pl}$ . This term hops a unit vortex loop parallel to its unit square (Fig 5). Both the loop growing term and the  $V_{pl}$  term can only correlate in the direction perpendicular to the loop. The correlation function for these two operators are,

$$\langle G_{31,xy}(0,0,0,0)G_{31,xy}(\tau,0,0,z) \rangle$$

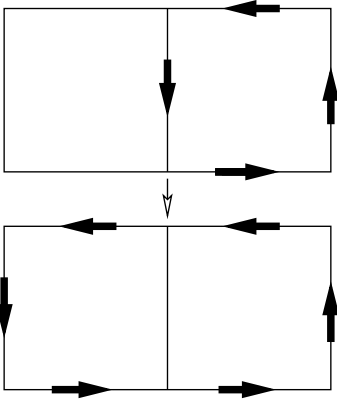


FIG. 4: The loop growing vertex operator can enlarge a vortex loop. The arrow is the direction of the vortex.

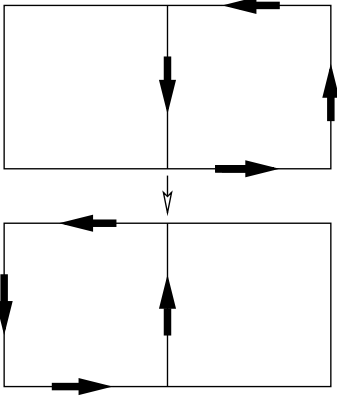


FIG. 5: The parallel jump vertex operator can jump a unit vortex loop to its adjacent unit square. The arrow shows the direction of the vortex.

$$\begin{aligned}
 & \sim \frac{1}{|\tau^2 + z^2|^{K\eta_1}}, \\
 \langle G_{13,xy}(0, 0, 0, 0) G_{13,xy}(\tau, 0, 0, z) \rangle & \\
 & \sim \frac{1}{|\tau^2 + z^2|^{K\eta_2}}, \\
 \langle V_{pl}(0, 0, 0, 0) V_{pl}(\tau, 0, 0, z) \rangle & \sim \frac{1}{|\tau^2 + z^2|^{K\eta_3}}.
 \end{aligned} \tag{27}$$

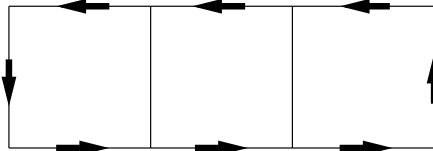


FIG. 6: The “large” loop creation operator  $G_{13,ab}$ , which creates a “large” vortex loop covering three unit squares.

Again,  $\eta_i$  are coefficients which depend on the lattice scale physics. The smallest of the three corresponds to the most relevant operator in the dual formalism. The numerical result gives that  $\eta_3 = 11.3$  is the smallest coefficient, therefore  $V_{pl}$  is the most relevant operator in the dual formalism.

## V. INSTABILITIES AND RESULTANT PHASES

### A. Instability

The algebraic phase has two dominant instabilities, which are expected when the dimensionless coupling constant  $K$  is very large or very small. If the pair tunnelling term in (11) is relevant, it breaks the symmetries listed in (13) and (12) down to  $\mathbb{Z}_2$  symmetries, which then gaps out the gapless excitations of the algebraic phases. The correlation functions between two pair-tunnelling terms  $\gamma \cos(2(\theta_a - \theta_b))$  vanishes within the algebraic bond liquid phase, but two adjacent pair-tunnelling terms can generate pair-bond operators such as  $B_{2x} = \cos(2\partial_x(\theta_x - \theta_y))$  and  $B_{2ab,a} = \cos(2\partial_a(\theta_a - \theta_b))$ . The correlation functions between pair-bond operators take the form,

$$\begin{aligned}
 d = 2 : \langle B_{2x}(0, 0, 0) B_{2x}(\tau, 0, y) \rangle & \sim \frac{1}{|\tau^2 + y^2|^{4/(\pi^2 K)}}, \\
 d = 3 : \langle B_{2xy,x}(0, 0, 0, 0) B_{2xy,x}(\tau, 0, y, 0) \rangle & \\
 & \sim \frac{1}{|y^2 + \tau^2|^{4\eta'/K}}.
 \end{aligned} \tag{28}$$

When the pair-bond terms become relevant and grow in magnitude under coarse graining, the  $\gamma$  term itself will become important too, and the system is expected to enter a  $\mathbb{Z}_2$  phase, which we discuss in the next subsection. The pair-bond terms are relevant when  $K > 2/\pi^2$  for  $d = 2$ , and  $K > 2\eta'$  for  $d = 3$ , so the  $\mathbb{Z}_2$  phase will occupy the large  $K$  region of the phase diagram as shown in Fig. 6.

The vertex operators in the dual representation can drive another instability when  $K$  is small. Once relevant, these vertex operators destabilize the algebraic phase, giving a Mott insulator which will break lattice symmetries due to the Berry’s phases. The most relevant vertex operators in the dual representation depend on the dimension and the average filling. When  $d = 2$  and  $\bar{n}_a$  are integers, the most relevant terms are  $V_x = \cos(2\pi\partial_x\varphi)$  and  $V_y = \cos(2\pi\partial_y\varphi)$ ; when  $d = 2$  and  $m$  is large enough, the most relevant terms are parallel jump terms  $V_{pl}$ . If the vertex operators are relevant, they will obviously break the symmetry in (20), since if we expand these vertex operators, they would read

$$\begin{aligned}
 V_x & \sim (\partial_x\varphi)^2, V_y \sim (\partial_y\varphi)^2, \\
 V_{pl} & \sim (\partial_x^2\varphi)^2 + (\partial_y^2\varphi)^2.
 \end{aligned} \tag{29}$$

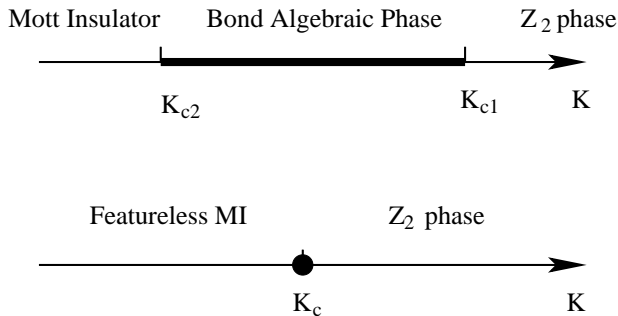


FIG. 7: The phase diagrams for the  $p$ -wave particles in both  $d = 2$  and  $d = 3$  case. The top diagram is for the case with  $\bar{n}_x - \bar{n}_y = 2/m$ , and  $m > 1$  in  $d = 2$ ; or  $\bar{n} \neq 3\mathbb{Z}$  in  $d = 3$ . In both cases nonzero Berry's phases enter the vertex operators, determining the crystalline order in the Mott insulator. The bottom diagram is for filling  $\bar{n} = d\mathbb{Z}$ , in which the Berry's phase vanishes and the Mott Insulator is featureless. Since the vertex operators are more relevant when  $\bar{n} = d\mathbb{Z}$ , the algebraic bond liquid does not exist as a stable phase.

These terms do not have the one dimensional symmetry any more.

When  $d = 3$  and  $\bar{n} = 3\mathbb{Z}$ , the most relevant terms are loop growing terms  $G_{ab} = \cos(2\pi(\partial_a\varphi_b - \partial_b\varphi_a))$ , and when  $d = 3$  and  $\bar{n} \neq 3\mathbb{Z}$ , the most relevant terms are again the  $V_{pl}$  terms. Based on the correlation functions in (22) and (27), the  $V_{pl}$  terms are irrelevant when  $K > 3/16$  for  $d = 2$  and  $K > 2/\eta_3$  for  $d = 3$ . The scaling dimensions of  $V_{1m}$  and  $G_{13}$  operators are higher than the  $V_{pl}$  terms as long as  $m$  is big enough. A numerical evaluation of the dimensionless numbers, which involves an integration over the full Brillouin zone, gives  $\eta_3 = 11.3$  and  $\eta' = 0.0950$ . Therefore there are finite parameter regions for the existence of a stable algebraic bond liquid phase in both  $d = 2$  and  $d = 3$  (Fig. 7):

$$\begin{aligned} d = 2 : 3/16 < K < 2/\pi^2, \\ d = 3 : 2/\eta_3 < K < 2\eta'. \end{aligned} \quad (30)$$

However, when  $\bar{n} = d\mathbb{Z}$  where the Berry's phase terms vanish, lower order vertex operators are allowed which are more relevant. In this case the bond algebraic liquid will not exist as a stable phase (Fig. 7). One notable aspect of this theory is that the calculation of the scaling dimensions depends on the behavior of the model on the lattice scale.

## B. $\mathbb{Z}_2$ phase

When the coupling  $K$  is sufficiently large the  $B_{2x}$  and  $B_{2ab,a}$  terms will be relevant, and the pair tunnelling term in (11) will also grow large and will drive the system into the  $\mathbb{Z}_2$  phase. What is the nature of this phase? First we notice that, even if the pair conversion term is relevant, the symmetries in (12) and (13) are still respected

as long as  $f_i$  and  $g_i$  take only two values, 0 or  $\pi$ . In the following, we will first focus on the  $d = 2$  case, which is technically much simpler. One can directly generalize to the  $d = 3$  system.

Since  $\gamma$  should be positive, when it becomes large it will lock  $\theta_x - \theta_y$  into two values,  $\pm\pi/2$ . Because  $n_x + n_y$  is a static variable in the projected Hilbert space, we can just focus on the variables  $n_x - n_y$  and  $\theta_x - \theta_y$ . Notice that if  $n_x + n_y$  is even (odd),  $n_x - n_y$  has to be even (odd) too.

Let us assume  $n_x + n_y$  is even, for simplicity. Two  $\mathbb{Z}_2$  variables can be introduced to describe the system:

$$\sigma^z = i \exp(i(\theta_x - \theta_y)), \sigma^x = \exp(i\pi(n_x - n_y)/2). \quad (31)$$

Using the formula

$$e^A e^B = e^{[A,B]} e^B e^A \quad (32)$$

we can see  $\sigma^x$  and  $\sigma^z$  anticommute with one another, so they have the same algebra as Pauli matrices. In terms of these "spin" variables, the original model (11) can be effectively written as,

$$H_{z2} = \sum_i -h\sigma_i^x - \sum_{\square} K\sigma_1^z\sigma_2^z\sigma_3^z\sigma_4^z. \quad (33)$$

This model was studied previously in reference [16], where it was introduced to describe a phase transition in an array of  $p + ip$  superconducting grains. For  $K > h$ , it was shown that the system is in a bond ordered phase, with the expectation value of  $\sigma_i^z\sigma_{i+\hat{x}}^z$  being independent of  $y_i$ . This phase has a large degeneracy,  $D$ , which can be estimated by counting the number of  $\mathbb{Z}_2$  symmetries of (12). The result is

$$D = 2^{L_x + L_y}. \quad (34)$$

The physical transformations within the degenerate ground state manifold only transform  $\theta_x - \theta_y$ . One of the classical ground states when  $h = 0$  is shown in Fig. 8<sup>17</sup>. When  $K < h$ , there is no bond order, and the ground state is not degenerate. The transition between these two phases happens exactly at  $K = h$ .

In  $d = 3$  with  $\gamma > 0$ , the three pair tunnelling operators are

$$H_v = \gamma(\cos(2(\theta_x - \theta_y)) + \cos(2(\theta_y - \theta_z)) + \cos(2(\theta_z - \theta_x))) \quad (35)$$

These three terms cannot be independently minimized, i.e. they are frustrated. Minimizing the sum of all three terms gives the following solutions,

$$\begin{aligned} \theta_x - \theta_y = \pi k_3/3, \theta_y - \theta_z = \pi k_1/3, \theta_z - \theta_x = \pi k_2/3, \\ k_1 + k_2 + k_3 = 0, \end{aligned} \quad (36)$$

where the  $k_i$  are not divisible by 3. The symmetry relating the manifold of solutions is the same as in (13) except that  $f_i + g_i$  only takes two values, 0 or  $\pi$ .



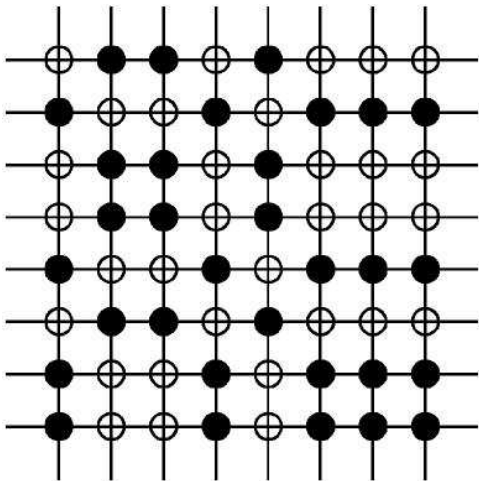


FIG. 8: One of the classical ground states of model (33) when  $h = 0$ . The ground states satisfy the condition that each unit square on the lattice has either  $(2+, 2-)$ , or  $(4+, 0-)$  or  $(0-, 4+)$  configurations.

As in  $d = 2$ , this  $\mathbb{Z}_2$  phase exhibits bond order, with spatially varying  $B_{ab,a}$  and  $B_{ab,b}$ . For instance, the expectation value of  $B_{xy,x} = \cos(\partial_x(\theta_x - \theta_y))(x, y, z)$  is independent of  $y$ . The expectation value can be  $\pm 1$  or  $\pm 1/2$ . Once again, this  $\mathbb{Z}_2$  phase has a large degeneracy. In the  $\mathbb{Z}_2$  version of the symmetry (13), the functions  $f_i$  and  $g_i$  correspond to physically different states in the ground state manifold, but this is not the case for the function  $\alpha$ . The total degeneracy  $D$  is obtained by counting all the physical symmetry transformations, which is

$$D \sim 2^{2(L_x L_y + L_y L_z + L_z L_x + L_x + L_y + L_z)} \quad (37)$$

When  $\bar{n} = 1$  the pair tunnelling term takes the system out of the projected Hilbert space. But acting in concert with hopping processes it will generate terms such as

$$H_e = \tilde{\gamma} \cos((\theta_x - \theta_y)_1 - (\theta_x - \theta_y)_2 + (\theta_x - \theta_y)_3 + (\theta_x - \theta_y)_4). \quad (38)$$

Here 1,2,3 and 4 label the four corners of one unit square. This process is shown in Fig. 9. The correlation function between two  $H_e$  operators is zero in the algebraic phase, but  $H_e$  terms can generate terms which have nonzero correlation functions along certain directions. If we check carefully, we will find that the leading term generated is still irrelevant enough for a stable algebraic liquid phase. Thus, with  $\bar{n} = 1$  the algebraic boson liquid can exist as a stable phase.

What kind phase will  $H_e$  drive the system into when it is relevant? Notice that equation (38) has the same  $\mathbb{Z}_2$  symmetry as (33), with the particle number of  $n_x - n_y$  conserved mod 2 along every row and column. Thus the order should again be bond order. Indeed, if we take (38)

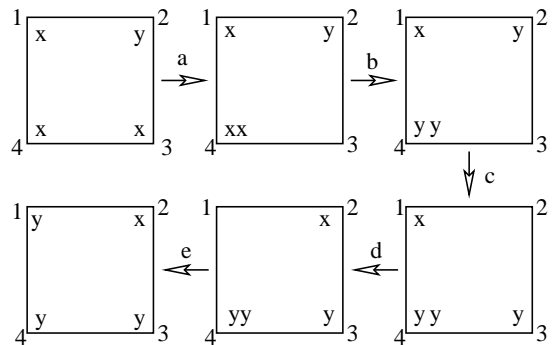


FIG. 9: The process in (38) which involves four intermediate states. Step (a),  $p_x$  particle at site 3 hops to site 4 (intermediate step); step (b), two  $p_x$  particles at site 4 can be transformed to two  $p_y$  particles due to the pair tunnelling term; step (c),  $p_y$  particle at site 2 hops to site 3; step (d),  $p_x$  particle at site 1 hops to site 2; step (e), one of the  $p_y$  particles at site 4 hops to site 1).

and the ring exchange term (11) together, the Hamiltonian involving  $\theta \equiv \theta_x - \theta_y$  reads,

$$H = -t \cos(\theta_1 - \theta_2 + \theta_3 - \theta_4) + \tilde{\gamma} \cos(\theta_1 + \theta_2 + \theta_3 - \theta_4) + \tilde{\gamma} \cos(-\theta_1 + \theta_2 + \theta_3 + \theta_4) + \tilde{\gamma} \cos(\theta_1 - \theta_2 + \theta_3 + \theta_4) + \tilde{\gamma} \cos(\theta_1 + \theta_2 - \theta_3 + \theta_4) \quad (39)$$

One of the minima of this Hamiltonian is  $\theta_i = \pi/2$ . All of the other minima can be obtained by transforming  $\theta_i \rightarrow \theta_i + \pi$  on each row and column. The symmetry and the ground state are thus the same as (33). The resulting phase is precisely the  $\mathbb{Z}_2$  phase found before.

### C. Mott Insulator phase

For small enough coupling  $K$ , when the vertex operators in the dual formalism are relevant, the system is driven into the Mott insulator phase. In the integer filling case with  $\bar{n} = d\mathbb{Z}$ , the Mott insulators are featureless and there is a unique ground state with no broken symmetries. With fractional filling, flavor-density ordered patterns which breaks space symmetries are expected. Again, let us first focus on  $d = 2$  case, and then generalize the results to  $d = 3$ .

In the 2 dimensional Mott Insulator phase, because the average filling of each flavor of bosons could be fractional, crystalline order is expected. However, the crystalline order is only in the  $n_x - n_y$  channel, since  $n_x + n_y$  is fixed to be  $\bar{n}$ . The dual action in the case with  $\bar{n} = 2\mathbb{Z}$  and  $\bar{n}_x - \bar{n}_y = 2/m$  reads,

$$L_{loop} = \frac{1}{2K} (\partial_\tau \varphi)^2 + \frac{1}{2K} (\partial_x \partial_y \varphi)^2 - \tilde{\gamma} \cos(2m\pi \partial_x \varphi) - \tilde{\gamma} \cos(2m\pi \partial_y \varphi) + \dots \quad (40)$$

The ellipses include other vertex operators, for instance the  $V_{pl}$  terms, however, these terms do not determine the

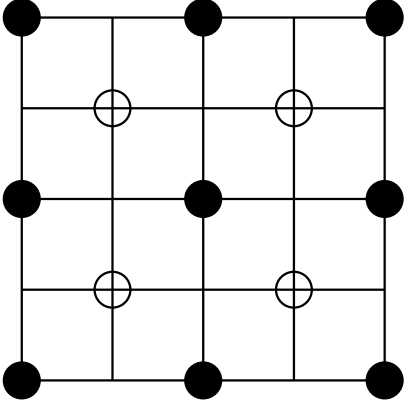


FIG. 10: The stripe order of  $n_x - n_y$  in the case of  $\bar{n}$  is odd integer and  $\bar{n}_x = \bar{n}_y$ . The full circles represent sites with  $n_x - n_y > 0$ , and the empty circles represent sites with  $n_x - n_y < 0$ .

ordered pattern. The double vertex operators induce degenerate ground state once they are relevant. The most natural phases have stripe order in the  $n_x - n_y$  density or a plaquette order. The order parameters can be represented in terms of low energy variables:

$$\begin{aligned}
 (n_x - n_y - \bar{n}_x + \bar{n}_y)e^{2\pi ix/m} &\sim \sin(2\pi\partial_y\varphi), \\
 (n_x - n_y - \bar{n}_x + \bar{n}_y)e^{2\pi iy/m} &\sim \sin(2\pi\partial_x\varphi); \\
 \cos(\partial_x\partial_y(\theta_x - \theta_y))e^{2\pi ix/m} &\sim \cos(2\pi\partial_y\varphi), \\
 \cos(\partial_x\partial_y(\theta_x - \theta_y))e^{2\pi iy/m} &\sim \cos(2\pi\partial_x\varphi). \quad (41)
 \end{aligned}$$

This representation can be obtained from either symmetry arguments or from renormalization group flow of the lattice theory. If the vertex operators in (40) are relevant, the stripe order parameters in (41) take nonzero values. If  $\tilde{\gamma}$  in (40) is positive, the plaquette stripe order is favored; if  $\tilde{\gamma}$  is negative, the  $n_x - n_y$  density stripe order is favored. If  $\bar{n}$  is odd integer and  $\bar{n}_x = \bar{n}_y$ , the dual action and the low energy representation of order parameter are similar to the equations (40) and (41) with  $m = 2$ . The  $n_x - n_y$  density stripe order with odd integer  $\bar{n}$  is depicted in Fig. 10. Both density stripe order and plaquette stripe order have been found numerically in a spin-1/2 model with a similar 1 dimensional gauge symmetry<sup>18,19</sup>. Notice that the crystalline pattern predicted from the field theory is only valid when the correlation length is much longer than the lattice constant, i.e. one is close to the phase transition between the Mott and algebraic liquid phases.

In  $d = 3$ , in the case of  $\bar{n} \neq 3\mathbb{Z}$  and  $\bar{n}_x = \bar{n}_y = \bar{n}_z = \bar{n}/3$ , the dual action including the vertex operators reads,

$$\begin{aligned}
 L_{loop} = & \\
 \sum_a \frac{1}{2K} (\partial_\tau \varphi_a)^2 + & \sum_{a \neq b, b \neq c, c \neq a} \frac{1}{2K} (\partial_a (\partial_b \varphi_c - \partial_c \varphi_b))^2
 \end{aligned}$$

$$- \sum_{a \neq b} g \cos(6\pi(\partial_a \varphi_b - \partial_b \varphi_a)) + \dots \quad (42)$$

The triple vertex operators gives rise to degenerate ground state when they are relevant. Again, because  $n_x + n_y + n_z$  is fixed at  $\bar{n}$ , the density modulation can only happen in the  $n_a - n_b$  variables. The most natural state is the stripe order, and the order parameter can be represented in terms of low energy variables as,

$$\begin{aligned}
 n_x - n_y &\sim \exp(2\pi ix/3) \sin(2\pi(\partial_y \varphi_z - \partial_z \varphi_y)) \\
 &\quad - \exp(2\pi iy/3) \sin(2\pi(\partial_z \varphi_x - \partial_x \varphi_z)), \\
 n_y - n_z &\sim \exp(2\pi iy/3) \sin(2\pi(\partial_z \varphi_x - \partial_x \varphi_z)) \\
 &\quad - \exp(2\pi iz/3) \sin(2\pi(\partial_x \varphi_y - \partial_y \varphi_x)), \\
 n_z - n_x &\sim \exp(2\pi iz/3) \sin(2\pi(\partial_x \varphi_y - \partial_y \varphi_x)) \\
 &\quad - \exp(2\pi ix/3) \sin(2\pi(\partial_y \varphi_z - \partial_z \varphi_y)). \quad (43)
 \end{aligned}$$

The plaquette stripe order parameters can be obtained in a similar way. These order parameters take nonzero expectation value once the vertex operators in (42) are relevant. Because the order parameters in (43) are not invariant under the space symmetry group, the ground states will have spontaneously broken symmetries.

A small off-site density-density interaction between bosons can in principle exist, e.g. the  $\delta H$  and  $\delta H'$  terms discussed in the second section. In the bond liquid phase, a term of this form renormalizes the field theory parameter  $K$ , but it does not introduce any new vertex operators. Thus a perturbation of off-site density interaction does not destabilize the liquid phase.

#### D. Other instabilities

Throughout the paper, we have assumed that the hopping amplitudes,  $t_{a,b,c}$  in (4), are only non-zero when  $a = b = c$ . In a real system,  $t_{a,b,c}$  will also be non-zero when this condition is not satisfied, but are expected to be very small. Once non-zero, these hopping processes will lead to new instabilities of the bond algebraic liquid phase. For instance, at the order of  $t_{x,x,x}t_{y,y,x}/U$ , bond operator  $B_{xy,x} = \cos(\partial_x(\theta_x - \theta_y))$  is generated. Once present this term will be relevant in the bond algebraic liquid phase. The system will be driven into a phase where  $e^{i(\theta_x - \theta_y)} = \pm i$ , the ground state is two fold degenerate. But so long as these other hopping amplitudes are very small, this instability will only occur at very low temperatures, and at higher temperatures the system will behave as if it were in the algebraic liquid phase. For instance, at small temperature the system can be in the quantum critical region controlled by critical point at which these perpendicular hopping terms are tuned to zero, this quantum critical point is described by the Gaussian actions (14).

## VI. OTHER PHASES

### A. Doping with $s$ -wave particles

The results in the previous sections have been obtained under the assumption that all the particles are in  $p$ -wave states. But what behavior is expected if some of the particles are in the  $s$ -wave ground state? Let us take  $d = 2$  as an example. Let us suppose that the total lattice filling is still an integer  $\bar{n}$ . For strong repulsive interactions which preclude superfluidity, we again project into the constrained Hilbert space with  $\sum_a n_a + n_s = \bar{n}$  on every site of the lattice. Here  $n_s$  is the density of  $s$ -wave particles, and we will denote the conjugate phase field as  $\theta_s$ .

Since the  $s$ -wave particles can readily hop in all directions, the low energy effective hopping within the projected Hilbert space will allow a term of the form (assuming  $d = 2$  for simplicity):

$$H_h = -t_1 \cos(\partial_x \theta_s - \partial_x \theta_x) - t_1 \cos(\partial_y \theta_s - \partial_y \theta_y). \quad (44)$$

The presence of the  $s$ -wave particles can thus strongly modify the dynamics of the  $p$ -wave particles. Moreover, in addition to the allowed pair tunneling between the  $p_x$  and  $p_y$  flavor particles, it will also be possible to exchange  $p$ -wave and  $s$ -wave particles. Although a single particle tunnelling is still forbidden by parity symmetry, pair tunnelling between  $s$ -wave and  $p$ -wave particles will be allowed. Upon including these processes, the full Hubbard-like Hamiltonian in the rotor representation will take the form,

$$H_r = -t \cos(\partial_x \theta_s - \partial_x \theta_x) - t \cos(\partial_y \theta_s - \partial_y \theta_y) \\ - \sum_{a=x}^y u_p (n_a - \bar{n}_p/2)^2 + u_s (n_s - \bar{n}_s)^2 \\ - \sum_{a=x}^y g_a \cos(2(\theta_s - \theta_a)) - g_{xy} \cos(2(\theta_x - \theta_y)) \quad (45)$$

Here  $\bar{n}_s$  and  $\bar{n}_p$  are the total fillings for the  $s$ -wave and  $p$ -wave particles, with  $\bar{n}_s + \bar{n}_p = \bar{n}$ . We are implicitly assuming that  $\bar{n} > 1$ .

Consider first the situation in which the  $s$ -wave particles are at integer filling and have a charge gap, which will be the case for large enough  $u_s$ . The above hopping term will then take the system out of the low energy sector, but can generate an effective ring exchange term,

$$H_{ring} = -t \cos(\partial_x \partial_y (\theta_x - \theta_y)) + \dots \quad (46)$$

This term is the same as the ring exchange term in (11). Therefore when the  $s$ -wave particles are gapped, the system behaves similarly to the undoped case, as expected on physical grounds.

To address the more interesting situation in which  $\bar{n}_s, \bar{n}_p$  are non-integer, it will be helpful to first consider the symmetries of the above Hamiltonian in the absence

of the pair tunneling terms,  $g_a = g_{xy} = 0$ . In this case the model is invariant under,

$$\begin{aligned} \theta_x &\rightarrow \theta_x + f_1(y) + g_1(x) + \alpha(x, y), \\ \theta_y &\rightarrow \theta_y + f_2(x) + g_2(y) + \alpha(x, y), \\ \theta_s &\rightarrow \theta_s + g_1(x) + g_2(y) + \alpha(x, y). \end{aligned} \quad (47)$$

We now search for a stable algebraic phase by first setting  $g_a = g_{xy} = 0$ , and then expanding the cosine terms in  $H_r$  to obtain the following quadratic Lagrangian,

$$L_d = \sum_{\mu=x,y,s} \frac{K}{2} (\partial_\tau \theta_\mu)^2 + \sum_{a=x}^y \frac{K}{2} (\partial_a (\theta_s - \theta_a))^2. \quad (48)$$

Upon diagonalization, one obtains 3 modes, which at small  $\vec{k}$  are given by,

$$\begin{aligned} \omega_0^2 &= 0, \\ \omega_1^2 &\sim k_x^2 + k_y^2 - \sqrt{k_x^4 - k_x^2 k_y^2 + k_y^4} \sim \frac{k_x^2 k_y^2}{k^2}, \\ \omega_2^2 &\sim k_x^2 + k_y^2 + \sqrt{k_x^4 - k_x^2 k_y^2 + k_y^4} \sim k^2. \end{aligned} \quad (49)$$

Again  $\omega_0$  corresponds to the unphysical gauge mode resulting from the hard constraint  $\sum_a n_a + n_s = \bar{n}$ . The  $\omega_2$  mode corresponds to the gapless Goldstone mode resulting from the ‘‘condensation’’ of  $\theta_s$ . The  $\omega_1$  mode vanishes along the coordinate axis in momentum space, and it gives rise to quasi-1d physics.

The relevance of the pair tunnelling terms can be estimated by considering their scaling dimension in the Gaussian phase action (48). Based on the above symmetries of the Gaussian theory, it is concluded that the operators  $\cos(2(\theta_s - \theta_a))$  can only correlate in one dimension. Moreover, a calculation shows that it has a power law correlation function, due to the contribution from  $\omega_1$  in (49). The parallel jump term generated by  $\cos(2(\theta_a - \theta_b))$  is not as relevant as  $\cos(2(\theta_s - \theta_a))$ . When  $\cos(2(\theta_s - \theta_a))$  is irrelevant, the system is in another type of algebraic liquid phase different from the bond liquid phase, which is described by the Gaussian action (48). The operators which have power law correlations are the tunneling operators  $\cos(2(\theta_s - \theta_a))$  instead of bond operators.

### B. 1d liquid phase and superfluid phase

So far everything we have considered is under the constraint  $\sum_a n_a = \bar{n}$  on every site of the lattice. What will happen if we soften this constraint? Softening the constraint will lower the gap for the charged excitations, and eventually superfluid order can develop which breaks the global  $U(1)$  symmetry of the phase  $\sum_a \theta_a$ . This superfluid phase has been discussed in reference<sup>11</sup>. But as we now show it is also possible to have a stable quasi one-dimensional liquid phase.

Let us take the  $d = 2$  system as an illustrative example. The original Hamiltonian before projection is (9). We notice that, with the pair tunnelling term, the symmetry of the system is the  $\mathbb{Z}_2$  version of the 1d symmetry, and there is moreover a global 2 dimensional  $U(1)$  symmetry  $\theta_x \rightarrow \theta_x + \theta_0$ ,  $\theta_y \rightarrow \theta_y + \theta_0$ , for a spatially independent  $\theta_0$ . In the case with small interaction  $U$ , upon passing to the rotor representation, and expanding the cosine hopping terms, the action associated with the Hamiltonian in (9) becomes,

$$L = \frac{1}{2K} \sum_{a=x,y} [(\partial_\tau \theta_a)^2 + (\partial_a \theta_a)^2] - \gamma \cos(2\theta_x - 2\theta_y). \quad (50)$$

When  $\gamma = 0$  this Lagrangian describes a type of one-dimensional liquid phase, which can be visualized as a set of decoupled 1 dimensional Luttinger liquids on every row and column of the 2 dimensional square lattice. To examine the stability of these Luttinger liquids one must check to see when the vortex tunneling events in the dual representation are irrelevant<sup>20</sup>. If we assume  $\bar{n}$  is odd, the average filling for each flavor is half-integer, and the single vortex operator in the dual theory has a nonzero Berry's phase. Double strength vortices are irrelevant provided<sup>20</sup>  $K < 2\pi$ . For  $K > 2\pi$  the proliferation of these vortices will destabilize the quasi 1d liquid phase and drive the system into a Mott insulator with broken translational symmetry.

Stability of this quasi-1d liquid phase also requires the irrelevance of  $\gamma$ . First we note that the two-point correlation function of the operator  $\cos(2\theta_x - 2\theta_y)$ , when evaluated in the quadratic theory, vanishes unless the operators are on the same site. We can thus focus exclusively on the time dependence of this correlator at (any) one given point on the 2d lattice, say  $x = y = 0$ . This site lies on only two of the Luttinger liquids, one running in the  $x$ -direction and the other in the  $y$ -direction. This problem can then be viewed as a zero dimensional point contact between two Luttinger liquids, which has been studied in detail<sup>20</sup>. Following this work we find that  $\gamma$  is irrelevant when  $K > \pi$ . We thus conclude that when  $\bar{n}$  is odd, there will be a range of parameters,  $\pi < K < 2\pi$ , within which the quasi-1d liquid exists as a stable phase.

What if  $\gamma$  is relevant? In this case we should minimize the cosine potential, which will lock the two fields  $\theta_{ix} = \theta_{iy} + \eta_i \pi/2$ .  $\eta_i = \pm 1$ , and can be represented as  $\eta_i = (-1)^{f(i_x)+g(i_y)}$ .  $f$  and  $g$  are both integer valued functions. In the simplest case wherein  $f$  and  $g$  are both constants, then the quadratic part of the Lagrangian when re-expressed in terms of a new field,  $\theta = \theta_x + \theta_y$ , will be simply  $L \sim \sum_{\mu=x,y,\tau} (\partial_\mu \theta)^2$ . This is the Lagrangian of a conventional 2d superfluid, found in previous work<sup>11</sup>. We thus conclude that the  $\gamma$  term tends to drive the system from the quasi-1d liquid phase into a 2 dimensional superfluid phase.

## VII. EXPERIMENTAL IMPLICATIONS

Here we discuss how one might try to detect experimentally the presence of the algebraic bond liquid phase. We will consider three different types of correlation functions which can in principle be measured in cold atom experiments. First, we will consider the particle density-density correlator, which can in principle be extracted via light scattering experiments. Secondly, we will consider the momentum distribution function,  $n_k = \langle \psi_k^\dagger \psi_k \rangle$ . In cold atom experiments this quantity can be extracted by releasing the atoms from the trap, and at some later time measuring their spatial distribution function in real space,  $\tilde{n}(R)$ , with  $R$  the radial coordinate measured from the center of the trap. At time  $t$  after releasing the atoms, these two quantities are related as,  $n_{k=mR/t} = \tilde{n}(R)$ . Finally, we consider the momentum density - momentum density two point function,  $\langle n_k n_{k'} \rangle$ , which can be measured by spatial noise correlations<sup>21,22</sup>.

### A. Particle density-density correlation function

The particle density-density correlation function  $\langle \rho(0)\rho(r) \rangle$  can be used to distinguish between various phases. For example, the crystalline Mott insulator phases manifest themselves as Bragg peaks, whereas quasi-1d liquid phase results in a power law behavior. The Bragg scattering techniques have been applied to atoms trapped in an optical lattice<sup>22</sup>, and the periodicity of optical lattice was extracted. The particle density-density correlation function depends on the form factor of  $p$ -wave particles, and the flavor density-density correlation function (for instance  $\langle n_x(i)n_y(j) \rangle$ ). In the phase of primary interest - the algebraic liquid phase - the correlator between total flavor density operators provides no useful information since the total particle density on each site is fixed at  $\bar{n}$ . Thus, the correlation functions between the total density operator  $n_x + n_y$  will be quite trivial. However, since  $p_x$  and  $p_y$  particles have different form factors, the particle density-density correlation function  $\langle \rho(0), \rho(r) \rangle$  contains nontrivial information about this algebraic phase.

We first briefly consider here the behavior of the correlator of the flavor-density operator, which is non-trivial in the algebraic liquid phase. From the  $d = 2$  duality we have,

$$\begin{aligned} n_x - \bar{n}_x &= \partial_x \partial_y \phi, \\ n_y - \bar{n}_y &= -\partial_x \partial_y \phi. \end{aligned} \quad (51)$$

In terms of the coarse grained field  $\varphi$ , the representation of  $n_x$  and  $n_y$  are

$$\begin{aligned} n_x &\sim \bar{n}_x + \partial_x \partial_y \varphi \\ &+ c(e^{2\pi i x/m} \sin(2\pi \partial_y \varphi) + e^{2\pi i y/m} \sin(2\pi \partial_x \varphi)), \\ n_y &\sim \bar{n}_y - \partial_x \partial_y \varphi \end{aligned}$$

$$-c(e^{2\pi ix/m} \sin(2\pi \partial_y \varphi) + e^{2\pi iy/m} \sin(2\pi \partial_x \varphi)). \quad (52)$$

The correlation functions can then be readily computed giving,

$$\langle (n_x(0, 0, 0) - \bar{n}_x)(n_x(x, 0, \tau) - \bar{n}_x) \rangle \sim e^{2\pi ix/m} \frac{1}{|x^2 + \tau^2|^{4K}} + a \frac{1}{|x^2 + \tau^2|},$$

$$\langle (n_x(0, 0, 0) - \bar{n}_x)(n_x(0, y, \tau) - \bar{n}_x) \rangle \sim e^{2\pi iy/m} \frac{1}{|y^2 + \tau^2|^{4K}} + a \frac{1}{|y^2 + \tau^2|},$$

$$\langle (n_x(0, 0, 0) - \bar{n}/2)(n_x(x, y, 0) - \bar{n}/2) \rangle \sim \frac{1}{x^2 y^2}. \quad (53)$$

When the vector separating the two  $n_x$  operators is parallel to one of axes,  $x$  or  $y$ , the correlation function has two contributions. The first comes from the sine term, the second from the derivative term in (52).

When the vector separating the two  $n_x$  operators is not along an axis, only the derivative term in (52) contributes. Notice that  $n_x$  not only correlates in the  $x$  direction, but also in the  $y$  direction. This is due to the fact that, the ring exchange term in (11) has no asymmetry between  $x$  and  $y$ . However, in the correlation function (53) the two dimensional rotational symmetry is not restored even when the scale is much larger than the lattice constant, which is quite different from other algebraic liquid phases.

The particle density correlation functions are supposed to be measurable with light scattering experiment. The result is usually written in momentum space  $\langle \rho_k, \rho_{-k} \rangle$ , which contains information of both the form factor of  $p$ -level state wave functions, and the particle density correlation functions. In the case of the bond-algebraic phase, the result at small momentum is

$$\begin{aligned} \langle \rho_k, \rho_{-k} \rangle &\sim \sum_{i,j} \sum_{a,b=x}^y \langle n_{ia}, n_{jb} \rangle \times \int d^2 x d^2 y e^{i\vec{k} \cdot (\vec{x} - \vec{y})} |\phi_{ia}(\vec{x})|^2 |\phi_{jb}(\vec{y})|^2 \\ &\sim F(k_x, k_y) \times (k_x^2 - k_y^2)^2 + \sum_i c_1 \delta(\vec{k} - \vec{G}_i). \end{aligned} \quad (54)$$

Here  $\vec{G}_i$  are the vectors in the reciprocal lattice of the optical trap potential, and  $F(k_x, k_y)$  is a function of  $k_x$  and  $k_y$ ,

$$F(k_x, k_y) = |k_x k_y| + \sum_{a=x}^y c(k_a - 2\pi/m)^{8K-1}. \quad (55)$$

The  $\phi_{ia}(\vec{x})$  is the wave function of the atoms at state  $p_a$  on site  $i$ .

In the case of Mott Insulator phase with a crystalline pattern of flavor difference  $n_x - n_y$ , the result is

$$\langle \rho_k, \rho_{-k} \rangle \sim \sum_i \delta(\vec{k} - \vec{G}'_i) (k_x^2 - k_y^2)^2 + \sum_i c_2 \delta(\vec{k} - \vec{G}'_i). \quad (56)$$

Here  $\vec{G}'_i$  are the crystalline order wave vectors. Therefore we conclude that using light scattering experiment, one can detect the bond algebraic phase.

## B. Momentum distribution function

The momentum distribution function has been used to detect both the Bose-Einstein condensate, being characterized by a sharp peak at  $k = 0$ , and the Mott insulator where there is no sharp peak<sup>23</sup>. We can calculate the momentum distribution function of various phases as follows. Focussing for simplicity on 2 dimensional systems, the momentum distribution for the superfluid phase of  $s$ -wave particles is

$$\begin{aligned} n(\vec{k})/N &= 1/N \int d^2 r_1 d^2 r_2 \langle \psi^\dagger(\vec{r}_1) \psi(\vec{r}_2) \rangle e^{i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)} \\ &\sim 1/N \int d^2 r_1 d^2 r_2 \exp(i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)) \\ &\quad \times \sum_{i,j} \langle \phi_{si}(\vec{r}_1) d_i^{(s)\dagger} \phi_{sj}(\vec{r}_2) d_j^{(s)} \rangle \\ &= 1/N \sum_{i,j} e^{i\vec{k} \cdot (\vec{R}_i - \vec{R}_j)} |\phi_s(\vec{k})|^2 \langle d_i^{(s)\dagger} d_j^{(s)} \rangle \\ &\quad \sim \sum_i |\phi_s(\vec{k})|^2 \delta(\vec{k} - \vec{G}_i) \end{aligned} \quad (57)$$

Here  $\vec{G}$  is the vector in the reciprocal lattice of the optical lattice,  $\phi_s(\vec{k})$  is the Fourier transformation of the real space  $\phi_s(r)$ , which roughly behaves as  $\phi_s(\vec{k}) \sim const.$  at small  $k$ . The momentum distribution function of the superfluid phase of  $p$ -wave particles has been calculated in<sup>11</sup>. The result is complicated by the fact that in this phase there is large set of  $\mathbb{Z}_2$  degeneracies, as shown in (34). The result of the calculation depends on which of the degenerate states the system is in.

For a Mott Insulator phase of  $s$ -wave particles, the momentum distribution is

$$n(\vec{k})/N \sim |\phi_s(\vec{k})|^2. \quad (58)$$

For a Mott insulator of  $p$ -wave particles, one finds,

$$n(\vec{k})/N \sim (a|\phi_x(\vec{k})|^2 + b|\phi_y(\vec{k})|^2). \quad (59)$$

Here, the magnitudes of  $a$  and  $b$  depend on the relative abundance of  $p_x$  and  $p_y$  particles. At small momentum,  $\phi_x(\vec{k}) \sim k_x$ ,  $\phi_y(\vec{k}) \sim k_y$ .

Although the algebraic bond liquid and the Mott Insulator phases are qualitatively different, the momentum distribution functions are the same,

$$n(\vec{k})/N \sim (|\phi_x(\vec{k})|^2 \bar{n}_x + |\phi_y(\vec{k})|^2 \bar{n}_y). \quad (60)$$

The reason for this is that the single particle correlation function is short-ranged in both the Mott insulator and the algebraic bond liquid nature, behaving in the latter case as  $\langle d_i^{(x)\dagger} d_j^{(x)} \rangle \sim \delta_{ij}$ . Thus the momentum distribution function cannot distinguish between these two insulating phases.

### C. Momentum density - momentum density correlator

Recently it has been proposed that the momentum density - momentum density correlation function

$$\langle n(R_1)n(R_2) \rangle \sim \langle n_{Q_1}n_{Q_2} \rangle, \quad (61)$$

with  $\vec{Q}_i = m\vec{R}_i/t$  can be extracted in cold atom systems by measuring spatial noise correlations<sup>21</sup>. Remarkably, this technique has been successfully implemented to detect crystalline density order<sup>22</sup>.

In the superfluid phase of  $p$ -wave particles, this correlator has been calculated in<sup>11</sup>, and the result again depends on which of the  $\mathbb{Z}_2$  degenerate ground states the system is in. As we now show, this correlator can in principle be used to distinguish between the algebraic bond liquid and the Mott insulator phases. The calculation is easier to carry out for the normal ordered operator,

$$\langle \psi_k^\dagger \psi_{k'}^\dagger \psi_k \psi_{k'} \rangle / N^2 = \langle n_k n_{k'} \rangle / N^2 + \langle n_k \rangle \delta(\vec{k} - \vec{k}') / N^2, \quad (62)$$

and  $\langle n_k \rangle / N$  is given in (60). The calculation of the correlation function in the algebraic bond liquid phase is lengthy but straight forward. At small  $k$  we find,

$$\begin{aligned} & \langle n_{\vec{k}} n_{\vec{k}'} \rangle / N^2 \\ & \sim a(\bar{n}_x k_x^2 + \bar{n}_y k_y^2)(\bar{n}_x k_x'^2 + \bar{n}_y k_y'^2) \\ & + \sum_i b(k_x k_x' \bar{n}_x + k_y k_y' \bar{n}_y)^2 \delta(\vec{k} - \vec{k}' + \vec{G}_i) \\ & + c_1(k_x^2 - k_y^2)(k_x'^2 - k_y'^2) / N \\ & + \alpha(k_x k_x' - k_y k_y')^2 F(k_x - k_x', k_y - k_y') / N \\ & - \gamma k^2 \delta(\vec{k} - \vec{k}') / N + \dots \end{aligned} \quad (63)$$

Here  $N$  is the total number of sites in the system, and the ellipses include all the terms smaller than  $O(1/N)$ .  $F$  is the function defined in (55). Vectors in the reciprocal lattice of the optical lattice are denoted as  $\vec{G}$ . The algebraic correlation function between flavor density give rise to the third and fourth line in equation (63).

In the stripe ordered Mott Insulator phase defined in (41), the momentum density - momentum density correlation function reads,

$$\langle n_{\vec{k}} n_{\vec{k}'} \rangle / N^2$$

$$\begin{aligned} & \sim a(\bar{n}_x k_x^2 + \bar{n}_y k_y^2)(\bar{n}_x k_x'^2 + \bar{n}_y k_y'^2) \\ & + \sum_i b(k_x k_x' \bar{n}_x + k_y k_y' \bar{n}_y)^2 \delta(\vec{k} - \vec{k}' + \vec{G}_i) \\ & + \sum_i c_2(k_x k_x' - k_y k_y')^2 \delta(\vec{k} - \vec{k}' + \vec{G}'_i) \\ & - \gamma k^2 \delta(\vec{k} - \vec{k}') / N + \dots, \end{aligned} \quad (64)$$

where  $\vec{G}'_i$  is the ordering wave vector of the stripe order. Therefore, this correlation function can be used to distinguish between the Mott insulator and the algebraic bond liquid phase. At the leading order, the two phases differ by the ordering wave vector  $\vec{G}'_i$ . There are also qualitative differences in the  $O(1/N)$  terms at small  $k$ .

A more complicated correlation function which can in principle be used to directly measure the power law correlations in the algebraic bond liquid phase is  $\langle n_k^2 n_{k'}^2 \rangle$ . We find,

$$\begin{aligned} \langle n_k^2 n_{k'}^2 \rangle / N^4 & \sim O(1) + \dots + \alpha(1/N^3) k_x^2 k_y^2 k_x'^2 k_y'^2 \\ & \times (|k_x - k_x'|^{2/(\pi^2 K)-1} + |k_y - k_y'|^{2/(\pi^2 K)-1} \\ & + |k_x + k_x'|^{2/(\pi^2 K)-1} + |k_y + k_y'|^{2/(\pi^2 K)-1}). \end{aligned} \quad (65)$$

The coefficient  $K$  is defined in the Gaussian action (14), and it directly encodes the power law decays in the algebraic bond liquid phase.

## VIII. CONCLUSIONS AND EXTENSIONS

In this work, a novel bosonic liquid phase has been predicted and explored in a system with strongly correlated bosons in different orbital levels of the wells of an optical lattice. In this algebraic boson liquid phase the correlation functions between two bond operators fall off as a power of the spatial separation. The reason for the existence of this new phase is the spatial anisotropy of the orbital wave functions: the  $p$ -level states extend and hop preferentially in one direction. Due to this orbital anisotropy, such a system in either  $d = 2, 3$  can behave similarly to a one dimensional system. This effect has been called “dimensional reduction”<sup>24,25</sup>. From the theoretical point of view, the phases studied in this work are new bosonic liquid phases in dimensions higher than one, which broadens our understanding of gapless “insulating” bosonic phases. Experimentally, it is an open challenge to see if such algebraic phases can be achieved in the cold atoms context.

A possible extension of this work is to consider a fermionic version of the model studied herein. Fermionic atoms such as <sup>40</sup>K and <sup>6</sup>Li trapped in an optical lattice have been studied extensively recently, both theoretically and experimentally. Various behaviors have been predicted - and some observed - in such fermionic atomic

systems. For instance, the normal fermi surface<sup>26</sup>, the condensate of molecules  $^{40}\text{K}_2$  and  $^6\text{Li}_2$ <sup>23,27</sup>, the fermionic superfluid state with and without imbalanced fermion populations<sup>28,29</sup> and the crossover between molecular condensate and the BCS like cooper pair condensate (for instance, see<sup>30</sup>).

If fermionic atoms are pumped to  $p$ -level states, when interaction between atoms keeps the filling on each well

close to  $\bar{n}$ , we can again effectively study the low energy projected Hilbert space with the constraint. The ring exchange term similar to (10) is generated at low energy. New fermionic phase different from all the previous studies is expected in this case, characterized by novel non-fermi liquid behaviors.

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