Onset of Superfluidity in Random Media

Daniel S. Fisher
Joseph Henry Laboratories of Physics, Jadwin Hall, Princeton University, Princeton, New Jersey 08544

Matthew P. A. Fisher
IBM Research Division, Thomas J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598
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The zero-temperature superfluid onset transition for a system of repulsively interacting bosons in a random potential is studied. The associated quantum critical behavior is characterized in d dimensions by exponents $\nu \geq 2/d$, η, and a dynamic exponent z. Static and dynamic scaling laws are derived: Some measurable exponents, such as $z=d$, are predicted exactly. Applications to experiments on ^{4}He absorbed in porous media are discussed.

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Perhaps the best studied phase transition of all is the transition from normal to superfluid ^{4}He as the temperature is decreased through the λ point. The corresponding transition in thin films is also well understood both theoretically[1] and experimentally.[2] Yet the problem of superfluid onset at *zero temperature* for helium in a random medium has received surprisingly little theoretical attention,[3-5] particularly in light of numerous experiments on such systems as ^{4}He absorbed in porous media[6,7] and on various substrates.[2] In these systems, it is found that a critical density of ^{4}He, n_c, is needed for the system to become superfluid at any positive temperature T. The implied transition at $T=0$ as the density is increased through n_c, represents an onset of superfluidity controlled entirely by quantum fluctuations. This *onset transition* is the subject of this paper. Although the onset transition only occurs strictly *at* $T=0$, and is thus multicritical, the properties of the transition will also control the low-temperature behavior for n in the vicinity of n_c.

Zero-temperature transitions occur in many physical systems: the transition from singlets to antiferromagnetism,[8] onset of superconductivity in granular materials,[9] and metal-insulator transitions in disordered Fermi systems.[10] The experience of studying the λ transition in ^{4}He suggests that the $T=0$ transition of ^{4}He in random media may lead to insights into more complicated $T=0$ transitions.

We will argue that repulsively interacting bosons in a random potential undergo an onset transition whose critical behavior is characterized by exponents $\nu \geq 2/d$, η, and a dynamic exponent z which we predict is equal to the spatial dimension d. Scaling laws for various static and dynamic properties for $n \approx n_c$ and T small are derived following Ma, Halperin, and Lee.[5] Many of these depend only on z and we can thus predict some exponents exactly.

Specifically, we consider a system of bosons with repulsive interactions, $g(\mathbf{x}-\mathbf{x}')$ of range a moving in a random potential $U(\mathbf{x})$ with the density controlled by a chemical potential μ. The partition function $Z = \int D\psi D\psi^* \exp(-S)$ in terms of the imaginary time action

$$S=\int d^dx \int_0^\beta d\tau [\psi^* \partial_\tau \psi + \tfrac{1}{2} |\nabla\psi|^2 - \mu |\psi|^2 + U(\mathbf{x}) |\psi(\mathbf{x},\tau)|^2 + |\psi(\mathbf{x},\tau)|^2 g(\mathbf{x}-\mathbf{x}') |\psi(\mathbf{x}',\tau)^2] , \tag{1}$$

yields the free-energy density $f=-(\beta V)^{-1} \ln Z$ where $\beta = 1/T$, V is the volume, and we have set $\hbar = m = 1$.

In the absence of interactions, all the bosons at $T=0$ will condense into the lowest-energy single-particle eigenstate of the potential $U(\mathbf{x})$ which will be localized. Thus, in contrast to Fermi systems, the repulsive interactions are needed to stabilize the system. If the characteristic magnitude of the random potential is weak compared to the interactions and the chemical potential μ adjusted so that the density $n \sim a^{-d}$, the system should be superfluid and the random potential merely a weak perturbation. Conversely, if the random potential is large or the density small, then one can fill the low-lying localized states of the potential almost independently with bosons until the energy ϵ_j (because of the repulsive interactions and random potential) is greater than μ. In this regime, the hopping between the localized states will be small compared to the typical excitation energies $\epsilon_j - \mu$ and, by analogy with strongly localized Fermi systems, the equal-time Green's function $G(x,x') = \langle \psi^+(x) \psi(x') \rangle$ will typically decay exponentially. The system will thus not be superfluid in this regime—indeed, it will be insulating at zero temperature. We refer to this phase as a *Bose glass*.

Let us consider schematically how the Bose-glass phase is destroyed as μ increases with $T=0$.[11] As the exchange between the "localized states" grows, the low-energy quasiparticle excitations will become less localized and the Bose statistics more important. When the localization length of the lowest quasiparticle excitations diverges, additional bosons can condense into the available extended state, thereby becoming superfluid.[12] This schematic argument suggests that a transition will occur *directly* from the Bose glass to the superfluid at some critical chemical potential μ_c.[13,14] We define $\delta \equiv \mu - \mu_c$ to be the distance from this transition.

From the above picture, it is apparent that the compressibility $\kappa = dn/d\mu$ will be nonzero in the Bose-glass phase since there will always be somewhere in the system where the (free) energy, $\epsilon_j - \mu$, required to add a quasiparticle will be arbitrarily small. In the superfluid phase, κ will also be nonzero, so we expect that $\kappa > 0$ at the transition.[13,14]

Investigation of the critical behavior at the onset transition is impeded by two factors, as discussed in detail in a longer paper[13]: First, the Bose-glass phase does not appear at all in an infinite-range-hopping mean-field limit about which one might hope to expand, and, second, since both disorder and interactions are necessary for the Bose glass to occur, there does not appear to be a natural Gaussian limit about which one can expand as attempted by Ma, Halperin, and Lee.[5] Indeed, we will argue that it is likely that there is no upper critical dimension for this problem. Fortunately, recent work[15] in one dimension does provide an important reference point with which we can compare the general scaling arguments given below.

The superfluid onset transition is driven by a competition between boson exchange, which tends to minimize the gradients of the phase of the order parameter, and the interactions and disorder which tend to minimize the number of fluctuations. Thus quantities which directly involve this competition should exhibit simple scaling behavior. In the superfluid phase, one such quantity is the fourth-sound (or in films third-sound) velocity, c_4, given at $T=0$ by $c_4^2 = \rho_s/\kappa$ where ρ_s is the superfluid density. At wavelengths much larger than the correlation length ξ in the superfluid, the (imaginary time) action for fourth sound can be expressed solely in terms of the phase ϕ of the order parameter[16]:

$$S_4 = \tfrac{1}{2} \int d^d x \int d\tau [\rho_s (\nabla\phi)^2 + \kappa(\partial_\tau \phi)^2] , \qquad (2)$$

where the second term, which involves the *full* compressibility, arises because the phase is conjugate to the *total* number density n.

Our basic scaling *Ansatz* is that, as $\mu \to \mu_c$ from above, the scaling of the fourth-sound fluctuations will be given by the fundamental characteristic correlation length ξ and correlation frequency $\Omega \sim \xi^{-z}$, where z is the dynamic critical exponent. In particular, this implies that $c_4 \sim \xi^{1-z}$ so that z is determined by the scaling of the static quantities ρ_s and κ. This is directly analogous to the *dynamic* scaling behavior at the usual λ point of ^{4}He as described by "Model F."[17] In our case, however, the fluctuations are purely quantum mechanical so that dynamic and static quantities are inexorably linked. Thus we expect that characteristic excitation energies and temperatures will scale in the same way as the characteristic frequency Ω.

Further scaling laws can be obtained by an argument analogous to hyperscaling at conventional thermal transitions. In particular, from Eq. (2) it is apparent that ρ_s and κ can be defined in terms of the change in energy under an appropriate change in boundary conditions[18] (which imposes, say, a twist of the phase by π in space or time, respectively). Then a straightforward application of finite-size scaling gives[18]

$$\rho_s \sim \delta^\zeta \text{ with } \zeta = (d+z-2)\nu , \qquad (3)$$

which is a generalization of Josephson's relation[18] to the $T=0$ onset transition as obtained previously by Ma, Halperin, and Lee[5] and

$$\kappa \sim |\delta|^{(d-z)\nu} . \qquad (4)$$

Here we have used $\xi \sim |\delta|^{-\nu}$. A similar hyperscaling argument yields that the singular part of the free-energy density given by $f_s \sim |\delta|^{2-\alpha} \sim |\delta|^{(d+z)\nu}$. Taking two derivatives with respect to μ yields a singular part of the compressibility

$$\kappa_s \sim |\delta|^{(d+z)\nu - 2} \sim |\delta|^{-\alpha} . \qquad (5)$$

We now have two alternatives: either (i) κ is dominated by its singular part so that $\kappa \sim \kappa_s$ implying $z\nu = 1$ or (ii) $\kappa \to$ const at μ_c and κ_s represents a singular correction. In this case we obtain $z=d$ from Eq. (4).

In order to distinguish between these two cases, we can apply the general inequality for the correlation-length exponent ν for random systems[19] $\nu \geq 2/d$. This implies that for any positive z, $\alpha < 0$ and hence κ cannot diverge. Thus the only way the singular part of κ can control the dynamics is if $\kappa \to 0$ at μ_c in which case (i) applies and $z \leq d/2$. This is *a priori* unlikely since it would imply a gap for particle-hole excitations (at $\mu = \mu_c$) which there is no reason to expect.[14] Moreover, the physical requirement that c_4 does not diverge as $\delta \to 0$, implies $z \geq 1$ for all d, or $z > d/2$ for $d < 2$. Thus, at least for $d < 2$, case (i) is definitely ruled out implying $z = d$.

In one dimension Giamarchi and Schulz[15] have investigated the onset transition in a random potential and found that κ and ρ_s both go to a constant at μ_c with[20] $\nu = \infty$ and $z = 1$, consistent with our scaling laws for case (ii). We note that a naive (and perhaps questionable) extrapolation[21] of the renormalization-group equations of Giamarchi and Schulz to $d = 1+\epsilon$ yields, to leading order, $z = 1+\epsilon$ also consistent with the equality $z = d$.

Before listing predictions of the scaling picture, we comment on the likely behavior in higher dimensions. In the absence of an explicit applicable mean-field or Gaussian calculation, we can try to guess the upper critical dimension, d_c, by asking whether $\nu=\frac{1}{2}$ and $\zeta=1$ in some dimension. This would suggest that $d_c=2$. However, from the correlation-length inequality[19] ν cannot be $\frac{1}{2}$ in $d=2$. This implies that either (a) the mean-field limit is unconventional, (b) the scaling law $z=d$ breaks down in some unknown way (implying the absence of fourth sound in scaling functions), or (c) the equality $z=d$ and hyperscaling is valid in *all* dimensions. Given the problems with obtaining a mean-field limit mentioned above and in detail in Ref. 13, it appears that possibility (c) is most likely and we will assume this henceforth. (We note that such an absence of a mean-field limit is also believed to occur for the Anderson localization transition of noninteracting fermions.[10])

A scaling law for the superfluid density can be obtained from (3) with temperatures scaling as $\xi^{-z} \sim |\delta|^{z\nu}$:

$$\rho_s(T,\mu)\sim\delta^{(d+z-2)\nu}P(T/|\delta|^{z\nu}). \quad (6)$$

In $d<2$, the crossover scaling function P is zero except when its argument $u=0$, while in $d\geq 2$, P is nonvanishing for $\delta>0$ and $u<u_c$, where u_c is some critical value of u. As $u\rightarrow u_c$ from below, $P(u)\sim(u_c-u)^{(d-2)\nu_T}$, where ν_T is the exponent for the thermal superfluid transition in the medium. Thus for $d\geq 2$, the superfluid transition temperature is $T_c\sim|\delta|^{z\nu}$. More directly measurable is the behavior of T_c as a function of $\rho_s(T=0)$: $T_c\sim\rho_s(0)^x$ with[5]

$$x=z/(d+z-2)=d/(2d-2). \quad (7)$$

As $T\rightarrow 0$, the specific heat C_P should vanish as

$$(T/c_4)^d\sim T^d\delta^{d\nu(1-z)}\sim T_c(T/T_c)^d$$

in the superfluid phase since the single-particle excitations will presumably be subsumed by the superfluidity. In the Bose-glass phase, on the other hand, localized quasiparticle and quasihole excitations should give rise to a constant density of states yielding a linear specific heat as in other disordered systems. At μ_c, scaling implies $C_P\sim T^{d/z}\sim T$. In addition, at μ_c we have the multicritical correlation length $\xi(\mu_c,T)\sim T^{-1/z}$.

Another important quantity is the linear (mass) conductivity σ of the system driven by a chemical-potential gradient. The conductivity should scale with the particle diffusivity, $D=\sigma/\kappa$, since κ is constant at μ_c. Moreover, since disorder is important at the fixed point, one expects D to scale as ξ^{2-z} implying

$$\sigma(T,\mu)\sim T^{(z-2)/z}\Sigma(T|\delta|^{z\nu}). \quad (8)$$

For $d\geq 2$ the scaling function $\Sigma(u)\rightarrow$ const for $u\rightarrow\infty$ (i.e., at μ_c) diverges at the superfluid transition as $u\rightarrow u_c$ from above for $\delta>0$; and vanishes for $u\rightarrow 0$ on the Bose-glass side, i.e., $\delta<0$. In the Bose glass, the low-temperature conductivity is presumably of a variable-range-hopping form as for the Fermi glass[10]: $\sigma(T)\sim\exp[-(T_H/T)^y]$, with $y<1$ and the characteristic temperature $T_H\sim|\delta|^{z\nu}$.

Applications to helium.—The most obvious application is to 3D systems of ^{4}He absorbed in various porous media, such as Vycor and silica gels.[6,7,22] The primary measured property is $\rho_s(T,n)$, and it is found that $\rho_s(0)=0$ for $n<n_c$ with n_c corresponding to about 2 monolayers of ^{4}He on the inside of the pores. Most previous work[4,6] has assumed that this is an inert layer and that for $n\gtrsim n_c$ the remaining helium atoms behave approximately like a dilute Bose gas with the rather questionable additional assumption that the random potential can be neglected. This picture yields $\rho_s(0)\sim n-n_c$ and the exponent in Eq. (7) with $x=\frac{2}{3}$, in $d=3$. However, provided the so-called inert layer has a nonvanishing compressibility (at $T=0$), as it most surely will, asymptotically close to the $T=0$ (quantum) onset transition the critical behavior should be controlled by the scaling theory described above.[23] Specifically, we predict in 3D, $x=\frac{3}{4}$ and $\zeta=2(d-1)\nu\geq\frac{8}{3}$, the latter in strong contrast to the inert layer model value of 1. (Note that a constant κ at μ_c enables one to scale with $n-n_c$ instead of δ.)

More generally, the scaling of $\rho_s(T,n)$ should be governed by Eq. (6) with finite-temperature (thermal) critical behavior near $T_c(n)$. Detailed comparison with experiment might be impeded, though, by the narrowness of either the ($T=0$) quantum or thermal asymptotic critical regimes. The apparent narrowness of the *quantum onset* critical region in Vycor[6] is presumably due to the low rate of exchange between the almost solid layers and the more mobile atoms. However, the (small) tail observed in $\rho_s(0)$ as n is decreased does, indeed, suggest a crossover into the nontrivial onset critical region with an exponent ζ of order 2.[6] A wider and more accessible quantum critical regime is expected in systems with *strong* disorder, uncorrelated down to atomic scale, for which the compressibility for $n\lesssim n_c$ will be large. Preliminary fits to the data on carbon black, which exhibits a wider apparent critical regime for $\rho_s(0)$ as a function of n, yield a value of $\zeta\approx 2.5$ and $x\approx 0.8$.[22]

By contrast, at the *thermal* transition, strong disorder is expected to *impede* the observation of true critical behavior. Indeed, the slow transient associated with the weakly irrelevant disorder ($\alpha_T\lesssim 0$) would probably lead to an *apparent* exponent *larger* than the expected asymptotic $\nu_T\approx\frac{2}{3}$ (as observed in experiments on ^{4}He in silica gels[6,22]). More experiments on both these and other related systems, such as ^{4}He films absorbed onto strongly disordered 2D substrates, would certainly be desirable.

In summary, we have produced scaling arguments for the critical behavior near the onset of superfluidity of bosons in a random potential at zero temperature. The critical behavior is determined (except for the hard to measure η) by the dynamic exponent $z=d$, which we conjecture holds in *all* dimensions, and $\nu \geq 2/d$ with $\nu \to \infty$ as $d \to 1$. Some of the concepts are likely to be applicable to disordered quantum spin systems, to the onset of superconductivity in amorphous and granular systems,[9] and perhaps will also lead to insights into the subtle effects near zero-temperature metal-insulator transitions[10] in interacting Fermi systems for which there is no simple order parameter. For disordered superconductors undergoing a transition from an insulator to a superconductor, we note that the present work should apply with a modification to account for the long-range $1/r$ forces between the Cooper pairs. This yields $z=1$ for all d rather than $z=d$.

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