Nonequilibrium Transport through a Point Contact in the $\nu = 5/2$ Non-Abelian Quantum Hall State

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We analyze charge-$e/4$ quasiparticle tunneling between the edges of a point contact in a non-Abelian model of the $\nu = 5/2$ quantum Hall state in the presence of a finite voltage difference using the time-dependent density-matrix renormalization group method. We confirm that, as the voltage decreases, the system is broken into two pieces. In the limits of small and large voltage, we recover the results expected from perturbation theory about the infrared and ultraviolet fixed points. We test our methods by finding the analogous nonequilibrium current through a point contact at $\nu = 1/3$.

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Introduction.—The $\nu = 5/2$ fractional quantum Hall state [1–3] is suspected to be a non-Abelian topological state. While theoretical evidence has been steadily accumulating over the years, there has been little experimental evidence—until now. Several transport measurements utilizing a point contact, one a low-frequency noise (''shot noise'') measurement [4], the other the tunneling current [5], indicate that the smallest quasiparticle charge at this plateau is $e/4$. This is consistent with two non-Abelian models of the $\nu = 5/2$ fractional quantum Hall state [6–9], but it also consistent with an Abelian model [10].

One limit to the success of these measurements is that the data are compared only to lowest-order perturbative calculations valid for small interedge tunneling at the point contact. However, interedge tunneling in these experiments is not so small and may have effects not described with low-order perturbation theory. Therefore, it is important to compute the expected I-V curve (and the low-frequency noise) beyond the perturbative regime. In so doing, we follow the crossover from the limit of weak quasiparticle tunneling across the point contact to the low-temperature, low-voltage regime.

In this Letter, we numerically compute the zero-temperature current through a point contact in the presence of a finite voltage bias in two fractional quantum Hall states. The first is the Abelian Laughlin state at $\nu = 1/3$, which allows us to check our numerical procedure and to confirm that indeed the Bethe ansatz computations [11] are applicable out of equilibrium. The second is the non-Abelian Moore-Read Pfaffian state with $\nu = 5/2$ [6]. We show that in both cases the droplet eventually breaks in two at low voltages. In the weak quasiparticle tunneling limit, the current follows the predicted power law. In the low-voltage limit, the conductance is obtained and is approached in a manner consistent with predictions [12].

We use the time-dependent density-matrix renormalization group (DMRG) method [13,14], because other approaches have difficulties. Namely, at finite bias, this nonequilibrium calculation is not amenable to a Monte Carlo simulation. (The Monte Carlo computation in the $\nu = 1/3$ case is applicable only to the linear-response regime [15].) As we discuss below, the problem can be mapped onto resonant tunneling between attractive Luttinger liquids. Since the ''leads'' are interacting, a Wilsonian numerical renormalization group cannot be used, unlike in the Kondo problem. Finally, when $\nu = 5/2$, the model is not integrable, and thus the Bethe ansatz is not applicable.

Models.—The edge excitations of the Laughlin state at $\nu = 1/3$ are described by a chiral Luttinger liquid. At a point contact, two edges come into close proximity so that a charge $e/3$ quasiparticle can be backscattered from one to the other. We describe each edge by a chiral boson ϕ_i, so that a charge $e/3$ Laughlin quasiparticle is created by the operator $e^{i\phi_i/\sqrt{3}}$. The effective Hamiltonian is

$$\mathcal{H}^{1/3} = \sum_{i=1,2} \frac{v_c}{4\pi} \int dx (\partial_x \phi_i)^2 + t e^{i[\phi_1(0) - \phi_2(0)]/\sqrt{3}} + \text{H.c.}$$

The tunneling amplitude has scaling dimension $[t] = 2/3$. Hence, at zero temperature the backscattered current $I_B \sim t^2 V^{-1/3}$ in the limit of a small tunneling current and large voltages. In the low-voltage limit, perfect backscattering occurs, and the Hall bar effectively breaks in two [16]. Charge transport between the two halves is due to electron tunneling, so for small V, $I_B - \frac{1}{3}\frac{e^2}{h} V \sim V^5$. The picture was confirmed by finding the full crossover from weak to strong backscattering via a Monte Carlo calculation of the linear-response current at nonzero temperature [15] and the Bethe ansatz solution for the full $I_B(V)$ curve [11].

To find the zero-temperature $I_B(V)$ curve numerically, we rewrite this problem as nonresonant tunneling between two semi-infinite *nonchiral* spinless Luttinger liquids. We define $\phi_a(x)$ on the half line $x < 0$ and $\phi_b(x)$ on the half line $x > 0$ as follows: $\phi_a(x < 0) = \phi_1(x) + \phi_1(-x)$ and

$\phi_b(x > 0) = \phi_2(x) + \phi_2(-x)$. The tunneling term in $\mathcal{H}^{1/3}$ then becomes

$$\mathcal{H}_{\text{nonres}} = t[\psi_a^\dagger(0)\psi_b(0) + \text{H.c.}] \tag{1}$$

where $\psi_{a,b}(x) = e^{i\phi_{a,b}(x)/2\sqrt{3}}$ are the two Luttinger quasiparticle creation operators (which are nonlocal combinations of the Laughlin quasiparticles). Since these operators have scaling dimension $1/6$, the Luttinger liquids have $g = 3$ in the conventions of Ref. [16]; a duality transformation maps this to a $g = 1/3$ Luttinger liquid perturbed by a δ-function impurity.

In our DMRG computations, we use a tight-binding model of spinless fermions with nearest-neighbor attractive interactions to describe each Luttinger liquid. This model is equivalent, under a Jordan-Wigner transformation, to an XXZ spin chain. Coupling the two liquids corresponds to including a link between the sites at the two ends, as illustrated in Fig. 1(a). Just as charge tunneling violates charge conservation of the individual edges, coupling the two chains violates the conservation of the individual magnetizations. We match parameters by noting that the scaling dimension of the staggered spin-raising operator in the spin chain $S_x^\dagger(-1)^x$ is equal to the Luttinger parameter g. The ferromagnetic XXZ spin chain anisotropy is then related to g by $J_z/J_\perp = -\cos(\pi/2g)$ [17]. Thus H_{nonres} can be realized with $J_z/J_\perp = -\sqrt{3}/2$.

Tunneling through a point contact for $\nu = 5/2$ also can be realized via coupled XXZ chains, by following the

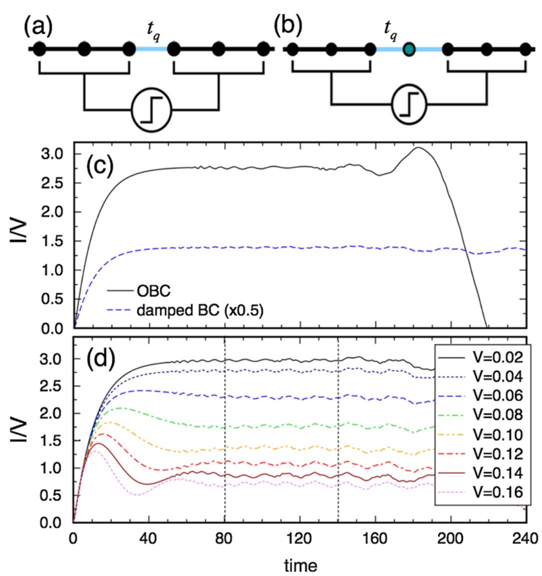

FIG. 1 (color online). (a) Junction model used to study the current through a quantum point contact at $\nu = 1/3$, consisting of a weak link connecting two interacting leads. (b) Quantum dot, or resonant level, system used for $\nu = 5/2$. (c) Current through a $\nu = 1/3$ junction after a step bias is applied. We show results for $L = 120$, $t_q = 0.1$, $V = 0.04$, and different boundary conditions. (d) Time-dependent current through the same system, with damped boundary conditions and different values of the bias V. Time is measured in units of the hopping t.

bosonization procedure of [12]. We assume that the $N = 0$ Landau level (of both spins) is filled and the $N = 1$ Landau level is in the half-filled Moore-Read Pfaffian state. The former are integer quantum Hall edge modes and are the outermost excitations of the system; we ignore them because we focus on tunneling across the interior of a Hall droplet. The gapless chiral theory describing the edge excitations of the Moore-Read state consists of a free boson (the charge sector) and a free Majorana fermion (the neutral sector). We study the interedge backscattering of the basic charge $e/4$ quasiparticle at a point contact at $x = 0$. Charge-$e/2$ quasiparticles can also tunnel, as can neutral quasiparticles, but the latter do not affect the electrical conductivity and the effects of the former are expected to be smaller. The effects of the latter, as well as the extension to the anti-Pfaffian and $(3, 3, 1)$ states, will be discussed elsewhere [18].

To bosonize this model requires a fairly elaborate computation, because the charge-$e/4$ quasiparticle has non-Abelian statistics. When the dust settles, the tunneling Hamiltonian can be written in terms of two bosons ϕ_ρ and ϕ_σ and a Kondo spin $\vec{S}$. The resulting Hamiltonian is [12]

$$\mathcal{H}_{5/2} = \int_0^\infty dx\left(\frac{v_c}{2\pi}(\partial_x\phi_\rho)^2 + \frac{v_n}{2\pi}(\partial_x\phi_\sigma)^2\right) + t(S^+e^{-i\phi\sigma(0)/2} + S^-e^{i\phi\sigma(0)/2})\cos[\phi_\rho(0)/2]. \tag{2}$$

The tunneling amplitude has scaling dimension $[t] = 3/4$. Therefore, in the limit of small current, $I_B \sim t^2V^{-1/2}$ at $T = 0$; this limit occurs for large voltage. In the opposite limit $V \to 0$, interedge tunneling becomes strong, and $I_B \to \frac{1}{2}\frac{e^2}{h}V$. Deviations from total backscattering vary generically as $I_B - \frac{1}{2}\frac{e^2}{h}V \sim V^5$. However, when the tunneling of neutral Majorana fermions is neglected, the system flows to the infrared fixed point along a special direction (even though Majorana fermion tunneling does not directly contribute to charge transport), so that $I_B - \frac{1}{2}\frac{e^2}{h}V \sim V^{15}$ [12]. In this Letter, we will compute $I_B(V)$ for arbitrary V.

The Hamiltonian (2) has the form of resonant tunneling between attractive Luttinger liquids; the reason that the Luttinger liquids are *attractive* ($g > 1$) is that the tunneling operator has scaling dimension $1/4$, which is highly relevant. We can make the relation to resonant tunneling between Luttinger liquids more apparent by rewriting (2) in the form

$$\mathcal{H}_{\text{res}} = \int_0^\infty dx\frac{v}{2\pi}[(\partial_x\phi_a)^2 + (\partial_x\phi_b)^2] + td^\dagger e^{i\phi_a(0)/\sqrt{g}} + td^\dagger e^{i\phi_b(0)/\sqrt{g}} + \text{H.c.}, \tag{3}$$

where $S^- = d$ and $S^+ = d^\dagger$ annihilate/create a particle on the resonant level. The Luttinger coupling is $g = 2$ with $\phi_{a/b} = (1/\sqrt{2})(\phi_\sigma \pm \phi_\rho)$.

As with the earlier case, we utilize two semi-infinite ferromagnetic XXZ chains for the two Luttinger liquids. Here the two chains couple through the resonant level, which in the lattice model corresponds to adding an extra site, as illustrated in Fig. 1(b). Since $g = 2$ here, we have $J_z/J_\perp = -1/\sqrt{2}$.

We apply a potential difference V between the two leads or $\pm V/2$ between lead a/b and the resonant level. This modifies the tunneling term in (3) to $td^\dagger e^{i\phi_a/\sqrt{g}} e^{ieVt/2} + td^\dagger e^{i\phi_b/\sqrt{g}} e^{-ieVt/2} + \text{H.c.}$ However, in the original 5/2 point contact problem, the tunneling term transfers charge $e/4$ between the two edges, so a potential difference V between the edges modifies the tunneling term in (2) to $t(S^+ e^{-i\phi_\sigma/2} + S^- e^{i\phi_\sigma/2})\cos[\phi_\rho(0)/2 + eVt/4]$. Because the current is proportional to the charge squared, the relation between the current in the original 5/2 point contact problem and in the Luttinger liquid resonant tunneling problem is $I_B^{\text{MR}} = (1/2)^2 I_{\text{res}}$. Likewise in the nonresonant case, we have $I_B^{1/3} = (1/3)^2 I_{\text{nonres}}$.

Time-dependent DMRG.—We find the *I*-*V* curves of the nonresonant and resonant tunneling problems using the time-dependent DMRG method. Two fundamental aspects of our calculation make it particularly unsuited to more conventional techniques such as Wilson's numerical renormalization group: (i) The conducting leads are interacting Luttinger liquids, and (ii) we are interested in the nonlinear regime, i.e., large voltage bias. In 1D metallic systems, correlations can drastically affect the density of states and transport properties. In particular, repulsive interactions suppress charge transport, while attraction "heals" the system, enhancing the conductance. The time-dependent DMRG is well suited for our systems, because it allows one to seamlessly incorporate interactions into the leads, and is not restricted to the linear-response regime.

Our technique consists of evaluating the time dependence of the current through the weak link or quantum dot, after a voltage bias is applied [19,20]. In a first step, the ground state is calculated using the conventional DMRG technique. Then the system is quenched: by applying a shift in the chemical potential $\delta\mu_L = V/2$ and $\delta\mu_R = -V/2$ to the left and right leads, respectively. The resulting nonequilibrium system is evolved in time by solving the time-dependent Schrödinger equation. As a response to the quench, a current starts flowing through the system. Typically, the current grows and a transient is observed in the beginning, followed by oscillations that tend to stabilize at a constant value, corresponding to the steady state [21,22]. Since the leads used in the calculation are finite, a reversing of the current is observed after the wave packet reaches the boundaries and is reflected back. This determines a time scale in which we expect the current to stabilize at a plateau value. As we show here, depending on the choice of parameters in the model, it is sometimes necessary to study large systems in order to achieve a steady state. In some cases, when the transient region is large and the system too small, this is hard to attain. In order to improve the behavior of the system, we used long leads and damped boundary conditions, by exponentially decreasing the coefficients in the Hamiltonian toward the end of the chains. As a result, removing a particle from these regions becomes energetically costly, and they effectively behave as reservoirs. As a consequence, the charge becomes trapped and accumulates without getting reflected, leading to longer plateaus [19,23].

Results.—The tunneling problem between $\nu = 1/3$ edges reduces to studying the current through a weak link connecting two interacting spinless leads, while the problem at $\nu = 5/2$ corresponds to a resonant level as seen in Figs. 1(a) and 1(b), respectively. These two models were studied in an early formulation of the time-dependent DMRG method [24]. The smaller the interlead hopping amplitude t_q, the larger the initial transient in the current, making it more difficult to reach a steady state in a finite system. We found it necessary to use long chains, up to 160 sites. In Fig. 1(c), we show results for $\nu = 1/3$, comparing the behavior of the current in systems with different sizes and boundary conditions. The damped boundary conditions, while yielding the same steady current for given values of t_q and bias V, extend the duration of the plateau, allowing one to reach the steady state in smaller systems. In Fig. 1(d), we show results for $t_q = 0.1$ and different values of the bias V, for a system with 120 sites and damped boundary conditions. Typical simulations extend to times of the order of 300 in units of the hopping, using a 3rd order Suzuki-Trotter decomposition of the evolution operator with a time step $\tau = 0.2$ and keeping the truncation error below 10^{-7} [25]. The current is averaged over an interval of time, and the error is calculated following the prescription discussed in Refs. [19,20]. The Suzuki-Trotter error associated to the finite time step was found to be much smaller than the error in the average.

In Fig. 2, we show the I_B-V characteristic curves for the $\nu = 1/3$ case, for different values of t_q. At small biases, the system exhibits a conductance $G_{\text{nonres}} = 3\frac{e^2}{h}$ (i.e., $G_{1/3} = \frac{1}{3}\frac{e^2}{h}$), as expected from the arguments of Ref. [16]. As the bias grows, the system departs from the linear-response regime and crosses over to the scaling behavior associated with quasiparticle tunneling at the ultraviolet fixed point $I_B \propto V^{-1/3}$ [16] (note that we plot I_B/V vs V). As seen in Fig. 2, not only the asymptotic power law but the full crossover follows the behavior predicted by the Bethe ansatz solution [11]. The exact Bethe ansatz expression has a free parameter, corresponding to the tunneling amplitude or t_q, which can be used to fit the numerical results. The agreement is excellent for small t_q, up until the very large biases at which the lattice model no longer accurately represents the quantum Hall edge due to curvature of the dispersion. At large t_q, the system remains near the infrared fixed point up to large biases, and the scaling behavior associated with the ultraviolet fixed point cannot be observed.

For the $\nu = 5/2$ case, we followed the same procedure described above but using the resonant level system shown in Fig. 1(b). The results are depicted in Fig. 3. At small bias, the system exhibits a conductance $G_{\text{res}} = 2e^2/h$

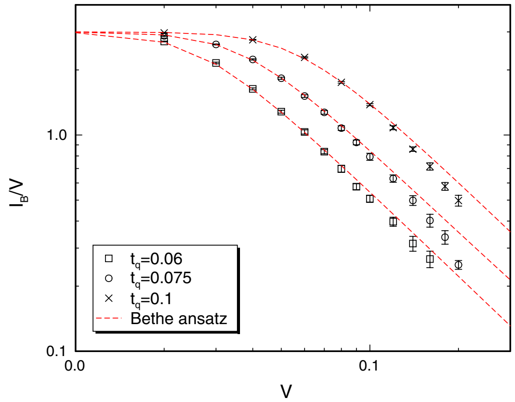

FIG. 2 (color online). I_B-V characteristics of quasiparticle tunneling between $\nu = 1/3$ quantum Hall edges, as modeled by a junction system modeling, for different values of the tunneling amplitude t_q, obtained using the time-dependent DMRG. Error bars correspond to the errors in averaging the current over an interval of time of the order of 40, in units of the hopping, and system sizes up to $L = 160$. Lines are fits to the data using the exact Bethe ansatz solution.

which corresponds to $G_{MR} = \frac{1}{2}e^2/h$, as expected from the arguments of Ref. [12]. As the bias is increased for small t_q, the system crosses over to the asymptotic power law associated with charge-$e/4$ quasiparticle tunneling, $I_B \propto V^{-1/2}$. Again, at large bias, the numerical results depart from the universal regime and exhibit the effects of the lattice. For small V, the deviations from $G_{res} = 2e^2/h$ are too small to be reliably fit to a power law; this may be an indication that they are, indeed, $\sim V^{15}$. In further work [18], we will investigate whether an additional marginal

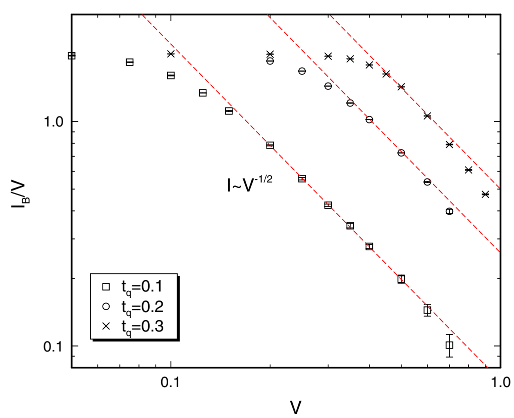

FIG. 3 (color online). I_B-V characteristics of quasiparticle tunneling between $\nu = 5/2$ quantum Hall edges, as modeled by a quantum dot system, for different values of the tunneling amplitude t_q, obtained using the time-dependent DMRG. Error bars correspond to the errors in averaging the current over an interval of time of the order of 40, in units of the hopping, and system sizes up to $L = 140$. Lines are fits to the data using an expression $I_B \sim V^{-1/2}$.

operator will lead to the generic flow into this infrared fixed point with deviations from the asymptotic value $\sim V^5$.

Discussion.—These results clearly demonstrate that weak interedge quasiparticle tunneling causes a Moore-Read quantum Hall droplet to split into two droplets which are coupled through weak electron hopping, as predicted in Ref. [12]. They further enable us to access the crossover regime of intermediate biases where we find quantitative deviations from power-law behavior. Further work will compute [18] the analogous I_B-V curves for the anti-Pfaffian [8,9] and (3, 3, 1) states [10] and compare all three to experimental measurements [5]. Such a comparison could pave the way to correctly identifying the $\nu = 5/2$ quantum Hall state.

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