

Boundary Conformal Field Theory and Tunneling of Edge Quasiparticles in non-Abelian Topological States

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We explain how (perturbed) boundary conformal field theory allows us to understand the tunneling of edge quasiparticles in non-Abelian topological states. The coupling between a bulk non-Abelian quasiparticle and the edge is due to resonant tunneling to a zero mode on the quasiparticle, which causes the zero mode to hybridize with the edge. This can be reformulated as the flow from one conformally-invariant boundary condition to another in an associated critical statistical mechanical model. Tunneling from one edge to another at a point contact can split the system in two, either partially or completely. This can be reformulated in the critical statistical mechanical model as the flow from one type of defect line to another. We illustrate these two phenomena in detail in the context of the $\nu = 5/2$ quantum Hall state and the critical Ising model. We briefly discuss the case of Fibonacci anyons and conclude by explaining the general formulation and its physical interpretation.

I. INTRODUCTION

The remarkable features of non-Abelian topological phases, including their potential use for quantum computation^{1,2,3}, stem from the non-integer number of internal degrees of freedom per quasiparticle. Namely, the number of states in the N -quasiparticle Hilbert spaces for N large grows as d^N , where d is the *quantum dimension*. For instance, the most promising models of the $\nu = 5/2$ quantum Hall state^{4,5,6,7,8} have charge- $e/4$ quasiparticles with $d = \sqrt{2}$ as do flux $hc/2e$ vortices in a $p + ip$ superconductor^{9,10,11,12}. Models supporting universal quantum computation, including one which may be relevant to the $\nu = 12/5$ quantum Hall state^{13,14}, have quasiparticles with $d = (1 + \sqrt{5})/2$, the golden ratio. In chiral topological phases, there are necessarily gapless excitations at the edge of the system. When a bulk quasiparticle is close to the edge, the degeneracy is lifted because of its interactions with these gapless excitations. In this paper, we uncover the dynamics by which the degeneracy of internal degrees of freedom is lifted.

A useful tool in our analysis is to exploit the equivalence of a quantum system in d spatial dimensions and a classical system in $d + 1$ spatial dimensions. The bulk-edge dynamics in a topological state then can be described by the flow between different conformally-invariant boundary conditions in an associated critical 2D statistical mechanical model. For example, the coupling to the edge of a charge- $e/4$ quasiparticle at $\nu = 5/2$ (or of a flux $hc/2e$ vortex in a $p + ip$ superconductor) is equivalent to the imposition of a magnetic field at the boundary of the critical 2D Ising model on the half-plane, causing a flow from free to fixed boundary conditions.

Backscattering between edges of a topological state at a point contact can also be understood as a flow between conformally-invariant boundary conditions. Namely, by “squashing” the edge of the system onto a line segment, which is then folded about the point contact (see Fig. 3 below), inter-edge backscattering can be understood in terms of two copies of the associated critical 2D statistical mechanical model coupled only at their boundary. Such a rephrasing allows us

to place our earlier work^{15,16,17} on charge- $e/4$ quasiparticle backscattering at $\nu = 5/2$ in a wider context. The squashing procedure also allows us to study more complicated topologies such as the annulus and situations in which there are multiple bulk quasiparticles.

In this paper, we focus mainly on the case of a chiral Majorana fermion edge mode, which is the edge theory of a $p + ip$ superconductor and is the neutral sector of the edge theory of the $\nu = 5/2$ quantum Hall state¹⁸. (Some of our results do not apply to a $p + ip$ superconductor because its vortices are essentially classical as far as their motion is concerned, but they do apply to a topological state which may be viewed as a quantum-disordered $p + ip$ superconductor resulting from the condensation of hc/e vortices. We will simply use Majorana fermion edge mode to refer to edge of this state and the neutral sector of the edge theory of the proposed non-Abelian $\nu = 5/2$ quantum Hall states^{4,7,8}.) The classical analog is the critical Ising model, whose boundary conditions¹⁹ and defect lines²⁰ have been analyzed in depth. We make a connection with these results, leading to a simple interpretation for a critical line of defect boundary conditions (‘continuous Neumann’) which has no simple interpretation in Ising language. We will discuss briefly the added complications arising from the presence of a charged mode in the $\nu = 5/2$ state. We will also mention how our results can be generalized to other conformal field theories, and will briefly discuss the case of the $k = 3$ Read-Rezayi state and \mathbb{Z}_3 parafermions.

In section II, we discuss the mapping between the edge theory of (the neutral sector of) the $5/2$ quantum Hall state and the critical 2D Ising field theory. In section III, we analyze the effect of Majorana fermion tunneling between a bulk vortex and the edge, showing that this is the same problem as a magnetic field applied to the boundary of a critical Ising model¹⁹. In section IV, we show how the effects of a point contact can be expressed in terms of a boundary problem by folding the system. The folding allows us to bosonize, and to utilize the results of Oshikawa and Affleck for the Ising model with a defect line²⁰. In section V, we analyze the effect of interedge backscattering at a point contact in a Majorana fermion edge

mode, and discuss in depth two critical lines of boundary fixed points. We also discuss the closely related case of the Moore-Read Pfaffian state. In section VI we analyze flows between these fixed points, and study the entropy drops. We extend our analysis to allow for an arbitrary number of quasiparticles in section VII. In section VIII, we generalize our results to a different topological state, supporting Fibonacci anyons. Finally, in section IX, we discuss our results in the larger context of topological phases and the transitions between them.

II. MAPPING A MAJORANA FERMION EDGE MODE TO THE CRITICAL 2D ISING MODEL ON A STRIP

Consider a very large quantum Hall droplet at filling $\nu = 5/2$ which we assume initially is in a Moore-Read Pfaffian state⁴. Circumnavigating the droplet are gapless chiral edge modes²¹: a bosonic charge mode, ϕ_c , and a neutral Majorana fermion, ψ .¹⁸ Initially we focus our attention exclusively on the neutral sector – a chiral Majorana fermion – which is formally equivalent to the edge of a $p + ip$ superconductor^{9,10}.

Taking the circumference of the droplet to be $2L$ it is convenient to “squash” this chiral system into an effectively one-dimensional model with both left and right movers. Reformulating the problem on a strip allows us to treat tunneling as a problem in 1+1-dimensional boundary field theory, or equivalently, a quantum impurity problem. There is an enormous literature on such problems, much of it following the seminal paper²². For the case of a single Majorana fermion, many results which are useful for us have already been obtained in this context, in particular Refs. 19,20. Thus, we introduce right- and left- moving fields, $\psi_R(x)$ and $\psi_L(x)$, which are functions of an x -coordinate lying in the interval $[0, L]$. The action describing the edge dynamics is

$$S_0 = \int dt \int_0^L dx \mathcal{L}_0, \quad (1)$$

with Lagrangian density,

$$\mathcal{L}_0 = i\psi_R(\partial_t + v_n\partial_x)\psi_R + i\psi_L(\partial_t - v_n\partial_x)\psi_L. \quad (2)$$

The edge modes have dispersion $\epsilon(k) = v_n k$ with the momenta $k \geq 0$ chosen to satisfy the appropriate boundary conditions which we shall discuss momentarily. Thus, the neutral sector of the edge of the Moore-Read Pfaffian state or, equivalently, the edge of a $p + ip$ superconductor is simply given by a non-chiral gapless Majorana fermion on the strip $x \in [0, L]$, $\tau \in [-\infty, \infty]$. (At non-zero temperature, the length in the Euclidean time direction τ is also finite.)

In order to complete the mapping to the strip, we must specify the boundary conditions at the two ends of the strip, $x = 0, L$. These are independent of the exact shape of the droplet, since the edge theory is conformally-invariant. They are, instead, determined by how the fermionic field behaves as one makes a circuit of the droplet. For the $p + ip$ superconductor (or Moore-Read Pfaffian state), this depends on the number of $hc/2e$ vortices (or charge- $e/4$ quasiparticles) in the bulk. When there are no vortices (or an even number),

the edge fermion behaves as fermions typically do under rotations of 2π : the fermionic field picks up a minus sign, so that it is antiperiodic. (This may be seen explicitly from the Bogoliubov-de Gennes equations in the $p + ip$ superconducting case or from the lowest Landau level wavefunctions in the Moore-Read Pfaffian case.) The presence of a single vortex (or an odd number) flips this to periodic; the vortex can be viewed as introducing a branch cut. An even number of vortices therefore leaves the fermionic field antiperiodic, while an odd number makes it periodic.

Once we have squashed to the strip, we need boundary conditions at the ends of the line segment which reflect left movers into right movers at $x = 0$ and right movers back into left movers at $x = L$. Instead of imposing these boundary conditions by hand, it is much more convenient instead to add boundary terms L_b to the action so that the boundary conditions are consequences of the equations of motion. This means that the original boundary conditions are treated on an equal footing as those induced by tunneling, making it much simpler to understand the flows between different boundary conditions. With this idea in mind, we modify the action to

$$S = S_0 + \int dt L_b \quad (3)$$

where the boundary terms in the absence of tunneling are

$$L_b = -iav_n\psi_L(0)\psi_R(0) + ibv_n\psi_L(L)\psi_R(L). \quad (4)$$

For Grassman variables α, β , we adopt the complex conjugation convention $(\alpha\beta)^* = \beta^*\alpha^*$, and for Majorana fermions we have $\psi_L^* = \psi_L$ and $\psi_R^* = \psi_R$. The term L_b then indeed obeys $(L_b)^* = L_b$ if a and b are real. Once L_b is included, the equations of motion for ψ_L and ψ_R are found by varying them independently. Varying ψ_R in (3) yields

$$\psi_R(0) = a\psi_L(0), \quad \psi_R(L) = b\psi_L(L) \quad (5)$$

for the equations of motion at the boundaries; the terms on the left-hand sides result from a surface contribution from S_0 . Varying ψ_L gives the same equations with L and R reversed, so consistency demands that $a^2 = 1$ and $b^2 = 1$.

The values of a and b in the boundary conditions depend on the number of vortices in the bulk. In the unsquashed geometry, the boundary conditions are antiperiodic when there is an even number N_v of vortices in the bulk. This means that we must have $ab = -1$ to reproduce this in the squashed geometry. It turns out to be more convenient, and to agree with the natural choice in conformal field theory¹⁹, to define ψ_L and ψ_R so that $a = -1$ and $b = 1$ when $N_v = 0$. We discuss the reasons for this below.

The introduction of bulk quasiparticles seems to complicate matters considerably, since the boundary conditions for the edge fermion depend on whether N_v is even or odd. However, a main point of one of our earlier papers is that key topological properties of the bulk quasiparticles can be taken into account by understanding edge properties¹⁷. In this paper we show that, equivalently, such effects can be incorporated via the boundary conditions.

We discuss this in depth below, but let us begin here with the simplest situation $e/4$ -quasiparticle pinned in the bulk of the sample. The boundary conditions on the chiral fermion in the original geometry are now periodic, because the vortex introduces a branch cut. This branch cut is very important: with it, the fermion has a zero mode on the edge, i.e. a solution to the equations of motion having zero energy²⁴. In the squashed picture, it is convenient to make the branch cut go through one of the points which becomes a boundary after squashing, so including the cut amounts to modifying the boundary conditions to $a = b$. The choice of whether $a = b = 1$ or $a = b = -1$ is equivalent to having the branch cut go through the left or right.

It is very useful to rephrase the preceding in the language of the Ising model. The Ising model can be described by a single non-chiral Majorana fermion: the post-squashing Lagrangian density (2) is precisely that of the transverse field Ising model at its quantum critical point on a line segment. If we continue to Euclidean time, this corresponds to classical Ising model at its (bulk) critical point on a strip. Equivalently, it corresponds to the $1D$ quantum transverse field Ising model on a finite chain at its critical point. There are two possible boundary conditions which preserve scale invariance, known as “free” and “fixed” in terms of Ising spins. The free boundary condition corresponds to having the end spin in the quantum transverse field Ising chain unconstrained, while the fixed boundary condition corresponds to fixing that spin to be a particular value for all time (or, in the classical $2D$ Ising model, to fixing all of the spins along the boundary).

Relating the fermionic boundary conditions (5) to Ising ones precisely is a little subtle. Having no bulk vortices ($-a = b = 1$) corresponds to having fixed boundary conditions at both $x = 0$ and $x = L$, even though $a = -b$ naively seems to imply that the boundary conditions are different at the two ends. This can be seen indirectly by comparing the partition functions for the boundary system^{22,23} with those for the topological state¹⁷. A direct proof¹⁹ follows by first considering Euclidean spacetime to be the upper-half-plane. Then one has $\psi_L = -\psi_R$ along the x -axis for fixed boundary conditions, and $\psi_L = \psi_R$ for free boundary conditions. To find the corresponding boundary conditions on the strip $x \in [0, L]$, we need to conformally transform it to the upper-half-plane. We map $z = e^{\tau+i\sigma}$, $\bar{z} = e^{\tau-i\sigma}$, which takes the strip $\sigma = \pi x/L \in [0, \pi]$, $\tau \in [-\infty, \infty]$ to the upper half-plane $\text{Im } z \geq 0$. Right-moving fields are then functions of z , while left movers depend on \bar{z} . Under this conformal transformation, a dimension D function of z transforms along the boundary $\text{Im } z = 0$ as $f(z) \rightarrow \left(\frac{\partial z}{\partial \tau}\right)^D f(\tau + i\sigma)$, and likewise $\bar{f}(\bar{z}) \rightarrow \left(\frac{\partial \bar{z}}{\partial \tau}\right)^D \bar{f}(\tau - i\sigma)$. This means that for $\sigma = 0$, this conformal transformation leaves the boundary condition unchanged, i.e. $a = -1$ for fixed and $a = 1$ for free. However, for $x = L$ (i.e. $\sigma = \pi$), the conformal transformation yields a factor $e^{i\pi D}$ for the right movers and $e^{-i\pi D}$ for the left movers. Since the fermions have dimension $1/2$, the fixed boundary condition here is modified to $\psi_L(L) = \psi_R(L)$ while free is now $\psi_L(L) = -\psi_R(L)$, so that $b = 1$ for fixed and $b = -1$ for free.

We have thus shown that when the boundary conditions at the end of the strip are both fixed, this corresponds in fermion language to having no vortices in the bulk. When one boundary condition is free and the other fixed, this corresponds to having a single vortex. These situations are illustrated schematically in figure 1.

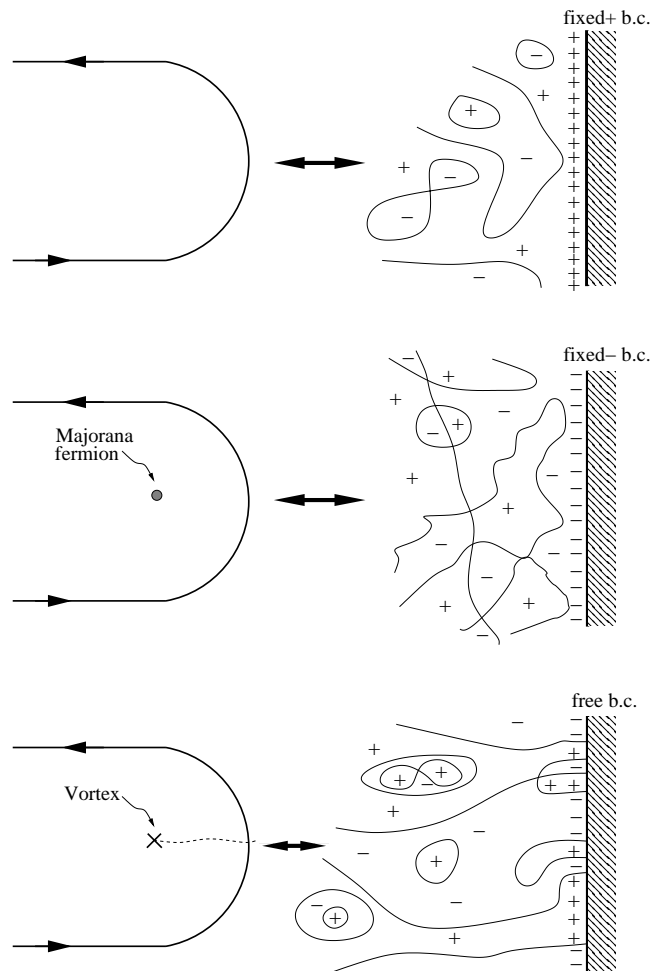


FIG. 1: The presence of different bulk excitations in a $5/2$ quantum Hall droplet is equivalent to different conformal boundary conditions in the critical Ising model. In all three cases, the boundary conditions on the other side of the strip are taken to be fixed up.

Note that there is some ambiguity in the translation from the fermionic edge theory to the Ising model. In the latter case, it is clear which end is fixed and which one is free while, in the former, a Z_2 gauge transformation can exchange the free and fixed ends (or even put branch cuts in the middle of the strip). The reason for this is the following. The gauge transformation which exchanges the free and fixed ends of the strip is $\psi_R \rightarrow -\psi_R$, $\psi_L \rightarrow \psi_L$. In Ising language, however, this is simply Kramers-Wannier duality since it flips the sign of the energy operator $\epsilon = \psi_R \psi_L$ and, under duality, free and fixed boundary conditions are switched. A free boundary condition for the order field is a fixed boundary condition for the disorder field and vice versa.

Another subtlety is that in the Ising model, there are ac-

tually two types of fixed boundary conditions: all boundary spins up, and all boundary spins down. These can be understood in the fermion problem by recalling that the fermion operator creates a cut in the spin field σ (the operator product $\psi(z)\sigma(0) \sim z^{-1/2}\sigma(0)$). When boundary conditions are fixed-up on one side of the strip and fixed down on the other, there must be an odd number of domain walls stretching throughout the system. When they are fixed the same on both sides, there are an even number of domain walls. Thus these two possibilities correspond respectively to odd and even fermion numbers at the edge. If the total number of electrons is fixed to be even, then the fermion number at the edge can only be changed by breaking a pair and putting one fermion at the edge and the other in the bulk, so even/odd fermion numbers in the bulk correspond to even/odd fermion numbers at the edge.

Acting with the vortex creation operator in the bulk changes either of the two fixed boundary conditions into free. Since a vortex is a non-Abelian quasiparticle, with quantum dimension $d = \sqrt{2}$, the free boundary condition has a higher entropy than fixed by $\ln \sqrt{2}$. We show in the next section III that in the case of a vortex in the bulk, bulk-edge coupling can cause the system to flow from free boundary condition back to fixed. This lifts the $\sqrt{2}$ -fold degeneracy of the vortex, leading to an entropy drop $\ln \sqrt{2}$. On the other hand, acting with a fermion creation operator leaves free boundary conditions invariant, and so does not change the entropy. Moreover, the fixed boundary condition is stable to bulk-edge coupling. This is what we expect since a Majorana fermion is an Abelian quasiparticle, i.e. it has quantum dimension $d = 1$; therefore fixed up and fixed down boundary conditions have the same entropy and are stable to perturbations.

Turning to the case of two vortices in the bulk, the fermions should again have anti-periodic boundary conditions around the unsquashed droplet. It would thus seem that we could either take both ends fixed, or both ends free, since these are related, in fermionic language, by a gauge transformation. However, they are not quite physically equivalent. If both ends are taken fixed, then the fermion number parity on the edge is also fixed – either to 0 or 1, depending on whether the spins on the two ends of the strip are fixed to the same or different values, respectively. If both ends are taken to be free, then the fermion number parity is not taken to be fixed and both 0 and 1 are allowed. Since the fermion number parity at the edge is equal to the value of the qubit formed by the two vortices, we conclude that both ends fixed is appropriate to the situation in which the qubit has a fixed value (in this basis) while both ends free is correct when the qubit does not have a fixed value and is in an entropy $\ln 2$ mixed state. We will discuss this further in section VII and generalize these results to an arbitrary number of vortices.

III. COUPLING OF THE EDGE TO A BULK VORTEX AND THE FLOW FROM FREE TO FIXED ISING BOUNDARY CONDITIONS

In this section and the following ones, we shall focus on various “tunneling” perturbations which act on the Majorana fields at $x = 0$. The first case we study is the effect of bringing a bulk vortex close to an edge. For simplicity, let us suppose that there is only a single vortex in the bulk. A vortex has a single Majorana zero mode localized at its core, which we denote as ψ_0 . Since the edge has a zero mode as well (recall that when there is a single vortex in the bulk, the boundary conditions at $x = 0$ and $x = L$ are the same), Majorana fermions can tunnel from the vortex to the edge. To study this within our boundary approach, we choose the point along the edge at which the tunneling occurs (i.e. the closest point to the vortex) to be one of the boundaries of the squashed system, say that at $x = 0$. The effect of the vortex on the edge then occurs entirely at $x = 0$, so it is convenient to place the branch cut associated with the vortex there as well. Thus in the absence of tunneling, we have $a = b = 1$ in (5); in Ising language we have a free boundary condition at $x = 0$ and a fixed boundary condition at $x = L$.

The zero-mode tunneling term in the Lagrangian resulting from bulk-edge coupling is therefore^{25,26}

$$L_h = i\psi_0\partial_t\psi_0 + ih\psi_0[\psi_R(0) + \psi_L(0)], \quad (6)$$

where h is the amplitude for tunneling between the edge and the zero mode associated with the vortex. Note that the relative sign between the two terms in the square brackets is consistent with the boundary condition $a = 1$, i.e. $\psi_R(0) = \psi_L(0)$, when $h = 0$. The i in front of the coupling to the vortex is necessary in order for the Hamiltonian to be hermitian, with h real. The magnitude of h is determined by the distance between the vortex and the edge; at large distance r from the edge, it should be $\sim e^{-\Delta r/v}$ where Δ is the bulk energy gap for Majorana fermions (which might be smaller than the charge gap in the $5/2$ quantum Hall state) and v is their velocity. We will comment below on the sign of h .

Since even with the perturbation (6) the action remains quadratic in the fermions, $\psi(x)$ and ψ_0 , it can easily be solved exactly. The equations of motion for ψ_0 , ψ_R and ψ_L at $x = 0$ become

$$\begin{aligned} 2\partial_t\psi_0 &= h[\psi_R(0) + \psi_L(0)], \\ v_n\psi_R(0) &= v_n\psi_L(0) + h\psi_0, \\ v_n\psi_L(0) &= v_n\psi_R(0) - h\psi_0. \end{aligned}$$

Going to frequency space gives then

$$\psi_R(x=0, \omega) = \frac{\omega + i\omega_0}{\omega - i\omega_0} \cdot \psi_L(x=0, \omega), \quad (7)$$

where the scale $\omega_0 \equiv h^2/2v_n$ grows with the bulk-edge coupling. Intuitively, it is helpful to view this as a scattering problem. Because of the quadratic Hamiltonian, each incident Majorana fermion is reflected one-by-one from the boundary, with an energy-dependent scattering amplitude given by the phase in (7).

The boundary condition (7) smoothly interpolates between the two boundary conditions we discussed previously. When there is no edge-vortex coupling ($h = \omega_0 = 0$), we recover the $a = 1$ boundary condition arising from a single bulk vortex. In the strong-coupling $\omega_0 \rightarrow \infty$ (or, equivalently, DC) limit, we obtain the boundary condition $a = -1$. This is the boundary condition in the absence of the vortex. Thus the presence of the relevant coupling between vortex and edge causes the cut to “heal: the zero mode at the vortex core is effectively absorbed into edge, and annihilates the edge zero mode. This situation is sketched schematically in Figure 2. In the language of boundary field theory, the relevant coupling causes a flow of boundary conditions from $a = 1$ to $a = -1$.

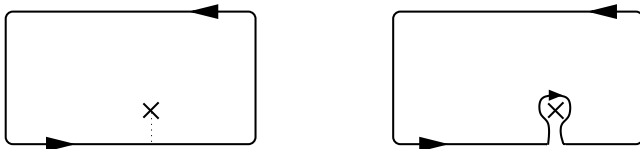


FIG. 2: Majorana fermion tunneling between the gapless chiral edge mode and a bulk quasiparticle/vortex causes the bulk zero mode to be absorbed by the edge.

In dynamical terms, when a single bulk vortex is brought close to the droplet edge at time $t = 0$, it is scattered strongly by the edge modes. At short times $t \ll \omega_0^{-1}$, it has little effect on the edge modes, but for $t \gg \omega_0^{-1}$ a π phase shift in the Majorana field is induced at the boundary, changing $a = 1$ to $a = -1$. Another important time scale is set by the system size, $t_L \equiv L/v_n$, i.e. the time it takes for an edge disturbance to circumnavigate the droplet. For $t \ll t_L$ the induced phase shift can be viewed as a “local” change and can have no influence on the other boundary at $x = L$. However, in the d.c. limit, $t \rightarrow \infty$ (taken before the thermodynamic limit, $t_L \rightarrow \infty$) the net effect of the crossover induced by coupling the vortex to the edge is a global change in boundary conditions on circumnavigating the droplet from periodic to anti-periodic. This change in boundary conditions disallows the zero mode: the vortex is no longer there as far as the edge is concerned. Of course, in the d.c. limit the point at which the fermion changes sign (in order to satisfy the anti-periodic boundary condition) can be placed wherever we like by a (static) gauge transformation; $x = 0$ is the most convenient choice in the squashed geometry.

A notable property of this crossover is that the phase shift in the DC limit is independent of the impurity strength h . This should be contrasted with the analogous problem of a chiral Dirac fermion, $\chi(x)$, scattering from an impurity level, with Lagrangian

$$\mathcal{L}_{imp}^D = id^\dagger \partial_t d + \lambda(\chi^\dagger(0)d + h.c.) + \epsilon_0 d^\dagger d, \quad (8)$$

where d and d^\dagger annihilate and create a particle on the resonant level. Here in the DC limit after squashing, $\chi_L(0) = e^{i\theta} \chi_R(0)$ with $\theta = 2 \tan^{-1}(\omega_0/\epsilon_0)$ and $\omega_0 = \lambda^2/2v_n$. It is only when the impurity energy is fine-tuned to zero, i.e. on resonance, that the phase shift for complex fermions becomes independent of the scattering strength, $\theta(\epsilon_0 = 0) = \pi$, as in

the Majorana case. Since $\psi_0^2 = 1$, a chiral Majorana fermion is *always* on-resonance, and no fine-tuning is required. A single Majorana fermion is not a physically meaningful object - they can only exist in pairs.

The absorption of the Majorana zero mode by the edge leads to a loss of entropy. For the case of Dirac fermions tuned to resonance, $\epsilon_0 = 0$, the entropy loss is simply $\ln 2$ since the d -level constitutes a two-level system ($d^\dagger d = 0, 1$) which is screened by the edge fermions. The entropy loss for the Majorana fermion case can be inferred by introducing a second identical copy of the Majorana edge plus impurity system. It is convenient to introduce a complex Dirac fermion for both the edge and impurity as $\chi = (\psi^\alpha + i\psi_\beta)/\sqrt{2}$ and $d = (\psi_0^\alpha + i\psi_0^\beta)/\sqrt{2}$, where α, β refer to the two copies. When re-expressed in terms of these complex fermions, the total Lagrangian $\mathcal{L}_\alpha + \mathcal{L}_\beta$ becomes identical to the Dirac case with ϵ_0 . Since the two Majorana copies are decoupled, the entropy drop is additive. Thus, the entropy change during the crossover induced by coupling the single Majorana zero mode to the edge is $\Delta S = -\frac{1}{2} \ln 2 = -\ln \sqrt{2}$.

Since the model is quadratic in fermions, much more than the entropy can be computed. In fact, the full partition function and explicit correlation functions have been computed along this whole flow by using the “boundary state”¹⁹.

The vortex-edge coupling h corresponds to a boundary magnetic field in the Ising language. This initially may seem strange, since a bulk magnetic field couples to the spin field σ , not the fermion ψ . However, the spin field in the Ising model is a product of left and right components of σ . At a boundary, one must identify these components of σ just like we identified ψ_L and ψ_R in (5). Thus at the boundary, the product $\sigma_L \sigma_R$ becomes the fusion $\sigma(0) \times \sigma(0) = I + \psi(0)$, so that a boundary magnetic field indeed couples to ψ as in (6)²². This picture in terms of the Ising model gives useful intuition. When the boundary magnetic field gets large ($|h| \rightarrow \infty$), the boundary spin is fixed to $+1$ or -1 , depending on the sign of h . Thus in Ising language, the coupling causes a flow from free to fixed boundary conditions; it has long been known that this causes a change $\Delta S = -\ln \sqrt{2}$ in entropy^{22,27}. It is also clear what sign h should have. If the total fermion number is even, then when the boundary at $x = 0$ flows to fixed boundary condition, the Ising spin must be fixed to the same value as at $x = L$; if the fermion number is odd, it must be fixed to the opposite value. Since the sign of h determines the sign of the Ising spin at $x = 0$, its sign is determined by the fermion number parity and the boundary condition at $x = L$.

For a boundary magnetic field to have any effect, we must start, at $h = 0$, with free boundary conditions, i.e. $a = 1$. If the boundary condition is fixed, then coupling to a boundary magnetic field can have no effect: there is nothing to couple to, since the boundary spin is not allowed to flip. This can also be seen from (6): if $\psi_R(0) = -\psi_L(0)$, then this term vanishes. In order to have a free boundary condition at $x = 0$, there must be present the Ising analog of a vortex, which in this context is called the “twist” operator²⁴. (The twist operator turns out, not surprisingly, to be the spin field.) The resulting zero mode is ψ_0 . ψ_0 can also be viewed from a formal perspective as the Klein field necessary to make the second term in the

Lagrangian (6) bosonic. Finally, it is also worth mentioning that the Majorana crossover is formally identical to that of an anisotropic 2-channel Kondo problem at its Toulouse point; the Majorana zero mode operator ψ_0 is mapped to the σ^x operator for the Kondo spin²⁸.

IV. POINT CONTACTS AND BOUNDARY CONFORMAL FIELD THEORY

In this section we explain how to understand the possible critical behavior of topological states in the presence of a point contact. The point contact allows backscattering of right movers on the top to left movers on the bottom. We use ‘backscattering’ to denote tunneling from one edge to the other across the point contact; we will use ‘tunneling’ generically to describe tunneling from the edge to a bulk quasiparticle or to another edge and, especially, from one droplet to another. Generically, backscattering destroys criticality, although in the next section V we will discuss a special case where criticality survives even in the presence of backscattering. Before studying backscattering, however, it is useful to understand how to deal with a point contact using boundary conformal field theory.

To turn a point contact into a boundary problem, we must squash and then “fold” the system around the point contact, so that effectively we have two copies of the system coupled at one of their boundaries²⁹. Namely, we place the point contact somewhere in the middle of the sample, far from the ends. Once we have squashed, the point contact is effectively an impurity in this non-chiral system, or a defect line in the equivalent two-dimensional classical system. By a conformal transformation, we can put the point contact/defect line at $x = 0$, and take $x \in [-L, L]$. This impurity/defect problem can in turn be turned into a boundary problem by folding the system at the impurity. What folding means is that the droplet has now been deformed into a horseshoe shape, with the point contact at the bend in the horseshoe, as depicted in Fig. 3. Since edge interactions are local, the top and bottom parts of the horseshoe are two copies of the system of length L , coupled only via the point contact.

For the Majorana fermion edge mode, folding turns out to simplify the problem considerably. The reason is that even though a single Majorana fermion cannot be bosonized, a pair can. In fact, this is the easiest way to compute explicit correlators in the critical Ising field theory: square the correlator, bosonize, compute the correlator, and then take the square root³⁰. This procedure is even more natural in our context, because the folding automatically doubles the degrees of freedom. Bosonizing the neutral sector of a $\nu = 5/2$ point contact allows to make contact with the detailed results of Ref. 20.

We have seen in section II that for the Majorana fermion edge mode with no vortices in the bulk, the boundary conditions in the squashed system are fixed at both ends (in Ising language), so that $\psi_L(-L) = -\psi_R(-L)$ and $\psi_L(L) = \psi_R(L)$. Labeling the two halves of the droplet by 1 and 2, folding results in two right-moving modes, $\psi_{1R}(x) = \psi_R(x)$, $\psi_{2R}(x) = \psi_L(-x)$ and two left-moving modes, $\psi_{1L}(x) =$

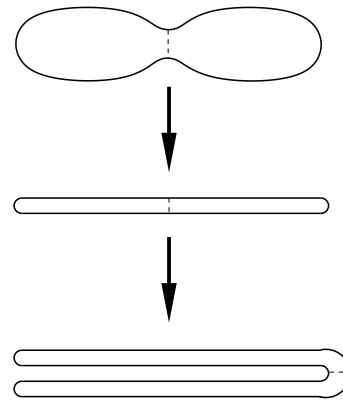


FIG. 3: Deforming a topological state with a point contact into a horseshoe shape so that it can be mapped onto a boundary problem.

$\psi_L(x)$, $\psi_{2L}(x) = -\psi_R(-x)$ with $x \in [0, L]$. The reason for the extra minus sign in the last of these is that, as explained in section II, a given boundary condition (free or fixed) at the right end of the line segment has the opposite sign from the same boundary condition at the left end (i.e. $a = -b$ in eqn. 5). Folding exchanges left and right ends, so the extra sign interchanges boundary conditions appropriately. Equivalently, we are choosing a gauge in which there is no branch cut at either end of the strip (which, after folding, translates to fixed Ising boundary conditions at both ends). In the gauge which we have chosen, the branch cut which is necessary in the absence of bulk vortices is at the point contact.

A. Conformal boundary conditions

A key component of our analysis is to understand the boundary fixed points, or in more formal language, the conformal boundary conditions. (We use the two descriptions interchangeably.) A fixed point of the renormalization group is scale invariant, and typically in two spacetime dimensions, conformally invariant as well. In a topological state, the edge is scale and conformally invariant in the absence of any tunneling. Once we allow tunneling at a point, scale invariance is typically broken at that point, but of course still remains valid in the rest of the edge theory. Thus the situation in the presence of tunneling is generally a conformal field theory with boundary conditions breaking the scale and conformal invariance. However, as we already saw in our analysis of vortex/edge tunneling in section III, at low temperatures and frequencies, the boundary conditions effectively flow to a new boundary fixed point. Thus before exploring which types of tunneling cause which flows, it is very useful to first understand the different possible boundary conditions which preserve scale and conformal invariance of the full system, including the boundary.

Some of the conformal boundary conditions for a point contact in a Majorana fermion edge mode are fairly obvious from both the Ising and fermionic points of view: these are “product” boundary conditions, in which the two copies decouple.

Physically, this corresponds to the point contact effectively splitting the system in two. We have already shown that the boundary condition (5) has $a = -1$ if there are no vortices in the bulk, and $a = 1$ if there is a single vortex. Thus, in the two copies, there are four possibilities: $(a_1, a_2) = (1, 1), (-1, 1), (1, -1),$ and $(-1, -1)$. In the Ising language, $a = 1$ and $a = -1$ correspond to “free” and “fixed” respectively. Thus if the point contact splits the system in two, the boundary conditions at $x = 0$ in the folded system are (fixed, fixed), (free, fixed), (fixed, free), or (free, free), depending on if and where vortices are located. (As discussed in section II, there are two possibilities for each fixed boundary condition.)

Obviously, product boundary conditions are not the end of the story. For example, one boundary condition corresponds to no defect at all, i.e. no backscattering occurring at the point contact. In this case, the point contact is effectively not there at all. Of course, we are still free to fold the system at this point, and treat the model as a boundary problem. We refer to the resulting boundary condition as “transmitting”. Before folding, this boundary condition simply ties the left-moving fields in the two copies together at $x = 0$. After folding, left movers in copy 2 become right movers, so

$$\text{transmitting: } \psi_{1L}(0) = \psi_{2R}(0), \quad \psi_{1R}(0) = -\psi_{2L}(0). \quad (9)$$

The minus sign in the latter term arises from the extra minus sign in the folding discussed above. This boundary condition (9) results from the equations of motion of the action (1) plus the boundary term

$$L_{\text{transmit}} = iv_n (\psi_{2L}(0)\psi_{1R}(0) - \psi_{1L}(0)\psi_{2R}(0)). \quad (10)$$

This term (10) is not the only quadratic boundary term one can add. In section V, we discuss how allowing fermion tunneling across the point contact results in a line of boundary conditions.

Understanding transmitting boundary conditions allows us to apply all our results to a topological state on an annulus as well as a disc. We have seen that after squashing and folding, the disc becomes a doubled system with the two copies coupled at $x = 0$, the location of the point contact. The boundary conditions at the far end $x = L$ turn into (fixed, fixed) boundary conditions, leaving the two copies decoupled there. For an annulus, we can still keep the point contact at $x = 0$, but now put transmitting boundary conditions at $x = L$. This sews the top edges of the two copies together, and the bottom edges together, but does not sew top to bottom. In the original unfolded case, this indeed corresponds to an annular geometry. Obvious physical arguments indicate that this change of boundary conditions at $x = L$ should have no effect on the behavior of the point contact, and indeed detailed computations confirm this. However, this change can and does effect global properties like the total entropy, and we will return to the annulus at the end of section VI.

B. Bosonization

To understand the conformal boundary conditions for the point contact in the Majorana fermion problem in more depth,

it is useful to bosonize the fields. In fact, using bosonization, all the boundary fixed points for two Ising conformal field theories coupled at the boundary have already been found²⁰. This will enable us to show that all these conformal boundary conditions (and then some) can be obtained in our problem, once we allow vortices to be present. We thus conclude this section by outlining some results of Ref. 20.

After folding, we have two Majorana fermions ψ_1 and ψ_2 , coupled only at the boundary $x = 0$. A single Dirac fermion χ can be formed out of these two Majorana fermions in the same way a complex number is formed out of two reals. A free Dirac fermion χ can be bosonized according to $\chi_R = e^{i\varphi_R}$ and $\chi_L = e^{i\varphi_L}$, where we normalize the bosonic fields so that the scaling dimension of $e^{i\alpha\varphi_L}$ or $e^{i\alpha\varphi_R}$ is $\alpha^2/2$. It is often useful to combine the chiral bosons into a single boson $\varphi = \frac{1}{2}(\varphi_L + \varphi_R)$. In this Ising model, φ has radius 1, meaning that we identify $\varphi \sim \varphi + 2\pi$. This is tantamount to saying that restricting α to be an integer results in only fermionic or bosonic operators. The Lagrangian for the folded system in the bosonic picture is then

$$L_0 = \frac{1}{2\pi} \int_0^L dx \left[\left(\frac{\partial\varphi}{\partial t} \right)^2 - v_n^2 \left(\frac{\partial\varphi}{\partial x} \right)^2 \right]. \quad (11)$$

The Lagrangian can equivalently be written in terms of a dual boson $\tilde{\varphi} = \frac{1}{2}(\varphi_R - \varphi_L)$. In the Ising model, $e^{i\varphi/2}$ is the product of the two Ising spin fields, while $e^{i\tilde{\varphi}/2}$ is the product of the two disorder fields.

Classifying the various possible conformally-invariant boundary conditions of the bosonic model (11), which include the different product boundary conditions as well as (16), is equivalent to classifying the different possible fixed points of the critical Ising model with a defect line. Indeed, Oshikawa and Affleck²⁰ analyzed this model by folding the Ising model about the defect, precisely as we folded our droplet above, and then bosonizing the two resulting copies of the Ising model. All of the conformal boundary conditions were found. Moreover, the boundary states were constructed explicitly, which allows correlators to be computed exactly for any conformal boundary condition.

The possible conformal boundary conditions are summarized in the table I. There are two different lines of boundary fixed points, dubbed ‘continuous Dirichlet’ and ‘continuous Neumann’ in Ref. 20. The former corresponds to setting the field $\varphi(0) = \varphi_0$ at the boundary, while the latter corresponds to setting the dual field $\tilde{\varphi}(0) = \tilde{\varphi}_0$. The remaining boundary conditions are of product type: either (fixed, fixed), (free, fixed), or (fixed, free). Table I also lists the contribution of each type of conformal boundary condition to the entropy; we discuss these values in depth in the section VI.

The continuous Dirichlet (CD) line of fixed points is easy to understand in the language of the classical Ising model with a defect. To move the model off of criticality, one varies the coupling between adjacent spins. In field theory, this corresponds to adding the energy operator $\epsilon(x, t)$ to the Lagrangian density. This operator turns out to have dimension one, so it is indeed a relevant perturbation in the bulk. However, if we add $\epsilon(x = 0, t)$ to the original (unfolded) quantum Ising chain at

Boundary condition	parameter	entropy
continuous Neumann	$2\varphi_0 = \delta + \frac{\pi}{2}$	$\ln \sqrt{2}$
continuous Dirichlet (free, fixed)	$2\varphi_0 = \delta + \frac{\pi}{2}$	0
(fixed, fixed)		$-\ln \sqrt{2}$
		$-\ln 2$

TABLE I: Summary of boundary conditions

a single point $x = 0$, this is an exactly marginal perturbation. In the two-dimensional classical field theory, this corresponds to a defect line at $x = 0$ for all τ . In the 2d classical lattice model, this amounts to a defect with deformed couplings between the adjacent spins across the defect. Varying φ_0 to move along the CD line therefore corresponds to varying the Ising coupling at a single link in the quantum transverse field model, or along the defect line in the classical lattice model.

The product boundary conditions (free, free) do not appear separately in the table, because they are a particular point on the CD line. This is obvious from the Ising defect interpretation: taking the limit of zero link coupling along the defect splits the system in two, and puts free boundary conditions on the spins on each side of the defect.

We will describe in detail in the next section how continuous Dirichlet boundary conditions correspond to allowing Majorana fermion backscattering at the point contact. We will also show there that the continuous Neumann (CN) boundary conditions arise in a topological state when a vortex is pinned in the bulk.

V. MAJORANA FERMION BACKSCATTERING AT A POINT CONTACT

In this section, we discuss in depth the simplest kind of backscattering at a point contact, that of Majorana fermions. This backscattering is identical for $p + ip$ superconductor, its quantum-disordered counterpart, and $\nu = 5/2$ fractional quantum Hall effect, because in the latter the Majorana fermion has no charge, so that the extra bosonic field does not affect its tunneling.

The effects of backscattering at a point contact can be treated as different boundary conditions at $x = 0$ in the folded picture. For Majorana fermion backscattering, these boundary conditions remain conformal. The reason is that free fermionic fields have dimension $1/2$, so any bilinear $\psi_1\psi_2$ has dimension 1. Such a bilinear backscattering operator at the point contact is an exactly marginal boundary operator, unlike the relevant backscattering in a Luttinger liquid^{31,32}. Tuning the strength of the backscattering results in a line of conformal boundary conditions. Such a line does not occur in the case of a single Majorana fermion (e.g. at one of the ends of the strip): because of the boundary condition (5), a bilinear $\psi_L(0)\psi_R(0)$ can be fused together, yielding the identity operator and thus a trivial boundary perturbation. Since the point contact results (after folding) in two copies of a Majorana fermion with a boundary, the bilinears $\psi_{1L}(0)\psi_{1R}(0)$ and $\psi_{2L}(0)\psi_{2R}(0)$ are

not trivial.

As we saw from the bosonization analysis, there are in fact two critical lines. We show in this section precisely how to move along either of these critical lines by tuning the backscattering strength across the point contact. The continuous Dirichlet line (of which transmitting and (free,free) are special cases) arises when there are no vortices present. The continuous Neumann line occurs when there is a single vortex pinned in the bulk.

A. Fermion backscattering in the absence of vortices

We start our analysis of fermion tunneling by considering a disk with no vortices. With no tunneling of any sort, the folded model has transmitting boundary conditions at $x = 0$. Allowing backscattering there perturbs the transmitting boundary condition by adding the operator

$$L_f = i\lambda_f(\psi_{1L}(0)\psi_{1R}(0) + \psi_{2L}(0)\psi_{2R}(0)) \quad (12)$$

to the Lagrangian. Because the transmitting boundary condition (9) relates the two terms in (12), fermionic anticommutation requires a relative plus sign to obtain a non-trivial perturbation.

Since L_f is exactly marginal, adding it to the Lagrangian results in a line of boundary fixed points parametrized by λ_f . This line is precisely the continuous Dirichlet line discussed at the end of the previous section. This is obvious from the Ising defect interpretation. Transmitting boundary conditions correspond to no defect at all, so in the lattice model this amounts to setting the link coupling across the defect to be the same as the link coupling everywhere else. This, therefore, is a point on the CD line. As is well known, the marginal energy operator ϵ is $\psi_L\psi_R$ in fermionic language. Perturbing by L_f is therefore the same as varying the link coupling across the defect, as illustrated schematically in figure 4. Thus, varying the Majorana fermion backscattering λ_f at the point contact corresponds to varying φ_0 to move along the CD line. Below, we relate λ_f to φ_0 explicitly.

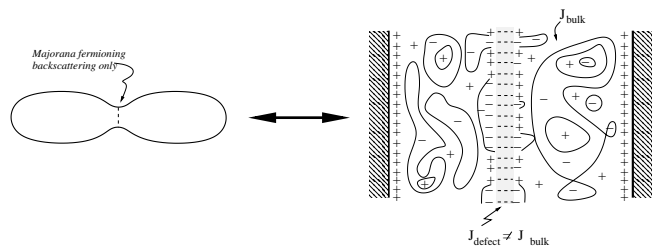


FIG. 4: A point contact at which Majorana fermion backscattering is the only non-zero tunneling process is equivalent to the Ising model with a column of bonds at which $J_{\text{defect}} \neq J_{\text{bulk}}$.

Since the boundary perturbation is quadratic, we can derive the exact equations of motion. The action of the folded system is of the form (3), where now $\mathcal{L}_{\text{edge}}$ is comprised of two copies of (2) for the two non-chiral Majorana fermions, and the boundary terms are (9) and (12). The solution can

be expressed most simply in terms of the Dirac fermions $\Psi_R = \psi_{2R} + i\psi_{1R}$ and $\Psi_L = \psi_{1L} - i\psi_{2L}$. The effect of the point contact is simply a phase shift:

$$\Psi_R(0) = e^{i\delta} \Psi_L(0) \quad (13)$$

where $\tan(\delta/2) = \lambda_f/2v_n$. (Another way to solve the system is to unfold and redraw the point contact with the left-movers flipped so that they became right-moving. Then the two right-moving Majorana fermions can be combined into a Dirac fermion, and again the effect of (12) is simply a phase shift.)

It is useful to re-express this in terms of a reflection matrix \mathcal{R} , which describes how the Majorana fermions behave when they bounce off the boundary. In the original, pre-folding, picture, this is the scattering matrix off the point contact. Since the Majorana fermions are real, the reflection matrix is necessarily real. One situation we have already discussed is transmitting, i.e. when there is effectively no point contact at all. In the folded system, perfect transmission (9) corresponds to reflection matrix

$$\mathcal{R}^{\text{transmit}} = -i\sigma^y. \quad (14)$$

Since the action remains quadratic in terms of the Majorana fermions, one can compute the reflection matrix directly or rewrite the solution from the Dirac fermion, yielding

$$\begin{aligned} \mathcal{R}(\lambda_f) &= \cos \delta (-i\sigma^y) + \sin \delta \mathbb{1} \\ &= \frac{1 - (\lambda_f/2v_n)^2}{1 + (\lambda_f/2v_n)^2} (-i\sigma^y) + \frac{\lambda_f/v_n}{1 + (\lambda_f/2v_n)^2} \mathbb{1} \end{aligned} \quad (15)$$

where λ_f controls the amount of backscattering. When λ_f is zero, we recover the transmitting case (14).

To make contact with the conventions of Oshikawa and Affleck²⁰, the Majorana fermions need to be combined in a slightly different fashion than the Ψ defined above. To be precise, we write $e^{i\varphi_R} = \chi_R = -\psi_{1R} + i\psi_{2R}$ and $e^{i\varphi_L} = \chi_L = \psi_{1L} + i\psi_{2L}$. Then, in terms of the Dirac fermion χ , the boundary conditions (13) or, equivalently, (15) take the form $\chi_R(0) = ie^{i\delta} \chi_L^\dagger(0)$. The Dirichlet boundary condition $\varphi(0) = \varphi_0$ discussed at the end of the last section then relates δ and φ_0 as³³

$$\begin{aligned} \varphi(0) &= \frac{\delta}{2} + \frac{\pi}{4} \\ &= \tan^{-1} \left(\frac{2v_n + \lambda_f}{2v_n - \lambda_f} \right). \end{aligned} \quad (16)$$

The backscattering strength λ_f therefore parametrizes a line of critical boundary conditions. To understand this line of critical points, it is useful to explore some special values of the phase shift δ .

At $\lambda_f = 2v_n$, we have $\delta = \pi/2$ and $\varphi_0 = \pi/2$. the reflection matrix simply becomes the identity matrix $\mathbb{1}$; the two copies are no longer coupled. From (15), we see that each resulting droplet has boundary condition (5) with $a = 1$, i.e.

$$(\text{free, free}): \psi_{1R}(0) = \psi_{1L}(0), \quad \psi_{2R}(0) = \psi_{2L}(0). \quad (17)$$

In Ising language, this is the product boundary condition (free, free). The system is effectively split into two. This split by fermion backscattering is smooth, without a crossover scale. This is unlike the case of vortex backscattering, which we discussed at length in two recent papers^{15,16} and will review in section VI below. Also unlike vortex backscattering, fermion backscattering results in droplets with free boundary conditions at $x = 0$ and fixed boundary conditions at $x = L$. Thus a pair of vortices has effectively been nucleated at the point contact, so that each of the two droplets then has a zero mode.

At $\lambda_f = -2v_n$, $\delta = -\pi/2$ and $\varphi_0 = 0$, so the reflection matrix is $-\mathbb{1}$. every fermion is backscattered, but this time neither droplet has a zero mode. One might be tempted to conclude that this is (fixed, fixed) boundary conditions, but this is not quite right. Changing the strength of Ising coupling or, equivalently, adding a fermion bilinear to the action cannot favor either up-spins or down. Hence, the boundary conditions of the two droplets cannot be fixed. When $\varphi_0 = 0$, the Ising coupling at the defect line is infinite. Therefore, the two Ising spins on either side of the defect are fixed to have the same value; however, this value is equally likely to be up or down. In fact, since the Ising coupling at the defect is infinite while the transverse field remains finite, the defect spins are not only equally likely to be up or down, but they are also not flipped by local dynamics. (In contrast, in the case of a free boundary condition, the boundary spin still fluctuates as a result of the transverse field, and only has entropy $\ln \sqrt{2}$.) Thus, we will call these boundary conditions (\pm, \pm) . They are almost (fixed, fixed), except that the value to which the two spins are fixed is a spin-1/2 degree of freedom (i.e. a two-level system).

Another boundary condition on the Dirichlet line is $\varphi_0 = 3\pi/4$, i.e. $\delta = \pi$. There is no backscattering at the point contact, so it is almost the same as transmitting boundary conditions, except for one thing: every fermion transmitted picks up a phase shift of π . In Ising language, this is an antiferromagnetic defect at which the Ising coupling is equal in magnitude to that in the bulk but opposite in sign. This follows from the result of Ref. 20 that flipping the sign of the link coupling sends $\varphi_0 \rightarrow \pi - \varphi_0$, so if the magnitude is unchanged from the transmitting boundary condition ($\varphi_0 = \pi/4$), then $\varphi_0 = 3\pi/4$.

The bosonic formulation of this critical line makes it possible to compute exact correlation functions along the edge for any value of δ . Fermionic correlators are of course trivial to find, but to find ones involving spin fields requires computing the boundary state. Detailed expressions can be found in [20]. One interesting result is the dimension Δ_b of the spin field along the defect, defined so that the correlator of two σ operators at the point contact at different times falls off as

$$\langle \sigma(0, t_1) \sigma(0, t_2) \rangle \sim \frac{1}{(t_1 - t_2)^{2\Delta_\sigma}}$$

The exact expression is²⁰

$$\Delta_b(\delta) = \frac{1}{8} \left(1 + \frac{2\delta}{\pi} \right)^2. \quad (18)$$

In Ising language, a boundary magnetic field couples to the spin field, so Δ_b is the dimension of the operator coupling to

this field. At the (free, free) point, $\delta = \pi/2$, we find $\Delta = 1/2$. This is not surprising since, as we described in the previous section, in a single Ising model with free boundary conditions, the left and right components of the spin field fuse to form the dimension-1/2 fermion so that a boundary magnetic field has scaling dimension 1/2. At the transmitting point $\delta = 0$, the left and right components of the spin field are decoupled, so Δ_b is the sum $1/16 + 1/16 = 1/8$. At the (\pm, \pm) fixed point $\delta = -\pi/2$, so we have $\Delta_b = 0$ here. This is a reflection of the fact that even an infinitesimal boundary magnetic field will favor either $(+, +)$ or $(-, -)$, thus splitting these two degenerate levels by adding a dimension-0 S_z term.

B. A Point Contact with a Localized Vortex in the Bulk: Continuous Neumann Boundary Conditions

The situations discussed above do not exhaust the possibilities for critical behavior at the point contact. Indeed, as seen in table I, there is another critical line of conformal boundary conditions, dubbed ‘‘continuous Neumann’’ (CN) in Ref. 20. CN boundary conditions do not occur in the classical Ising model with a defect line; in the quantum transverse field Ising model, they correspond to varying the Ising coupling J at a single link in the presence of a peculiar $\sigma_{-1}^z \sigma_0^x$ term across this link ($i = -1, 0$ are the sites on either side of the defect link)²⁰. Since the Ising spin field σ^z is the vortex creation operator, we expect that the CN line can be realized in the $p + ip$ superconductor/ $\nu = 5/2$ quantum Hall state with a single vortex in the bulk.

To prove our assertion, note that the state with a vortex in the bulk and no fermion backscattering has a higher entropy than one without a vortex by $\ln \sqrt{2}$ (we saw this in section III by deforming the droplet so that the vortex is at an endpoint of the strip and its presence or absence is simply the difference between free and fixed boundary conditions). But this is precisely the entropy difference between the CN and CD lines²⁰. Since the state without a vortex and no backscattering is the transmitting point (i.e. $\delta = 0$ or, equivalently, $\varphi_0 = \pi/4$) point on the CD line, we conclude that the state with a vortex and no backscattering – which has a higher entropy by $\ln \sqrt{2}$ – is on the CN line. We dub this the ‘‘Neumann-transmitting’’ point. We will show in this section that we can move away from this point along CN line by allowing (non-resonant) fermion backscattering.

In fermionic language, the CN case is not really substantially different from the CD case unless the vortex is pinned right at the point contact and we consider its coupling to the edge (which leads to a flow from Neumann to Dirichlet). Otherwise, the effect of the vortex can simply be absorbed into the boundary conditions at $x = L$; the boundary condition at the point contact will then be precisely the same as in the absence of the vortex. However, the translation to the Ising model and, especially, to the results of Ref. 20 can be made more directly if we keep the Ising boundary conditions in both copies fixed at $x = L$. Furthermore, this formulation of the problem allows us to consider the interesting situation in which the vortex is in the point contact and Majorana fermions can (resonantly)

tunnel between the vortex and both edges. We discuss this in the next section; for the rest of this section we discuss fermion backscattering and the CN line.

In the unfolded picture, we have periodic boundary conditions, so that $a = b = 1$, i.e. free at $x = -L$ end and fixed at $x = L$. We now fold the strip in half, as before, but now we define the two right-moving modes as $\psi_{1R}(x) = \psi_R(x)$, $\psi_{2R}(x) = \psi_L(-x)$, and the two left-moving modes as $\psi_{1L}(x) = \psi_L(x)$, $\psi_{2L}(x) = \psi_R(-x)$ with $x \in [0, L]$. The absence of a minus sign in the last of these is the difference with the CD case. When we fold the system, the free boundary condition at $x = -L$ becomes a fixed boundary condition for the second copy. Thus no extra minus sign is needed to ensure fixed boundary conditions at $x = L$ in both copies. Consequently, when there is no fermion backscattering, we do not need the branch cut which we put at the point contact at the transmitting point on the CD line. This amounts to modifying the transmitted boundary condition (9) to Neumann-transmitting or ‘N-transmitting’:

$$\text{N-transmitting: } \psi_{1L}(0) = \psi_{2R}(0), \quad \psi_{1R}(0) = \psi_{2L}(0).$$

so that the reflection matrix is

$$\mathcal{R}_{\text{N-transmit}} = \sigma^x,$$

as opposed to $-i\sigma_y$ for transmitting boundary conditions in the absence of the pinned vortex, eqn. (14).

Fermion backscattering is, again, a marginal perturbation:

$$L_{fv} = i\lambda_{fv}(\psi_{1L}(0)\psi_{1R}(0) - \psi_{2L}(0)\psi_{2R}(0)). \quad (19)$$

The solution can, once again, be expressed most simply in terms of a phase shift for a Dirac fermion:

$$\Psi_R(0) = e^{i\tilde{\delta}} \Psi_L(0), \quad (20)$$

where $\tan(\tilde{\delta}/2) = \lambda_{fv}/2v_n$. However, we must define Ψ a little differently as a result of the absence of a minus sign in the definition of ψ_{2L} : $\Psi_R = \psi_{2R} + i\psi_{1R}$ and $\Psi_L = \psi_{1L} + i\psi_{2L}$. Consequently, the reflection matrix now takes the form:

$$\begin{aligned} \mathcal{R}(\lambda_{fv}) &= \cos \tilde{\delta} \sigma^x + \sin \tilde{\delta} \sigma^z \\ &= \frac{1 - (\lambda_{fv}/2)^2}{1 + (\lambda_{fv}/2)^2} \sigma^x + \frac{\lambda_{fv}}{1 + (\lambda_{fv}/2)^2} \sigma^z \end{aligned} \quad (21)$$

To match the results with those from bosonization, we form the Dirac fermion χ precisely as in the last subsection: $\chi_R = -\psi_{1R} + i\psi_{2R}$, $\chi_L = \psi_{1L} + i\psi_{2L}$. The CN boundary condition is now $\chi_R(0) = ie^{i\tilde{\delta}} \chi_L(0)$. Bosonizing as before, we see that the dual boson $\tilde{\varphi} = \frac{1}{2}(\varphi_R - \varphi_L)$ now has CN boundary condition $2\tilde{\varphi}_0 \equiv 2\tilde{\varphi}(0) = \tilde{\delta} + \pi/2$. Once again, we have a critical line parametrized equivalently by λ_{fv} , $\tilde{\delta}$, or $\tilde{\varphi}_0$.

When there is no backscattering we have the Neumann-transmitting boundary condition, $\tilde{\varphi}_0 = \pi/4$. For $\tilde{\delta} = \pi/2$ or $\tilde{\varphi}_0 = \pi/2$ on the other hand, every Majorana fermion incident on the point contact is backscattered:

$$\text{(free, } \pm): \psi_{1R}(0) = \psi_{1L}(0), \quad \psi_{2R}(0) = -\psi_{2L}(0). \quad (22)$$

Thus the droplet is broken in two. From (21), we see that one droplet has free boundary condition while the other has fixed boundary condition, but with either fixed value equally likely. Like the $\tilde{\delta} = -\pi/2$ case on the CD line, this is not a product boundary condition. In analogy with the notation, we call this (free, \pm). For $\tilde{\delta} = -\pi/2$ or $\tilde{\varphi}_0 = 0$, this is reversed, and we obtain (\pm , free) boundary conditions.

C. The Moore-Read Pfaffian state

Thus far, we have focused on the Majorana fermion edge mode, ignoring the charged mode which is present in the Moore-Read Pfaffian state⁴. The edge theory of the Moore-Read state involves a chiral boson ϕ_c as well as the Ising fields I, σ and ψ ¹⁸. At filling fraction $\nu = 1/m$, the “electron” on the edge is created by the operator $\psi e^{i\sqrt{m}\phi_c}$. Setting $m = 2$ gives the case which may be relevant to the $\nu = 5/2$ quantum Hall state, so that the “electron” is the physical electron. The topological field theory is defined as the full theory “mod an electron”; fusing with an electron leaves the same topological state. Each different type of quasiparticle corresponds to a primary field in a rational conformal field theory: at $m = 2$ they comprise the Neveu-Schwarz sector of the superconformal theory with central charge $c = 3/2$, and are the identity, $\sigma e^{\pm i\phi_c/2\sqrt{2}}$, ψ , and $e^{\pm i\phi_c/\sqrt{2}}$. (If one instead considers the $m = 1$ bosonic Moore-Read state, one obtains the $SU(2)_2$ conformal field theory, which has primaries $\sigma e^{i\phi_c/2}$ and ψ .)

The boundary conditions of a squashed MR state are therefore combinations of the three boundary conditions of the critical $2D$ Ising model with those of the charge boson. The key condition is that all allowed boundary conditions should be left invariant by adding an electron, since an electron carries no topological charge. The chiral boson here has m different primary fields and, therefore, m different conformal boundary conditions, corresponding to the different possible electrical charges at the edge, modulo the charge of the electron. Therefore, the allowed (Ising, Charge) boundary conditions are $(+, n) + (-, n + m)$, $(-, p) + (+, p + m)$, $(f, q/2) + (f, q/2 + m)$ where $n, p = 0, 1, \dots, m - 1$ and $q = 1, 3, 5, \dots, 2m - 1$. All of the $2m$ boundary conditions $(+, n) + (-, n + m)$, $(-, p) + (+, p + m)$ are analogous to the fixed boundary conditions, and they can result at the boundary of either of the two droplets formed when a droplet is broken in two by vortex backscattering. The key difference is that electron tunneling is an allowed (although irrelevant) perturbation, whereas only fermion bilinears can tunnel from fixed boundary conditions in the Ising model. Similarly, $(f, q/2) + (f, q/2 + m)$ is analogous to the free boundary condition.

A similar analysis applies to the anti-Pfaffian state^{7,8}. At the edge of this state, the Majorana fermion mode ψ propagates in the opposite direction to the charged mode ϕ_c and there is an extra counter-propagating neutral bosonic mode ϕ_n . The latter can be fermionized, thereby forming, together with ψ , a triplet of Majorana fermions. Consequently, there is a triplet of electron operators, and all boundary conditions must be in-

variant under adding any of these three electron operators.

Because the fermion ψ has no charge, our earlier analysis of zero-mode tunneling and fermion backscattering goes through without modification. However, vortex backscattering is no longer the single-channel Kondo problem discussed in the next section, but instead becomes a variant of the two-channel Kondo problem¹⁶.

VI. FLOWS FOR A MAJORANA FERMION EDGE MODE

We have described in depth in the previous sections the different boundary fixed points possible for a Majorana fermion edge mode. In this section we describe the flows between these fixed point occurring when interactions at the point contact break scale and conformal invariance. A valuable tool in understand these flows is “boundary” entropy, a sub-leading term in the entropy which depends on the boundary conditions²⁷. At boundary fixed points, it can be computed directly from conformal field theory²². Moreover, it must decrease (at least in perturbation theory) in these flows, so the resulting constraints allowing us here to understand essentially all the different types of flows. Some of these flows have been understood in the Ising context^{20,34}, but new insight is gained by considering the topological context.

A. Flowing from CN to CD by zero-mode tunneling

We have already discussed one flow between different boundary conditions. In section III the coupling of the edge to a bulk vortex causes a flow from free to fixed boundary conditions, with resulting entropy change $\Delta S = -\ln \sqrt{2}$. The obvious generalization to the point-contact case is in agreement with the change from (free, fixed) to (fixed, fixed) in the table, which is the flow for bulk-edge coupling in one of two decoupled droplets. Likewise, the flow from CD to (free, fixed) must have the same entropy drop, since (free, free) is a point on the Dirichlet line.

As noted above, the entropy along the CN line is higher than along the CD line by $\ln \sqrt{2}$. When the vortex is coupled to the edge of the system via resonant Majorana fermion tunneling, the boundary condition flows from a point on the CN line to a point on the CD line. The value of δ in the latter is not $\tilde{\delta}$ in the former, unless the vortex is coupled to only one side of the point contact.

To explore this further, we now couple the zero mode of the pinned vortex to the edge. The interesting difference with the calculation in section III is that we can couple the vortex to the bottom and top edges of the point contact, with couplings λ_T and λ_B respectively. The Lagrangian in folded language is

$$L_{pv} = i\psi_0\partial_t\psi_0 + i\lambda_T\psi_0[\psi_{1R}(x=0) + \psi_{2L}(x=0)] + i\lambda_B\psi_0[\psi_{1L}(x=0) + \psi_{2R}(x=0)]. \quad (23)$$

Note that because $\psi_0^2 = 1$, when L_{pv} is present in the Lagrangian, non-resonant fermion backscattering (19) is gener-

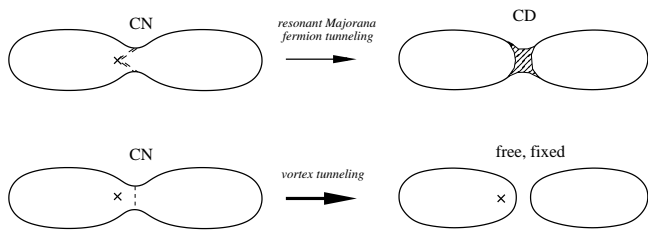


FIG. 5: (a) Inter-edge *Majorana fermion* backscattering through the zero mode on a localized quasiparticle takes the system from a point on the continuous Neumann line to a point on the continuous Dirichlet line. The region between the two droplets is not vacuum since inter-droplet Majorana fermion tunneling is an allowed perturbation. (b) Inter-edge *vortex* backscattering takes the system from a point on the continuous Neumann line to the (free, fixed) point. There is vacuum between the two droplets into which the system is broken.

ated at order $\lambda_T \lambda_B$, i.e. a Majorana fermion can tunnel to the vortex and then from there to the other side.

For simplicity, let us suppose that we are at the N -transmitting point on the CN line. If we were to set either λ_T or $\lambda_B = 0$, then L_{pv} would reduce precisely to (6). As we saw in section III, the bulk vortex is effectively absorbed by the edge to which it is coupled, with an entropy drop of $\ln \sqrt{2}$. The resulting boundary condition at the end of the flow is thus the transmitting point on the CD line. The entropy drop indeed is the difference between Neumann and Dirichlet entropies, as seen in table I.

For non-zero $\lambda_T \lambda_B$, the resulting flow is from Neumann-transmitting to a point on the Dirichlet line with $\delta \neq \tilde{\delta}$. Since the action remains quadratic, we can diagonalize it explicitly to compute the reflection matrix for any λ_T and λ_B . In the DC limit (or equivalently, large λ_T and/or λ_B), we find

$$\mathcal{R}_{pv}(\theta) = \sin(2\theta)\mathbb{1} + \cos(2\theta)i\sigma^y, \quad (24)$$

where $\theta = \arctan(\lambda_T/\lambda_B)$. In the resonant case, $\lambda_T = \pm\lambda_B$, we have $\theta = \pm\pi/4$ so that

$$\mathcal{R}_{res} = \pm\mathbb{1}, \quad (25)$$

corresponding to complete backscattering. These are the (free, free) and (\pm, \pm) boundary conditions discussed above. For arbitrary λ_T, λ_B , we end up at the point on the CD line with $\delta = 2\theta$ or, equivalently, $\varphi_0 = \theta + \pi/4$. If we start at a point on the CN line with $\tilde{\delta} \neq 0$, then we end up at the point $\delta = \tilde{\delta} + 2\theta$ on the Dirichlet line since the phase shifts add.

B. Vortex backscattering

In the previous section, we have analyzed the continuous Dirichlet and Neumann boundary conditions in terms of Majorana fermion backscattering at a point contact. In order to discuss the remaining boundary conditions, (free, fixed) and (fixed, fixed), and the flows to them from CD and CN boundary conditions, we need to recall a few facts about inter-edge vortex backscattering.

The problem of vortex backscattering across a point contact is not as easily solvable as fermion backscattering is. In two earlier papers^{15,16}, however, we exploited bosonization to map this problem onto the anisotropic single-channel Kondo problem. The effective spin-1/2 Kondo spin \vec{S} arises to take account of the non-Abelian statistics of the vortex operators. The vortex backscattering Lagrangian (in the normalization used above) is

$$L_v = \lambda_v (S^+ e^{i\varphi(0)/2\sqrt{2}} + S^- e^{-i\varphi(0)/2\sqrt{2}}). \quad (26)$$

Since the dimension of $e^{\pm i\varphi(0)/2\sqrt{2}}$ is 1/8, vortex backscattering is relevant; it causes the system to flow away from the Dirichlet line. (Where we start on the Dirichlet line depends on the strength of fermion backscattering.) Independent of where we start, this is a strongly relevant perturbation to the free Majorana edge modes and there is a crossover to strong coupling where all edge modes are completely backscattered and the system breaks into two droplets. The resulting entropy drop is $\ln 2$, due to the screening of the emergent spin-1/2 degree of freedom. We conclude that, in Ising language, the flow for vortex backscattering is from Dirichlet to (fixed, fixed). This is the most stable boundary condition: vortex backscattering splits the droplet in two so completely that all allowed perturbations coupling the two droplets are irrelevant^{15,16}. Thus, in the language of defect lines in the Ising model, vortex backscattering is a magnetic field at the two columns of sites on either side of a particular column of bonds, which leads to fixed boundary conditions for both columns of spins. Indeed, one can see to all orders in perturbation theory that the Kondo interaction (26) is equivalent to adding a defect magnetic field in the Ising field theory³⁵. In contrast, Majorana fermion backscattering is, as we have seen, a weakening (or strengthening) of that column of bonds.

By the same logic that led us to conclude that vortex backscattering leads from the CD line to the (fixed, fixed) point, we conclude that when there is a single vortex in the bulk, decoupled from the edge, vortex backscattering causes the flow from CN to (free, fixed) boundary conditions. Of course, unless the bulk vortex is right at the point contact, this is really the same, as far as local physics near the point contact is concerned, as the Dirichlet to (fixed, fixed) flow since the branch cut associated with the bulk vortex can be moved to $x = L$ by a gauge transformation. When the bulk vortex is right at the point contact, however, the problem is quite subtle and depends on the precise backscattering paths and how they wind around the bulk vortex.

A simple picture explains the difference between the (free, free) and (fixed, fixed) points heuristically. At both fixed points, the system divides into two droplets. Imagine tunneling a fermion from one droplet to the other; the tunneling term would look like:

$$H_{\text{tun}} = t_f (\psi_{1R}(0) + \psi_{1L}(0)) (\psi_{2R}(0) + \psi_{2L}(0)) \quad (27)$$

In the (free, free) case, $\psi_{1R}(0) = \psi_{1L}(0)$ and $\psi_{2R}(0) = \psi_{2L}(0)$. Hence, it is possible to couple the two droplets with such a term. It is simply that t_f has been tuned to zero at the (free, free) point. By varying t_f , we move along the CD

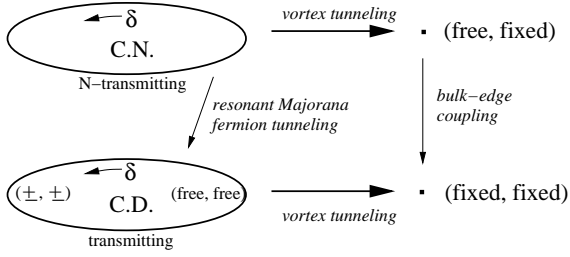


FIG. 6: The different fixed lines and fixed points for a point contact in the Majorana fermion edge mode. The entropy drop associated with resonant Majorana fermion backscattering (CN to CD and (free, fixed) to (fixed, fixed)) is $\ln \sqrt{2}$. The entropy drop associated with vortex backscattering (CN to (free, fixed) and CD to (fixed, fixed)) is $\ln 2$.

line. In the (fixed, fixed) case, however, $\psi_{1R}(0) = -\psi_{1L}(0)$ and $\psi_{2R}(0) = -\psi_{2L}(0)$. Therefore, H_{tun} vanishes at the fixed point, and single fermion tunneling (which would be a marginal perturbation) is not possible. Instead, the leading perturbation of the fixed point is a Cooper pair tunneling term^{15,16}. The same argument applies to the (free, fixed) point. Meanwhile, the (\pm, \pm) point and the continuous Neumann analogues of (free, free) and (\pm, \pm) can all be perturbed by fermion tunneling.

C. Entropy drops

We have shown in this section that there are a variety of flows between the various boundary fixed points. We summarize these flows in the figure 6. In this subsection we discuss these flows in more depth by analyzing the entropy drops in these flows.

In Refs. 15,17, it was noted that the entropy drop which takes place as a result of vortex backscattering (i.e. as the system flows from the Dirichlet line to (fixed, fixed) or from the Neumann line to (free, fixed)) could be understood simply as the result of one droplet breaking into two. Namely, the sub-leading term in the thermodynamic entropy that we have discussed here has an intriguing correspondence with a different object, the entanglement entropy between regions in a topological state^{38,39}. The topological entanglement entropy of a droplet with trivial total topological charge and perimeter L is $S = \alpha L - \ln \mathcal{D}$, where α is temperature-dependent and \mathcal{D} is the total quantum dimension of the particular topological state of matter. For a $p + ip$ superconductor, $\mathcal{D} = 2$. When a droplet breaks into two droplets, each of which has trivial topological charge, the entropy of the two droplets is $S = \alpha L_1 + \alpha L_2 - 2 \ln \mathcal{D}$. Hence, if the total length of the edge(s) remains unchanged, $L = L_1 + L_2$, the entropy drop is simply $\ln \mathcal{D} = \ln 2$. This $\ln 2$ entropy drop as is the same as the drop in the thermodynamic entropy resulting from vortex backscattering across a point contact effectively splitting the droplet in two. This correspondence is not a coincidence, but has been proved to hold for arbitrary topological states¹⁷.

In light of this correspondence, it is natural to wonder why there is no entropy drop associated with breaking the sys-

tem in two by tuning from the transmitting point $\lambda_f = 0$ to $\lambda_f = \pm 2v_n$, i.e. the (free, free) and (\pm, \pm) points. The difference with the (fixed, fixed) case, which does have an entropy drop, is seen most easily at the (\pm, \pm) point. At the (\pm, \pm) point, the system is broken in two, just as in the (fixed, fixed) case, so there is a naive entropy drop of $\ln 2$. However, the spins at the defect line are equally likely to both be fixed up as to be fixed down. Hence, there is an extra $+\ln 2$ entropy associated with the choice of up or down spin. This cancels the naive entropy drop, so the total entropy is unchanged. To what does this choice of up or down spin correspond in fermionic language? Let us suppose that the fixed boundary conditions at $x = L$ are up (i.e. +) in both copies, so that the total fermion number is even (as discussed in sections II and III). Then, if the spins at $x = 0$ are both up, the fermion number in both droplets is even. If the spins at $x = 0$ are both down, then both droplets have odd fermion number. Both possibilities are equally likely at the (\pm, \pm) point. In effect, Majorana fermion backscattering, when tuned to $\delta = -\pi/2$, generates a qubit, just as if we had a pair of vortices in each half of the original droplet (with the total topological charge of all four vortices being trivial). Since the fermion number parity is the same in each droplet, the two droplets are not really decoupled. However, vortex backscattering causes the fermion number parity in each droplet to be fixed; the $\ln 2$ entropy drop can be interpreted as the loss in the uncertainty in the fermion number parity as the two droplets become truly decoupled.

At the (free, free) point, each droplet has a zero mode. Let us call the corresponding operators ψ_0^L, ψ_0^R , and consider the operator $(-1)^{N_F^R}$, the parity of the fermion number on the right droplet. This operator satisfies the anti-commutation relation $\{(-1)^{N_F^R}, i\psi_0^L\psi_0^R\} = 0$. Representing this algebra requires a two-dimensional ground state Hilbert space²⁴. Hence, there is, once again, a $+\ln 2$ entropy. However, unlike in the (\pm, \pm) case, this entropy can be split into a $\ln \sqrt{2}$ entropy ascribed to each droplet. When vortex backscattering is turned on, it is as if a magnetic field has been applied to the free boundaries of both edges: they both flow to fixed, losing entropy $2 \ln \sqrt{2}$.

On the CN line, we similarly have a naive entropy drop of $\ln 2$ when the droplet is split into two at the (free, \pm) and (\pm, free) points. However, there is, once again, an uncertainty in the fermion number parity of each droplet. One of the droplets contains the vortex and, therefore, has a zero mode. The other droplet can have either even or odd fermion number, as if it contained a pair of vortices, and this uncertainty leads to an extra entropy of $\ln 2$.

The CN to CD flow is caused by the coupling of a bulk vortex to the edge, which causes an entropy drop of $\ln \sqrt{2}$. The flow from CD to (fixed, fixed) is caused by inter-edge vortex backscattering, as is the flow from CN to (free, fixed); both lead to an entropy drop of $\ln 2$. The flow from (free, fixed) to (fixed, fixed) is again caused by the coupling of a bulk vortex in the left droplet to the edge, accompanied by an entropy drop of $\ln \sqrt{2}$. The only remaining flow is from CD to (free, fixed), which must also be accompanied by an entropy drop of $\ln \sqrt{2}$. At the (free, free) point on the CD line, it is simply the flow of the $x = 0$ boundary of one of

the droplets from free to fixed boundary conditions. Since the droplets have zero modes, this flow is presumably caused by the coupling of this zero mode to the edge.

The following is one way to view the results of this section. When vortex backscattering is allowed, the droplet breaks completely in two and there is vacuum between the two droplets. Thus, as suggested in Refs. 15,17, the entropy drops by $\ln 2$ since the number of droplets increases by one (or the Euler characteristic increases by one, as alluded to above). However, when only Majorana fermion backscattering is allowed, even when the system is tuned to the (free, free) point and the droplets seemingly break in two, it isn't vacuum between the droplets. Instead, another topological phase is effectively nucleated which has Majorana fermions as an allowed (though gapped) excitation. The obvious choice is the toric code (the doubled \mathbb{Z}_2 phase), which occurs in the bulk when the Chern number of the Majorana fermion number excitations of the Ising phase changes³⁶ (see also Ref. 37). Since this phase has the same total quantum dimension \mathcal{D} , it is natural that there be no entropy drop when it is nucleated.

We conclude this section by noting that these arguments about entropy drops hold in arbitrary geometries. In particular, the correspondence with topological entanglement entropy still holds. As shown by Bonderson⁴⁰, the entanglement entropy of a topological fluid in any planar region R is equal to

$$S = \alpha L - \chi_R \ln \mathcal{D} \quad (28)$$

where χ_R is the Euler characteristic of the region R and \mathcal{D} is the total quantum dimension of the topological fluid. Splitting a disc into two causes χ_R to increase from 1 to 2, so indeed the entropy drop is $\ln \mathcal{D}$. A more complicated situation is if the topological fluid has an annular topology. As discussed in section IV, the annulus corresponds to choosing transmitting boundary conditions at $x = L$. In the $p + ip$ case, the classical analog is an Ising model with periodic boundary conditions and a defect line. At the point contact at which vortex backscattering is allowed, the flow effectively splits the system at the point contact. The presence of the transmitting boundary conditions at the other end does not change this, as makes sense physically. However, now splitting the system at the point contact does not split the system into two, but rather changes the annular geometry to that of a single disc, changing the Euler characteristic from 0 to 1. Thus even though the entropies are different due to the different geometries, the entropy drop in the flow between the remains the same $\ln \mathcal{D}$. In fact, whenever any geometry is split via a point contact, the Euler characteristic will always decrease by 1, in accord with our assertion that the dynamics of the point contact is not affected by the geometry of the full system (i.e. the other boundary conditions).

VII. MULTIPLE BULK QUASIPARTICLES

In the foregoing we have dealt exhaustively with the cases of no vortices and a single vortex in the bulk of a $p + ip$ superconductor. We saw how in the absence of tunneling and

backscattering, the effect of a bulk vortex could be taken into account by changing a boundary condition in the squashed system. Zero-mode tunneling between this vortex and the edge then causes a change in this boundary condition. We devote this section to explaining how to account for the effect of multiple bulk quasiparticles on the edge modes. This makes it possible to consider different types of tunneling processes, including zero-mode tunneling between vortices.

In this section we again focus on a Majorana fermion edge mode, but it is straightforward to generalize the results: one of our earlier papers¹⁷ contains most of the necessary analysis. There we showed how to compute the partition function of the chiral conformal field theory describing the edge modes of a topological fluid. This partition function depends on how many of each type of quasiparticle are in the bulk. We dubbed this the ‘‘holographic’’ partition function, because it encodes the topological properties of the full system, not just the edge. For example, the universal part of the full topological entanglement entropy can be extracted from these partition functions. Squashing the system of course does not change the partition function, so our earlier computations still apply.

As before, the goal is to squash the droplet with chiral edge modes down to a line segment having both left and right movers. However, in order to be able to treat the many different possible tunneling process in terms of (perturbed) boundary conformal field theory, we must make multiple copies of the system. There is not a unique way of doing this, but one convenient way of doing so is illustrated in figure 7. When there are n quasiparticles, then we have n copies of the system. The $x = 0$ boundary of each of the n th copy corresponds to the point on the edge closest to the n th quasiparticle. The boundary condition on each of these copies then depends on the type of the corresponding quasiparticle. For a vortex in the $p + ip$ case, this is of course the free boundary condition. The $x = L$ boundary is where we couple the copies with a ‘‘fixed-transmitting’’ boundary condition. This means that we couple the left movers in the i th copy to the right movers in the $i + 1$ st copy (mod n) at $x = L$.

For the $p + ip$ superconductor with n vortices, putting this together means

$$\psi_{iL}(0) = \psi_{iR}(0), \quad (29)$$

$$\psi_{iR}(L) = (-1)^{i+1} \psi_{(i+1)L}(L). \quad (30)$$

for all i . The effect of zero-mode tunneling here between the i th vortex and the edge then results simply in adding to the Hamiltonian the term (6) for the i th copy.

For one vortex, these boundary conditions reduce to those considered previously. For two vortices, we can simplify the problem somewhat. As discussed at the end of section II, one doesn't need two copies here: one can impose free boundary conditions at both ends of a single copy. This can be recovered from the above by treating the boundary condition (30) at $x = L$ for $n = 2$ as transmitting. Then unfolding indeed gives a system with free boundary conditions at both ends of a system of length $2L$. The advantage of using two copies instead of one is that then one can consider tunneling between the two vortices as a boundary condition, in the same framework as all our calculations: it is a dimension-0 perturbation which is,

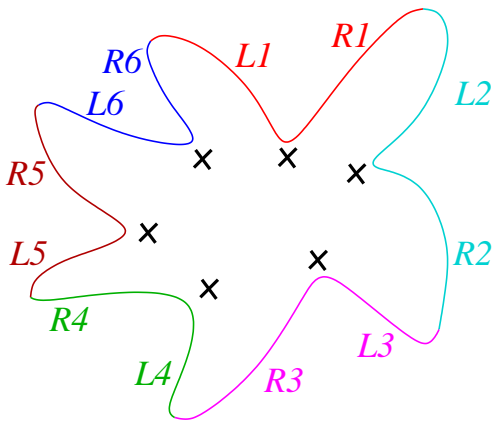


FIG. 7: A droplet with 6 quasiparticles, each marked by a cross. This can be mapped to 6 non-chiral systems on the strip, so the each line segment labeled by R_i or L_i becomes the right or left movers in the i th copy. The successive copies are coupled at the ends of the “fingers”, while the left and right movers within the i th copy are coupled at the point on the edge closest to the i th vortex. For a Majorana fermion edge mode, these boundary conditions are given explicitly in (29,30).

therefore, strongly relevant and causes a flow to the state with trivial total topological charge and entropy diminished by $\ln 2$.

Including a point contact is not much more difficult. One just needs two of the “inside” points in figure 7, and the point contact couples the two systems at $x = 0$, in the fashion as before. The system can then be split into two across this point contact, just as before. Similarly, a hole can be put in the surface by changing the boundary conditions at $x = L$ so that e.g. $\psi_{iR}(L) = \psi_{(i)L}(L)$ and $\psi_{(i-1)R}(L) = \psi_{(i+1)L}(L)$, just like we did for the annulus.

VIII. FIBONACCI ANYONS AND THE 3-STATE POTTS MODEL

It has long been known that each primary field in a rational conformal field theory corresponds to a particular conformal boundary condition²². In the context of a topological state, this means that there is a boundary fixed point corresponding to each quasiparticle type. This correspondence has a very nice application in this context: it means we can determine how boundary conditions change when additional bulk quasiparticles are included. This is the topological analog of fusion by a boundary operator^{22,27,41}; the corresponding chiral partition functions can be found in ref. 17.

We now illustrate this in a somewhat more complicated example of a topological state. The Read-Rezayi (RR) states¹³ generalize the MR state by replacing the ($k=2$) Ising CFT with the \mathbb{Z}_k parafermionic model, while simultaneously changing the radius of the charge boson in order to keep the scaling dimension of the electron operator fixed. In this section, we will briefly consider the first state after the Moore-Read state in this sequence, the $k = 3$ Read-Rezayi state. If a quantum Hall state were observed at $\nu = 13/5$, this state might be re-

alized there. The particle-hole conjugate of the Read-Rezayi state, the anti-RR state¹⁴, might be realized at the observed plateau at $\nu = 12/5$. The anti-RR state is related to the RR state in the same way as the anti-Pfaffian state is related to the MR state. Thus, much of what we have to say about the $k = 3$ RR state applies as well to the $k = 3$ anti-RR state.

The neutral sector of the edge theory of the $k = 3$ RR state is the \mathbb{Z}_3 parafermionic CFT. This conformal field theory describes the critical ferromagnetic 3-state Potts model. The 3-state Potts model is a lattice model of spins s_i which take the values $s_i = A, B, C$. The Hamiltonian is

$$H = -J \sum_{\langle i,j \rangle} \delta_{s_i s_j} \quad (31)$$

This model undergoes a second-order phase transition at a temperature T_c below which the spins develop a non-zero expectation value. The conformal theory for the critical point has 6 primary fields, $1, \psi, \psi^\dagger, \sigma, \sigma^\dagger, \varepsilon$. The fields $1, \psi, \psi^\dagger$ have a \mathbb{Z}_3 structure to their fusion rules: $\psi \times \psi = \psi^\dagger$, $\psi^\dagger \times \psi^\dagger = \psi$, $\psi \times \psi^\dagger = 1$. The field ε is a Fibonacci anyon: $\varepsilon \times \varepsilon = 1 + \varepsilon$. Then σ, σ^\dagger are formed by combining ε with the \mathbb{Z}_3 fields: $\sigma = \varepsilon \times \psi$, $\sigma^\dagger = \varepsilon \times \psi^\dagger$.

In the $k = 3$ RR quantum Hall state, there is also a charge boson, and the quasiparticles of the theory are products of the critical Potts quasiparticles with exponentials of the charge boson. This state can be interpreted as a quantum Hall state of triplets of electrons. The parafermions ψ, ψ^\dagger can then be viewed as a single electron or a pair of electrons (modulo 3). The Potts spin operator σ can be viewed as a flux $hc/3e$ vortex, and $\sigma^\dagger, \varepsilon$ can be viewed as such a vortex with one or two parafermions in its core, respectively. When the electrical charge is included, σ has electrical charge $1/5$, as we would expect for a flux $hc/3e$ vortex at $\nu = 3/5$; σ^\dagger has electrical charge $3/5$, as we would expect for a vortex with a charge $2/5$ parafermion; ε is neutral, as we would expect for a vortex with two parafermions.

All the conformal boundary conditions in the three-state Potts model are known⁴². There are six boundary conditions corresponding to the six quasiparticles (ignoring extra multiplicities due to the different possible electric charges). These are the three possible fixed boundary conditions, denoted A, B and C , and three more in which one of the spin values is forbidden at the boundary, denoted not- A , not- B , and not- C ⁴³. With the not- A boundary condition, each boundary spin is independently allowed to take either the value B or C while the value A is forbidden. These boundary conditions are called mixed boundary conditions, and sometimes not- A is written BC .

Consider a droplet which has been squashed, as in our Ising discussion, and suppose that the boundary condition at $x = L$ is fixed to A . When there are no quasiparticles in the bulk, the boundary condition at $x = 0$ will also be fixed to A . If there is a ψ or ψ^\dagger in the bulk, it will, instead, be fixed to B or C , respectively. (In the quantum Hall context, a ψ must be accompanied by an electric charge and a corresponding boson exponential in order to be local with respect to electrons. However, the charge part of this quasiparticle plays no role in the present

discussion.) All three of these possible fixed boundary conditions are completely stable (assuming that the bulk quasiparticle is pinned). Since ψ and ψ^\dagger are Abelian quasiparticles, this is not surprising. If there is an ε in the bulk, the boundary condition at $x = L$ will be not- A . If there is a σ or σ^\dagger in the bulk, the boundary condition will be not- B or not- C , respectively. These boundary conditions have a higher entropy than the fixed boundary condition by $\ln \tau$, where $\tau = (1 + \sqrt{5})/2$ is the quantum dimension of σ , σ^\dagger , and ε . Thus, mixed boundary conditions are unstable to bulk-edge coupling. The three types of vortices have a ε zero mode which couples with the ε field on the edge. This is a dimension-2/5 perturbation of the edge and is, therefore, highly relevant. It leads to the vortex being absorbed by the edge and an entropy drop $\ln \tau$.

Because the 3-state Potts model has a non-diagonal partition function, the six boundary conditions described above do not exhaust the conformal boundary conditions: there are in fact eight of them^{42,44}. The free boundary condition is obviously one of the missing ones. Because it does not correspond to one of the quasiparticle types, it does not occur simply by adding a quasiparticle in the bulk. (This is clear from the preceding discussion since we have already exhausted all of these possibilities.) It does not result from fusion with one of the primary fields of the Potts model but, instead, from fusion with one of the primaries of a $c = 4/5$ CFT with more primary fields (the tetracritical Ising model)⁴².

Thus, the free boundary condition and the eighth boundary condition, simply called ‘new’ by Affleck *et al.*⁴², are slight oddities in the boundary conformal field theory context. However, in the quantum Hall context, free boundary conditions arise quite naturally in a manner analogous to their appearance in the analysis of a point contact in the Ising case. Namely, consider a $k = 3$ RR droplet with a point contact in the middle. Let us suppose that there is non-zero backscattering of ϵ quasiparticles but no other tunneling is allowed so that the backscattering Hamiltonian is of the form

$$H_{\text{tun}} = \lambda_\epsilon \epsilon_R(0) \epsilon_L(0) \quad (32)$$

The analogous Ising operator, $\psi_R(0)\psi_L(0)$, has scaling dimension 1 and, therefore, is an exactly marginal perturbation. Here, this operator has scaling dimension 4/5, so that λ_ϵ is relevant and flows to strong coupling. To what does it flow in the strong coupling limit? In the 3-state Potts context, (32) is the energy operator, so on the lattice it is a local change of J on a column of bonds (precisely as in the Ising case with $\psi_R\psi_L$). If $\lambda_\epsilon > 0$, then this is a local *decrease* of J , which flows to $J = 0$, decoupling the Potts model into two halves and leaving free boundary conditions on each half at the column of vanishing bonds. The boundary entropy of (free,free) boundary conditions is less than zero⁴², so this flow from transmitting from boundary conditions (where the boundary entropy is zero) indeed has an entropy drop.

If, on the other hand, $t_\epsilon < 0$, then this is a local *increase* of J , which flows to $J = \infty$. As in the Ising case, the system is again cleaved in two with the same fixed boundary condition on each half and all three possible values of the fixed boundary condition equally likely. We thus call this boundary condition ($A/B/C, A/B/C$). The parafermion number of each droplet

is not fixed, so any of the possible fixed boundary conditions can occur. Since there are three possible values for the fixed boundary condition, there is an entropy $\ln 3$ larger than that of fixed boundary conditions on both droplets. Since this entropy is shared equally between the two droplets, each has an entropy $\ln \sqrt{3}$ higher than the fixed boundary condition.

(Free,free) boundary conditions have the same entropy as ($A/B/C, A/B/C$). Although there is not as simple an argument as in the case of the latter, the underlying reason is the same, namely that the parafermion number is not fixed. (We note that the interpretation of the entropy in terms of the parafermion number, as with the fermion number in the Ising case, makes sense at the fixed point in the limit of finite L . While the fixed point can be reached by fine-tuning, it is generically reached by flowing to the infrared, $T \rightarrow 0$. However, the entropy is ordinarily computed in the opposite limit: $L \rightarrow \infty$ first and then $T \rightarrow 0$. Thus, our intuitive argument applies to the opposite of the ordinary order of limits. The result is the same, however, as may be seen from direct computation in the $L \rightarrow \infty, T \rightarrow 0$ limit⁴². The equality of these two orders of limits may be related to the fact that we are discussing integer-valued degeneracies.) The computation of the partition function²³ shows that the free boundary condition on a single system indeed has entropy $\ln \sqrt{3}$ relative to the fixed boundary condition⁴². Since $\sqrt{3} > (1 + \sqrt{5})/2 > 1$, the free boundary condition has higher entropy than either mixed or fixed and the addition of a perturbation can lead to a flow to either one. In both the (free, free) and ($A/B/C, A/B/C$) cases, parafermions can tunnel from one droplet to the other, so it is clear that the two droplets are not separated by vacuum. The natural guess is that they are separated by a topological phase described by the deconfined phase of \mathbb{Z}_3 gauge theory, but this will be discussed elsewhere.

The ‘new’ boundary condition does not have a simple interpretation in the language of the 3-state Potts model. However, the ‘new’ boundary condition is known to be dual to the mixed boundary condition(s), just as the free boundary condition is dual to the fixed boundary condition(s)⁴². Thus, just as the mixed boundary condition is obtained from the fixed one by creating an additional ϵ quasiparticle in the bulk (or σ or σ^\dagger for its \mathbb{Z}_3 partners), the ‘new’ boundary condition is obtained from the free boundary condition by creating an additional ϵ quasiparticle in the bulk. Since the free boundary condition is \mathbb{Z}_3 -invariant, it does not matter whether we create an ϵ , a σ , or a σ^\dagger . Alternatively, we could begin with an RR droplet with an ϵ in the bulk (analogous to N-transmitting) and then turn on (32). Either way, we see that the ‘new’ boundary condition has entropy $\ln \tau$ larger than the free boundary condition.

One can handle other conformal boundary conditions in a point contact in a $k = 3$ RR droplet in a similar fashion. However, the number of the possible boundary conditions increases rapidly as k is increased. In Ising, there are just three basic transmitting-type boundary conditions: the basic one ($\delta = 0$ on the CD line), the antiferromagnetic defect ($\delta = \pi$ on the CD line), and the N-transmitting one ($\tilde{\delta} = 0$ on the CN line). (The latter two defects can be fused to give $\tilde{\delta} = \pi$ on the CN line.) For the three-state Potts model, even ignoring the charge mode, there are already 16 of them⁴⁵. Thus there

will be an array of unstable boundary fixed points here, even without considering fixed points such as $(A/B/C, A/B/C)$, which is neither transmitting nor a product boundary condition. However, there are no fixed lines for $k = 3$, because there are no dimension-1 tunneling operators.

IX. DISCUSSION

The preceding discussion can be generalized to other topological states. We can squash a droplet as before so that we formulate the edge effective theory in terms of conformal field theory on a strip. The presence of a quasiparticle in the bulk changes the boundary condition at one end of the strip. The boundary condition with no quasiparticles in the bulk is stable. If the added quasiparticle is Abelian, then the new boundary condition has the same entropy as the no-quasiparticle boundary condition and is also stable. However, if the added quasiparticle is non-Abelian, then the new boundary condition has higher entropy. Such a boundary condition is unstable to coupling to the edge. The non-Abelian quasiparticle has a zero mode to which edge quasiparticles with scaling dimension $\Delta < 1$ can tunnel resonantly.

The general framework, which the Ising and 3-state Potts models exemplify, makes the notion of squashing precise. For a 2+1-dimensional topological theory on a disk geometry, the topologically-distinct excitations correspond to the primary fields of the associated edge conformal field theory. If there is an excitation labeled by a in the bulk, then in the chiral conformal field theory of the edge, which is defined on a cylindrical

spacetime, the partition function χ_a results¹⁷. The results of Ref. 22 then allow one to find boundary conditions on the strip which give the same partition function χ_a . Thus any quantity in a given sector of the chiral conformal field theory can be computed in the non-chiral theory on the strip by imposing the appropriate boundary conditions.

We discussed these ideas in detail in the context of the Ising model, where there are three primary fields, $\sigma, 1, \psi$, which correspond to free, fixed +, and fixed -, respectively. In the corresponding 2 + 1-dimensional topological state, these correspond to a state with a vortex in the bulk, and then the states in which the vortex has been absorbed by the edge, with either 0 or 1 unpaired fermions in the bulk.

These ideas can be generalized to the description of point contacts in topological states. Inter-edge quasiparticle backscattering generically splits a droplet into two, with one of the aforementioned conformal boundary conditions on each of the resulting droplets. (Here, the Ising model is non-generic because it has two fixed lines.) One interesting feature which arose in our analysis is that the region between the two droplets is generically not the vacuum but, rather, a *different* non-trivial topological phase. This may be a zero-dimensional analogue of the condensation phenomena discussed in Refs. 37,46.

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