Correspondence between two-dimensional bosons and a bulk superconductor in a magnetic field

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We study the partition function of two-dimensional lattice bosons at $T=0$ and derive a dual representation which is isomorphic to a bulk superconductor with fluctuating-gauge field in an applied magnetic field. This allows us to relate boson ground states to thermodynamic phases of the superconductor. A density-wave Bose insulator corresponds to an Abrikosov flux-lattice phase, whereas boson superfluidity implies a nonsuperconducting flux-line liquid phase. By analogy with a boson supersolid, we suggest the possibility of an exotic Abrikosov flux-lattice phase with no superconducting long-ranged order.

A fundamental quantum feature of boson systems is that the phase of the superfluid order parameter ϕ and the boson density n are noncommuting conjugate variables[1] $[\phi,n]=i$. A theoretical description is then possible either in terms of a basis of states which is diagonal in the phase or diagonal in the density. Although the standard approach is in terms of the phase representation,[2] Haldane[3] has introduced a beautiful alternate framework for discussing one-dimensional (1D) bosons involving only the long-wavelength *density* modes. This framework has recently been applied successfully to study the $(T=0)$ insulator-superfluid transition in a disordered Bose system.[4] Attempts have been made to analyze this same transition in high dimensions by working in terms of the superfluid order parameter but are fraught with difficulties.[5,6] In light of this, it seems worthwhile to generalize Haldane's framework to higher dimensions. In this paper we describe a general scheme for doing this, valid in arbitrary dimension d, and work out the details explicitly for a system of 2D bosons.

More specifically, we study the partition function of a model of disordered lattice bosons at $T=0$ and derive a *duality* mapping from an order parameter representation to a dual density representation. In 2D the dual model is isomorphic to a bulk anisotropic lattice superconductor with fluctuating-gauge field in an applied (random) magnetic field. This isomorphism allows us to relate various possible ground states of the 2D bosons to thermodynamic phases of the classical superconductor (see Table I). For example, a density-wave boson insulator corresponds to an Abikosov flux-lattice phase whereas superfluidity in the boson system is isomorphic to a nonsuperconducting phase consisting of a *liquid* of flux lines.[7] This phase was recently predicted[8] to be present in the new Cu oxide superconductors, just above H_{c1}. From a symmetry point of view, though, this phase is indistinguishable from the normal phase with $H>H_{c2}$. The so-called "supersolid" phase,[9] if possible in the boson system, would correspond to an exotic Abrikosov vortex lattice with no superconducting off-diagonal long-ranged order (ODLRO).

TABLE I. Correspondence between 2D bosons and bulk superconductor obtained from duality mapping described in the text.

2D bosons ($T=0$)	Bulk superconductor
Chemical potential μ	Applied field H
Bose density n	Total field B
Mott insulating phase	Meissner phase
Density-wave insulator	Abrikosov flux lattice
Superfluid	Nonsuperconducting flux-line liquid
Supersolid	Nonsuperconducting flux lattice
Bose glass insulator	Superconducting glass

To be explicit, we consider a model Hamiltonian describing interacting bosons hopping on a d-dimensional cubic lattice: $\hat{H}=\hat{H}_0+\hat{H}_1$ with

$$\hat{H}_0=\frac{u}{2}\sum_i \hat{n}_i^2-\sum_i \mu_i\hat{n}_i\,, \tag{1a}$$

$$\hat{H}_1=-t\sum_{i,\nu}\cos(\Delta_\nu\hat{\phi}_i) \tag{1b}$$

where $\hat{n}_i$, which represents the deviation of the Bose number on site i from a mean value, is conjugate to the order-parameter phase, $[\hat{\phi},\hat{n}]=i$. Here u is a repulsive interaction between bosons and μ_i a random on-site chemical potential. In (1b), Δ_ν denotes a lattice derivative $\Delta_\nu\hat{\phi}_i\equiv\hat{\phi}_{i+\nu}-\hat{\phi}_i$. Typically,[2] the partition function for $\hat{H}$ in (1) is expressed as a path integral over a basis of states *diagonal* in the phase of the order parameter $Z=Tr'_\phi\exp(-S_\phi)$ ($\hbar=1$)

$$S_\phi=\frac{1}{2u}\int_\tau\sum_i\dot{\phi}_i^2(\tau)+i\sum_i\int_\tau\mu_i\dot{\phi}_i(\tau)+\int_\tau H_1[\phi_i(\tau)]\,. \tag{2}$$

Here, the imaginary time integration runs from 0 to β and the prime on the Tr_ϕ indicates a sum over all paths with a constraint $\phi_i(\beta)=\phi_i(0)+2\pi N_i$ for all integers N_i. Note that for noninteger μ_i, $\exp(-S_\phi)$ is *complex*. For integer μ_i, the action is essentially a classical Hamiltonian for a $(d+1)$-dimensional XY model. Since the 3D XY model is known[10,11] to be dual to a lattice superconductor with gauge field, it is natural to explore the dual representation of the full boson model (2) in 2D, with inclusion of the complex chemical potential term. As shown below, in the dual representation this complex term corresponds to a magnetic field applied to the lattice superconductor. This allows the myriad of possible thermodynamic phases of a bulk superconductor in a *magnetic field* to be related to 2D boson ground states.

The most direct way to obtain the dual representation for the partition function above, is by working in a basis of states *diagonal* in the density. To this end, the partition function $Z = Tr\exp(-\beta H)$ is split into infinitesimal time slices with size ϵ, and the identity operator $\hat{1}$, resolved in a basis diagonal in $\hat{n}$, is inserted between adjacent time slices. As in Ref. 10, it is convenient to replace the nondiagonal contribution to each time slice by a Villain form $\exp[\epsilon t\cos(\Delta\phi)] \to \exp[V(\Delta\phi;\epsilon t)]$ with

$$\exp[V(\Delta\phi;K)] \equiv \sum_m e^{im\Delta\phi}\exp(-m^2/2K) . \tag{3}$$

Since $\exp[V]$ preserves the required periodicity in the phase ϕ, the slight change in functional form should not alter the physics, although hereafter we denote the modified partition function by Z_V. One thereby obtains

$$Z_V = Tr_{\mathbf{J},n}\delta(\Delta_\perp\cdot\mathbf{J}+\Delta_\tau n)\exp(-S_n) , \tag{4a}$$

$$S_n = \sum_\tau \epsilon H_0[\{n_{i\tau}\}] + \frac{1}{2t\epsilon}\sum_{i,\tau}|\mathbf{J}_{i,\tau}|^2 , \tag{4b}$$

where $n_{i\tau}$ and $\mathbf{J}_{i\tau}$ denote an integer field and an integer d-dimensional vector field, respectively, defined on each discrete space-time point (i,τ). Notice that H_0 is diagonal in the density basis $\hat{n}$. The constraint in (4a) at each space-time point, $\Delta_\perp\cdot\mathbf{J}+\Delta_\tau n=0$, is a lattice continuity equation reflecting conservation of boson density. Thus $\mathbf{J}$ can be correctly interpreted as a lattice current.

Equation (4) is valid in general dimensions. In the following we specialize to $d=2$. Then the integer vector field $\mathbf{m}\equiv(\mathbf{J},n)$ is three dimensional and the constraint in (4a) can be removed by introducing an auxiliary integer field $\mathbf{z}$ such that $\Delta\times\mathbf{z}=\mathbf{m}$. The trace over $\{\mathbf{m}\}$ can then be replaced by a trace over $\{\mathbf{z}\}$, up to a multiplicative constant to the partition function. The β-periodic boundary condition on $\mathbf{m}$ is replaced, in turn, by the boundary condition $\Delta\times\mathbf{z}_{i,\tau}=\Delta\times\mathbf{z}_{i,\tau+\beta}$. It is convenient, at this stage, to convert the trace over the integer field $\mathbf{z}$ to a trace over a real continuous field $\mathbf{a}$ via the Poisson summation formula. The partition function then reads

$$Z_V = Tr_{\mathbf{a},l}\exp\left[i\sum_{i,\tau} l_{i,\tau}\cdot\mathbf{a}_{i\tau} - S_n(\Delta\times\mathbf{a})\right] , \tag{5}$$

with l an integer vector field.

In order to obtain a more instructive dual representation, we now soften the integer constraint on $\mathbf{a}$ by introducing the generalized Villain representation of the above partition function,[10] namely, $\tilde{Z}_V = Tr_{\mathbf{a},l}\exp(-S)$, with

$$S = -i\sum_{i,\tau} l_{i,\tau}\cdot\mathbf{a}_{i\tau} + S_n(\Delta\times\mathbf{a}) + \tfrac{1}{2}\sum_{i,\tau}[y_\perp|l^\perp_{i,\tau}|^2 + y_\tau(l^\tau_{i,\tau})^2] . \tag{6}$$

Here $y_\perp$ and y_τ are core energies for the space and time components, respectively, of the vortex loops[10] in the order parameter phase [see (2)]. From (2) it is easy to convince oneself that a sensible $\epsilon\to 0$ limit requires $y_\perp \propto (\epsilon)^{-1}$ and $y_\tau \propto \epsilon$. We thus take $y_\perp = (J\epsilon)^{-1}$ and $y_\tau = K\epsilon$.

In terms of the lattice bosons, the approximation in (6) is equivalent to softening the integer constraint on the eigenvalues of the number operator $\hat{n}_i$. Although this will change quantitative features in the phase diagram, such as the precise location of phase boundaries, it should not affect the long-distance, low-energy structure of the phases themselves. It does, however, allow the l summation to be conveniently performed using (3), which yields $\tilde{Z}_V = Tr_{\mathbf{a}}\exp(-S_{\mathbf{a}})$ with

$$S_{\mathbf{a}} = S_n(\Delta\times\mathbf{a}) + \sum_{i,\tau}[V(a^\perp_{i\tau};J\epsilon) + V(a^\tau_{i,\tau};1/K\epsilon)] . \tag{7}$$

It remains to take the time continuum limit $\epsilon\to 0$. Before doing so, it is useful to exploit the gauge invariance of S_n by shifting $\mathbf{a}\to\mathbf{a}-\Delta\theta$ and then performing a functional integral over the real field θ. This simply introduces a multiplicative overcounting in the partition function. Care is now needed as $\epsilon\to 0$, since the last term in (7) forces the constraint $a^\tau_{i,\tau}-\Delta_\tau\theta_{i,\tau}=2\pi m_{i,\tau}+O(\epsilon)$, with integer m. These integers, though, can be conveniently absorbed into θ, provided the β periodic boundary conditions on θ are replaced by $\theta_{i,\tau+\beta}=\theta_{i,\tau}+2\pi N_i$, for all integers N. Then upon defining $A^\tau_{i,\tau}=\epsilon^{-1}a^\tau_{i,\tau}$ and $\mathbf{A}^\perp_{i,\tau}=\mathbf{a}^\perp_{i,\tau}$ the $\epsilon\to 0$ limit can be easily taken, giving the final desired dual representation

$$\tilde{Z}_V = Tr'_{\mathbf{A},\theta}\exp[-S(\mathbf{A},\theta)] , \tag{8}$$

with $S=S_0+S_1$ and

$$S_0 = \sum_i\int d\tau\left[\frac{1}{2t}|(\nabla\times\mathbf{A})^\perp|^2 + \frac{u}{2}|(\nabla\times\mathbf{A})^\tau|^2 - \mu_i(\nabla\times\mathbf{A})^\tau\right] , \tag{9a}$$

$$S_1 = \frac{1}{2K}\sum_i\int d\tau(A^\tau-\dot{\theta})^2 + J\sum_{i,\nu}\int d\tau\cos(\Delta_\nu\theta_i - A^\nu_i) . \tag{9b}$$

Here $\mathbf{A}$ and θ are time-continuous fields satisfying the boundary conditions $(\nabla\times\mathbf{A})^\tau_{i,\tau+\beta}=(\nabla\times\mathbf{A})^\tau_{i,\tau}$ and $\theta_{i,\tau+\beta}=\theta_{i,\tau}+2\pi N_i$ for all integers N_i. In (9a) the gradient operator is discrete in space yet continuous in time $\nabla\equiv(\Delta_\perp,\partial_\tau)$. For notational clarity the Villain function has been replaced by a cosine in (9b). Note that the dual action (9) is entirely real, in contrast to the phase-diagonal action S_ϕ in (2).

Since μ couples to $(\nabla\times\mathbf{A})^\tau$ in (9a) it is apparent that this operator is effectively the boson density. The second term in (9a) is then simply the repulsive on-site interaction present in the original Hamiltonian (1a), whereas the first terms in (9a) and (9b) take into account the noncommutativity of $\hat{\phi}$ and $\hat{n}$. The cosine term in (9b), upon tracing out θ, gives a field acting on $(\nabla\times\mathbf{A})^\tau$ which favors integer values. Thus this term reflects the discreteness of the underlying boson field. Indeed, in its absence ($J=0$) the action is quadratic and at $T=0$ (even in the presence of a random μ_i) S describes a third sound mode, $\omega(k)\sim(ut)^{1/2}k$, indicative of superfluidity in the boson system. Destruction of superfluidity is in fact only possible when a symmetry is broken and $\langle\exp(i\theta)\rangle\neq 0$. Then $\mathbf{A}$ picks up a (Higgs) mass, destroying the third sound and superfluidity. Note, by contrast, that in the phase representation (2) a symmetry breaking *implies* superfluidity.

At $T=0$ the action (9) closely resembles the Hamiltonian of a classical anisotropic 3D superconductor,[10,11] coupled to a fluctuating-gauge field **A**. More precisely, the analog superconductor has a lattice structure in two space dimensions yet is continuous in the third direction (as in an array of superconducting cylinders with principle axis in the $\hat{\tau}$ direction, Josephson coupled with strength J). Notice that the chemical potential μ, which was the coefficient of a complex term in the phase representation (2), now plays the role of an applied magnetic field H (in the $\hat{\tau}$ direction), whereas the Bose site density $[\langle(\nabla\times\mathbf{A})_i^\tau\rangle]$ corresponds to the local magnetic field B_i actually piercing the ith plaquette. Thus, the flux lines penetrating the classical superconductor are equivalent to the world lines of the Bose particles.[12] Moreover, boson superfluidity (at $T=0$) implies an absence of ODLRO in the superconductor, since the third sound mode indicates a massless **A**. Likewise, a normal (insulating) boson ground state, wherein density fluctuations (and **A**) acquire a mass, implies a thermodynamic superconducting state with ODLRO.

To explore the consequences of this isomorphism, consider first the pure case $\mu_i=\mu$. Since the boson compressibility $dn/d\mu$ is related to the superconductor susceptibility dM/dH, via $dn/d\mu\rightarrow 1-4\pi dM/dH$, the incompressible Mott insulating phase of the bosons (with $\langle n\rangle=$integer) is isomorphic to the Meissner (superconducting) phase in which $dM/dH=-1/4\pi$. Likewise, superfluidity in the boson system with $\langle n_i\rangle$ noninteger, corresponds to a bulk nonsuperconducting phase consisting of a liquid of flux lines. Such a phase was recently predicted to be observable just above H_{c1} in bulk Cu oxide superconductors. From a symmetry point of view this phase is indistinguishable from the normal phase with $H>H_{c2}$, exhibiting, for example, a nonzero dc resistance.

In the presence of a longer-range boson-boson interaction $\hat{H}_{0'}=\frac{1}{2}\sum_{i,j}u_{ij}n_in_j$ in (1a), a Mott insulating phase in which the Bose density exhibits a commensurate superlattice structure can be stabilized. This amounts to introducing a flux-flux interaction term in the superconductor and, hence, enhances the type-II character. Indeed, the boson superlattice Mott state is isomorphic to the Abrikosov vortex lattice phase in the superconductor. In light of this, an extremely interesting question arises: What is the analog phase of the supersolid state postulated to exist in the boson problem,[9] wherein ODLRO and diagonal (density) long-range order coexist? In this phase the classical superconductor would exhibit an Abrikosov flux lattice yet be normal with no ODLRO. A finite dc resistance would be observed (even in the presence of a periodic pinning potential acting on the flux lattice). In an array of superconducting cylinders, this phase could only occur provided (i) the mutual inductance between nearby plaquettes was strong, and (ii) the temperature was below the bulk transition temperature of the cylinders yet above the phase-coherence temperature of the array.

In the presence of disorder the Bose system can exhibit yet another possible ground state:[6] the Bose glass insulator wherein there is no ODLRO yet the compressibility is finite. In this phase the Bose density is effectively pinned by the random potential. The corresponding phase in the classical superconductor is a superconducting glass state, where the magnetic field penetrates the system yet superconductivity persists since the flux lines are frozen in and pinned by the disorder. It should be emphasized that in the classical analog system the disorder is perfectly correlated in the $\hat{\tau}$ direction. It has recently been argued that in the more realistic case of uncorrelated disorder, a superconducting (or vortex) glass phase nevertheless exists.[13]

When the bosons are at $T>0$, the time integration in (9) is of finite extent (0 to β), and the action apparently describes a superconducting film with finite thickness $L\sim\beta$. Extreme caution is necessary in this case, though, due to the β periodic boundary conditions on $(\nabla\times\mathbf{A})^\tau$ which are essential in describing correctly the Bose statistics (at $T\neq 0$), yet are exceedingly unnatural in terms of the flux tubes penetrating the classical superconductor. Indeed, with free boundary conditions appropriate to the superconducting film, the distinction (at $T\neq 0$) between the normal fluid and superfluid phases of the boson system would cease. In a real superconducting film, we expect only one flux-line fluid state is possible,[14] in contradiction to the speculations by Nelson.[8]

In addition to the mapping to a 3D classical superconductor, the interacting boson problem in the pure limit is isomorphic to (2+1)-dimensional lattice quantum electrodynamics. To see this we note that in the Coulomb gauge $\nabla_\perp\cdot\mathbf{A}^\perp=0$, S_0 in (9a) becomes

$$S_0=\sum_i\int d\tau\left[\frac{1}{2t}|\nabla_\perp A^\tau|^2+\frac{1}{2t}|\partial_\tau\mathbf{A}^\perp|^2+\frac{u}{2}|(\nabla_\perp\times\mathbf{A}^\perp)|^2\right], \tag{10}$$

which describes photons in 2+1 quantum electrodynamics with the identification of A^τ and $\mathbf{A}^\perp$ with the scalar and vector potentials, respectively.

Finally, we remark that when the steps used to obtain (9) are repeated for 1D lattice bosons the dual action is given by $S_0=S_1+S_2$, with

$$S_0=\sum_i\int d\tau\left[\frac{1}{2t}\dot{A}_i^2+\frac{u}{2}(\Delta_xA_i)^2-\mu_i\Delta_xA_i\right], \tag{11a}$$

$$S_1=J\sum_i\int d\tau\cos(A_i), \tag{11b}$$

with $A_i(\tau)$ a real scalar field. The action arrived at heuristically by Haldane for 1D bosons is precisely a long-wavelength coarse-grained version of (11).

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[1]See P. W. Anderson, in *Lectures on the Many Problem,* edited by E. Caianiello (Academic, New York, 1964), Vol. 2, p. 113.

[2]F. W. Wiegel, in *Path Integrals,* edited by G. J. Papadopoulos and J. T. Devreese (Plenum, New York, 1978), p. 419; L. Jacobs, J. V. José, and M. A. Novotny, Phys. Rev. Lett. **53**, 2177 (1984), and references therein.

[3]F. D. M. Haldane, Phys. Rev. Lett. **47**, 1840 (1981).

[4]T. Giamarchi and H. J. Schulz, Europhys. Lett. **3**, 1287 (1987).

[5]M. Ma, B. I. Halperin, and P. A. Lee, Phys. Rev. B **34**, 3136 (1986).

[6]M. P. A. Fisher, P. Weichman, G. Grinstein, and D. S. Fisher (unpublished).

[7]The analog phase in 2D films has been discussed by D. S. Fisher, Phys. Rev. B **22**, 1190 (1980).

[8]D. R. Nelson, Phys. Rev. Lett. **60**, 1973 (1988).

[9]K. Liu and M. E. Fisher, J. Low Temp. Phys. **10**, 655 (1973).

[10]C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. **47**, 1556 (1981).

[11]P. O. Thomas and M. Stone, Nucl. Phys. **B144**, 513 (1978).

[12]This equivalence was noted and employed in Ref. 8, independent of our work.

[13]M. P. A. Fisher, IBM Research Report No. RC 14143 (63371), 1988 (unpublished).

[14]It can be shown easily that in the continuum model studied in Ref. 8, when interactions between flux lines are ignored and free boundary conditions at 0 and β are taken, the partition function $Z(\beta)$ (which factorizes into a product of single flux-line contributions) is everywhere *analytic* in β. Thus, only *one* fluid phase exists, in contrast to noninteracting bosons which (when $d>2$) exhibit both a superfluid and normal-fluid phase.