The return of a hysteretic Josephson junction to the zero-voltage state: *I-V* characteristic and quantum retrapping

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We study the behavior of a hysteretic current-biased Josephson junction in the vicinity of its return to the zero-voltage states, with primary though not exclusive emphasis on the limit of weak damping ($\beta_J \ll 1$), and under the assumption that the zero-point and thermal energies are both small compared to $I_c\phi_0$ so that fluctuation effects are important only very close to the return point. We consider in detail the resistively shunted junction (RSJ) and quasiparticle-tunneling models, and also make predictions for more general models. Denoting the value of imposed current I at which return to the zero-voltage state would take place in the absence of fluctuations by I_r, we study in particular (a) the dc current-voltage characteristic in the running state for $I - I_r \ll I_r$, and (b) the first-passage-time statistics of the return to the zero-voltage state induced by both classical and quantum fluctuations. With regard to (b), we express our results in the form of a prediction of the width σ of the distribution of retrapping events as a function of imposed current; this prediction extends down to zero temperature and can be compared directly with the experimentally measured widths. Our two principal results are as follows: (a) In the running state, for $I \ll I_r$, the current-voltage characteristic should be given quite generally by the formula $(I - I_r)/I_r = [(AV_0/V) + B]\exp - V_0/V$, where A and B are constants specific to the model, and V_0 is a characteristic voltage which for the simplest models is given in the weak-damping limit by $V_0 = \omega_J\phi_0 + 0(\beta_J^2)$, with ω_J the junction plasma resonance frequency at zero current bias. (b) The square σ^2 of the width of the retrapping distribution plotted as a function of I/I_r is given to within logarithmic factors by $\sigma^2(T) = \text{const}\,\mu f(T)$, where $\mu \equiv \hbar\omega_J/I_c\phi_0$, the constant is of order 1, and $f(T)$ is a function which tends to 1 as $T \to 0$ and is proportional to T in the limit of high T; it is computed explicitly for the RSJ model. We also suggest an explanation (other than lead effects) of the "forbidden voltage regions" which appear to be a characteristic of many high-quality junctions. We discuss the application of our results to the determination of the parameters of Josephson junctions necessary for the investigation of quantum effects on the macroscopic level.

I. INTRODUCTION

A Josephson junction biased by a fixed external current is the prototype of a macroscopic multistable system. In such a system not merely the averaged effect of fluctuations but the statistics of individual events can be observed, and as a result it is of great interest from the point of view of classical and quantum statistical physics. Moreover, in recent years the current-biased Josephson junction and closely related systems have acquired additional significance in the context of fundamental tests of quantum mechanics (see, e.g., Ref. 1). For such applications it is often essential to be able to extract the parameters of the system from experiments conducted in the regime where quantum effects are unimportant. Most often (though not always—see, e.g., Ref. 2) the dc current-voltage characteristic has been used for this purpose, since it is the most routinely measured property of a junction. Unfortunately, there seems to be considerable disagreement in the literature about how to interpret this characteristic, and in particular what significance, if any, should be attached to its slope in various regions.

With the above considerations in mind, we present in this paper a detailed study of the behavior of a current-biased junction at and close to the return to the zero-voltage state, with primary though not exclusive attention to the weak-damping case. Specifically, we study (1) the dc current-voltage characteristic in the running state for bias currents I such that $I - I_r$ is small compared to the return current I_r, and (2) the distribution of values of I at which the system jumps back to the zero-voltage state. To concentrate on this rather small piece of the complete *I-V* characteristic might at first sight seem rather parochial, but we shall see that it (a) permits unambiguous determination of some of the junction

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parameters, and (b) is usually the only region of the characteristic, other than the well-studied region near $I = I_c$ on the outward branch of the characteristic, where classical and quantum fluctuation effects are significant. (We exclude in this paper the case where fluctuation effects are so strong as to effect the whole curve appreciably: see Sec. II.)

Needless to say, both questions (1) and (2) above have been studied in numerous papers in the literature. As regards (1), however, the vast majority of these papers have studied a much larger portion of the characteristic, and have therefore inevitably been mainly computational in nature; moreover, most of them have studied very specific models of the junction. As a result, it does not seem to be generally appreciated, either by theorists or by experimentalists trying to extract parameters from their data, that in the immediate neighborhood of the return to the zero-voltage state the characteristic should have a rather simple and largely model-independent behavior. In this paper we shall try to extract and present this behavior with a minimum of mathematical obfuscation.

As regards question (2), there is a considerable literature on the phenomenon of thermal-fluctuation induced return to the zero-voltage state ("retrapping"): we note in particular Refs. 3 and 4, and the various series of papers by Ben-Jacob, Cristiano, Mel'nikov, and their respective collaborators (see, respectively, Refs. 5–7). Again, however, these papers tend to be rather general in nature and it is sometimes rather difficult to see the wood for the trees: the results are often left in complicated integral forms from which it is nontrivial to extract quantities directly relevant to experiment, such as the width of the retrapping distribution. More importantly, the question of retrapping due to *quantum* fluctuations has been barely touched in the existing literature, and then[8] only in the high-temperature limit where it is a small correction; contrast the situation with regard to the escape from the zero-voltage state, where the corresponding phenomenon (macroscopic quantum tunneling or MQT) is by now the subject of a considerable experimental and theoretical literature. In this paper we shall fill this lacuna.

The plan of the paper is as follows: In Sec. II we introduce the general problem of a current-biased Josephson junction, discuss some commonly used models for the system, and define the dimensionless parameters which determine the qualitative features of the characteristic. In Sec. III we calculate the running-state characteristic for $I - I_r \ll I_r$ under the assumption that both classical and quantum fluctuation effects can be neglected; we present results for the standard resistively shunted junction (RSJ) model, for the quasiparticle-tunneling model, for a mixture of the two, and finally for a quite general model of the junction. In Sec. IV we consider the effects of noise in both the classical and quantum regimes: we show that the effect on the running-state characteristics calculated in Sec. III is negligible, and calculate the widths of the retrapping distributions as observed in a conventional ramping experiment. We present results for the weakly damped RSJ model which we believe are essentially asymptotically exact in the weak-fluctuation limit, and show that for more general weakly damped models the corrections to the RSJ results enter only through logarithmic factors and a change in the details of the temperature dependence, while for the more strongly damped case we make it plausible that the difference reduces to an overall factor which has little effect on the temperature dependence of the distribution widths. Section V is a conclusion in which we discuss the significance and applicability of our results. Appendixes A and B discuss repectively the deterministic motion across the barrier top for arbitrary shunting admittance and the status of the quantum Langevin approach used in Sec. IV; those who find the conclusions of either of these discussions obvious should omit it.

II. GENERAL CONSIDERATIONS, MODELS, AND DIMENSIONLESS PARAMETERS

A good general introduction to the subject of Josephson junctions may be found in Ref. 9. For present purposes we may define a Josephson junction as an electrical circuit element which can carry without dissipation a current of the form

$$I = I_c f(\phi), \quad \max|f(\phi)| = 1, \tag{1}$$

where the quantity ϕ, which normally represents the difference in phase of the wave function of the Cooper pairs in the bulk superconductors on the two sides of the junction, satisfies the Josephson relation

$$\frac{d\phi}{dt} = \frac{2eV(t)}{\hbar}, \tag{2}$$

$V(t)$ being the voltage developed across the junction. The function $f(\phi)$ is assumed periodic in ϕ with period 2π. The nonlinear element described by Eq. (1) may be shunted by an arbitrary complex admittance Y which may depend on ϕ, V, and frequency. We will consider such an element to be part of a circuit in series with a very large impedance Z_l, so that to the extent that a classical description of the leads is valid, the current fed into the region of interest is a constant controlled by the experimenter. The results obtained for the dc characteristic in this "current-biased" situation may of course be adapted to the more realistic case of finite lead impedance by a standard load-line construction[10]; they will be valid in their simple form to the extent that Z_l is much larger than the *differential* resistance of the junction region for all relevant frequencies (including zero), a condition we shall assume in this paper unless otherwise stated. Thus, the model considered here is represented by the generic circuit diagram shown in Fig. 1, with the constitutive equations (1) and (2).

For pedagogical clarity we shall first consider in this section the so-called RSJ model, and subsequently discuss the more general situation. The RSJ model is the special case of the above one obtained by setting (a) $f(\phi) = \sin\phi$ and (b) $Y = R^{-1} + i\omega C$. Thus, in this case the "black box" shunting the junction in Fig. 1 consists of a capacitance C and ohmic resistance R in parallel. In the original, and simplest, version of the RSJ model the "resistance" in question was taken to be the "normal-state" resistance R_N, that is, the resistance of the junction when the two bulk metals on the two sides are above their transition temperature(s) T_c; how-

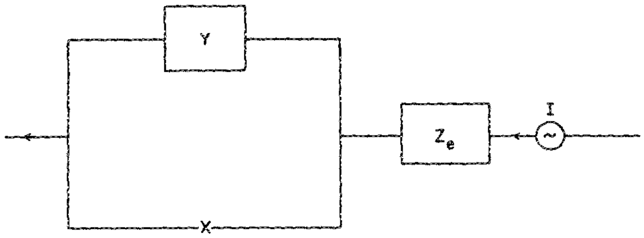

FIG. 1. Generic circuit diagram of a current-biased Josephson junction. The cross represents the junction.

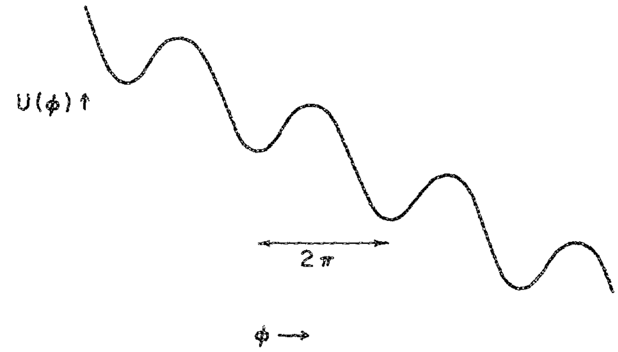

FIG. 2. "Washboard potential" of Eq. (5).

ever, it should be strongly emphasized that many real-life junctions may be described, *in the region of the I-V characteristic which we will study in this paper*, by an effective RSJ model with a resistance which at low temperatures may be several orders of magnitude higher than R_N. We will reserve the symbol R below exclusively for this so-called "subgap" resistance; if we need to refer to the normal-state resistance, we will always denote it explicitly by R_N.

The equation of motion of the RSJ model when biased with a constant current I is well known: see, e.g., Ref. 9, Chap. 6. For notational simplicity we choose for the moment a system of units in which $\hbar/2e \equiv \phi_0/2\pi \equiv 1$, where $\phi_0 \equiv h/2e$ is the superconducting flux quantum: ordinary units are then restored by the replacement $C \to (\phi_0/2\pi)^2 C$, $R \to (\phi_0/2\pi)^{-2} R$, $I_{(c)} \to (\phi_0/2\pi) I_{(c)}$, $V \to (\phi_0/2\pi) V$. In this system of units the equation of motion of the phase difference ϕ and of the voltage $V(t)$ read

$$C\ddot{\phi} + R^{-1}\dot{\phi} + I_c \sin\phi = I + I_n(t)\,, \tag{3}$$

$$V(t) = \dot{\phi}(t)\,, \tag{4}$$

where the quantity $I_n(t)$ is the (negative of the) fluctuating part of the current through the resistor: the mean value of $I_n(t)$ is zero, and its fluctuations are specified in accordance with the fluctuation-dissipation theorem (see Sec. IV). The equation of motion (3) represents, as is well known, a "particle" of mass C and with coordinate ϕ, subject to a friction coefficient R^{-1} and moving in the so-called "washboard potential" $U(\phi)$ given by

$$U(\phi) = -I_c \cos\phi - I\phi. \tag{5}$$

This potential is represented in Fig. 2. The problem is exactly isomorphic to that of the forced damped pendulum. Under the conditions which will be of interest in the present paper it is largely unnecessary (though see Sec. IV F) to address the vexed question[11] of whether or not states differing in ϕ value by $2n\pi$ should be identified; if an explicit decision on this point is required for notational purposes, we shall assume they are not.

The RSJ model contains, either explicitly or implicitly, a number of characteristic energies. For the moment let us ignore the noise term $I_n(t)$. Then there are two obvious energy scales: the barrier height $2I_c$ (or $I_c\phi_0/\pi$ in conventional units) and the energy dissipated per cycle when the system "rolls" in the washboard potential at bias currents small compared to I_c; for not too strong damping this is of order $(I_c/C)^{1/2}R^{-1}$. It is also convenient at this point to introduce $\Delta(T)$, the (order of magnitude of the) single-particle energy gap in the bulk superconductors on the two sides of this junction; although in the original RSJ model, which assumes that Eq. (3) is applicable for the *whole I-V* characteristic, the quantity Δ plays no role, it is important for the "effective RSJ model" discussed above, since we expect that for frequencies ω and voltages V comparable to or larger than $2\Delta/\hbar$ and Δ/e, respectively, the effective resistance switches over from the (often very high) "subgap" resistance R to something of the order of the normal-state resistance R_N.

There is also a characteristic energy associated with the noise current $I_n(t)$. In the classical limit this is obviously the thermal energy $k_B T$, while in the quantum case it is of the order of the zero-point energy $\hbar\omega_J$, where the Josephson plasma resonance frequency ω_J is defined by

$$\omega_J \equiv (I_c/C)^{1/2}\,. \tag{6}$$

We will be primarily interested below in the low-temperature regime where $k_B T \lesssim \hbar\omega_J$, and will therefore specify the regimes of interest in terms of the quantity $\hbar\omega_J$. For higher temperatures our conclusions for any particular regime will in general remain valid provided the inequalities defining the regime remain valid with $\hbar\omega_J$ replaced by $k_B T$. This point should be borne in mind when we refer, in subsequent sections, to "the limit $k_B T/\hbar\omega_J \to \infty$." Unless otherwise noted, we shall assume throughout this paper that both $\hbar\omega_J$ and $k_B T$ are small compared to Δ, and will work to lowest nonvanishing order in the ratios $\hbar\omega_J/\Delta$ and $k_B T/\Delta$. Moreover, we will assume except where otherwise stated that we are interested in the portion of the characteristic where the dc voltage V is appreciably less than Δ/e.

The general features of the current-voltage characteristic in this regime are determined, for $k_B T \lesssim \hbar\omega_J$, by three dimensionless ratios (of which only two are independent) of the energy scales other than Δ. It is convenient to choose them in conventional units as follows[12]:

$$\beta_J \equiv (\omega_J RC)^{-1} \equiv (\phi_0/2\pi I_c C R^2)^{1/2}\,, \tag{7a}$$

$$\mu \equiv \hbar\omega_J/I_c\phi_0 \equiv (2\pi\hbar^2/C I_c \phi_0^3)^{1/2}\,, \tag{7b}$$

$$\rho \equiv \beta_J^{-1}\mu \equiv R/R_Q, \quad R_Q \equiv h/4e^2 \cong 6.5\ \mathrm{k}\Omega\,. \tag{7c}$$

Roughly speaking, β_J is a measure of the relative energy loss per cycle, μ of the overall importance of quantum fluctuation effects relative to the classical dynamics of the junction, and ρ of the relative fluctuations in the dissipation per cycle: at zero temperature ρ is also prima facie a measure of the degree to which quantum phase coherence is preserved from one cycle to the next (see Sec. IV). In the high-temperature regime, as noted above, the appropriate changes in the definitions (7) are that β_J (which does not contain $\hbar$ explicitly) is unchanged, while μ and ρ are each multiplied by a factor $k_B T/\hbar\omega_J$; however, ρ is then no longer simply related to the quantum phase coherence (cf. Sec. IV F). Note that in general R may itself be a function of T.

We will assume throughout this paper that the parameter μ is much less than unity. This means that in general the dynamics of the junction is well described by classical deterministic dynamics, and the noise term $I_n(t)$ on the right-hand side of Eq. (3) has an appreciable effect only near the points, if any, corresponding to bifurcation of the characteristic. Since the constant multiplying $(CI_c)^{-1}$ in the definition of μ, Eq. (2b), is of order 10^{-23} SI units, it is very likely that most junctions examined to date have well satisfied the condition $\mu \ll 1$ in the low-temperature regime $kT \lesssim \hbar\omega_J$; for higher temperatures we must postulate this condition explicitly. For junctions for which the condition $\mu \ll 1$ is not fulfilled even at low temperatures a completely new approach is probably necessary: cf. Ref. 13. Regarding the parameter β_J, it is well known (see, e.g., Ref. 9, Chap. 6) that in the approximation of neglect of fluctuations this quantity determines the presence or absence of hysteresis in the characteristic; if $\beta_J > 1$ there is no hysteresis, while for $\beta_J \to 0$ the "return" current (see below) tends to zero. In this paper we shall always assume that β_J is less than 1, so that hysteresis occurs, and will mostly though not exclusively be interested in the case $\beta_J \ll 1$; we will in any case assume that β_J is not too close to 1, so that the critical and return currents are well separated. Note that these conditions still allow the third dimensionless parameter ρ to be either large or small compared to unity.

The general features of the "deterministic" current-voltage characteristic for $\beta_J < 1$, i.e., that which is obtained from Eq. (3) by neglecting the noise term $I_n(t)$, are well known and are discussed, e.g., in Chap. 6 of Ref. 9. In Fig. 3 we show a typical characteristic for the case $\beta_J \ll 1$, with the solid line representing the deterministic formula. As is well known, the point A represents the point at which the minima of the potential (5) become inflection points: at this point the system is tipped out of the well and starts rolling down the washboard potential with a considerable velocity. Generally speaking, the voltage at point A', which corresponds to the steady-state motion for $I = I_c$, is of order $I_c R_N$, where R_N is the normal-state resistance; this is because for a standard tunnel-oxide junction, the critical current is of order $\Delta/R_N e$ and hence this voltage is of order Δ/e, i.e., large enough that the effective resistance for the rolling motion is of the order of the normal-state value (cf. below). Quite generally, the behavior in the region of point A' (and *a fortiori* at higher voltages) will be determined by an effective resistance which may be quite different from (and in general considerably smaller than) the "subgap" value. Thus, the "outward" ($A \to A' \to C$) portion of the deterministic characteristic gives information only on the critical current I_c and not on the other parameters of interest (the capacitance C and subgap resistance R). On the other hand, the point B, at which the applied current is by definition equal to the "return" current I_r, corresponds to the point at which the energy gained from the applied current over a cycle is just equal to that dissipated in the resistance; provided that the condition $\omega_J \ll \Delta$ holds as we are assuming, the behavior in the neighborhood of this point [on the "return" (lower) curve] is determined entirely by the low-frequency parameters, and in particular, as we shall see, gives information not only on the subgap resistance R but also directly on the capacitance C. Thus we shall confine ourselves in this paper to the neighborhood of the return to the zero-voltage state, i.e., to the portion of the characteristic between points B and D in Fig. 3.

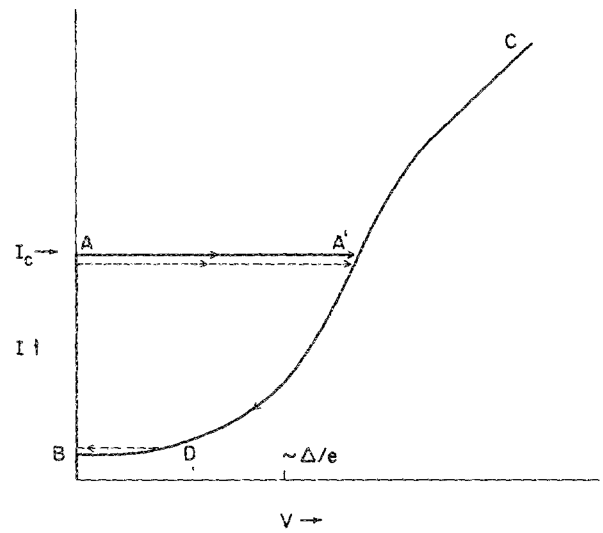

FIG. 3. Typical I-V characteristic of a strongly hysteretic junction ($\beta_J \ll 1$) in the RSJ model.

We have also indicated in Fig. 3, by dashed lines, the effect of fluctuations, assuming that the fluctuation parameter μ is small compared to unity. Strictly speaking, even in the presence of fluctuations we should expect that the system eventually attains a steady state for any fixed value of the external current and therefore that the current-voltage characteristic is deterministic. However, in practice the relevant experiments are always conducted by ramping the external current over a finite time scale (typically anything between a microsecond and a few minutes) and for $\mu \to 0$ the time taken to reach the steady state may be much longer than this. For example, suppose that we are ramping the current upwards from zero. At some point somewhat below A in Fig. 3 fluctuations may lead to the escape from the (still metastable) minimum of the washboard potential, and the system will jump across into the "running" (finite-voltage) state. Although thereafter there is strictly speaking some probability that a fluctuation will restore the system to the well, i.e., to the zero-voltage state, in practice for $\mu \to 0$ this probability is so small (cf. Sec. IV) that the ramping would have brought the current I far above I_c before it would occur. (Even if the experiment were performed by holding the current stationary at some value just below I_c, the "return" events are so rare that the time-averaged dc voltage would differ only negligibly from that associated with point A'). By a parallel argument, once the system is "retrapped" into the zero-voltage state at B, it will for all practical purposes stay there. Thus, as emphasized by Cristiano and Silvestrini,[6] in this limit the quantity of interest is the statistics of the "first passage time," that is, the time at which the system, started (e.g.) in the running state at $t = 0$, is observed to return to the zero-voltage state. We will calculate these statistics for the case of constant current: the distribution of values of the current at which the system is observed to return in an experiment conducted at finite ramping rate is then obtained by a standard convolution procedure (see Sec. IV C).

So far, we have assumed that all of the features of the I-V characteristic which we wish to study are well described by an effective RSJ model. Let us now consider possible more general models. The most obvious generalization of the RSJ model consists in choosing a potential more general than $I_c \cos\phi$ (but still periodic in ϕ) for the first term in $U(\phi)$ [Eq. (5)], and also allowing the capacitance C and/or the resistance R to be a function of ϕ. As we shall see below, such a generalization is probably sufficient to describe most junctions of the standard tunnel-oxide type except at very low temperatures ($kT \lesssim \hbar\omega_J$). It is clear that those results of the simple RSJ model which involve the motion over a whole cycle will now be changed; for example, the standard relation $I_r/I_c = (4/\pi)\beta_J$ for $\beta_J \to 0$ (see Sec. III) will in general no longer hold. However, many of the features of the results to be derived in the next two sections depend only on the motion near the top of the barrier, and will therefore be unchanged provided we replace I_c by an effective value $\tilde{I}_c$ defined by

$$\tilde{I}_c \equiv -\left(\frac{\partial^2 U}{\partial \phi^2}\right)_{\phi=\phi_{\max}}, \tag{8}$$

where $\phi_{\max}$ corresponds to the position of the barrier top, and define C and R as the values appropriate to this value of ϕ. This point is explored further in Sec. III E.

A much less trivial generalization, at least at first sight, is to the case of a tunnel oxide junction described by the standard Bardeen–Josephson tunneling Hamiltonian. We shall refer to this model for brevity as the "quasiparticle-tunneling" (QPT) model.[14] Its deterministic dynamics has been studied by Werthamer[15] and many others (see, e.g., Ref. 9, Chap. 2), and recently a series of papers[16] by Ambegaokar and co-workers has extended the formalism so as to permit the discussion of both classical and quantum fluctuation effects. They derive for the junction phase ϕ a quantum Langevin equation which in our reduced units takes the form

$$C\ddot{\phi}(t) - 4\int_{-\infty}^{t} dt'\left[\alpha_I(t-t')\sin\left(\frac{\phi(t)-\phi(t')}{2}\right) - \beta_I(t-t')\sin\left(\frac{\phi(t)+\phi(t')}{2}\right)\right] = I + I_n(t). \tag{9}$$

Here $I_n(t)$ is the noise term, which will be discussed in Sec. IV, and the quantities $\alpha_I(t)$ and $\beta_I(t)$ are the imaginary parts, respectively, of the functions $\alpha(t)$ and $\beta(t)$ defined by

$$\alpha(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{1-e^{-\beta\hbar\omega}} I_{\rm qp}(\omega), \tag{10a}$$

$$\beta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{1-e^{-\beta\hbar\omega}} I_c(\omega), \tag{10b}$$

where the quantities $I_{\rm qp}(\omega)$ and $I_c(\omega)$ are the in-phase parts, respectively, of the quasiparticle and pair contributions to the total current obtained at a dc voltage $\hbar\omega/e$. The general form of these functions is given for the simple QPT model (in which quasiparticle damping effects, etc., are ignored) in, e.g., Ref. 16, Eq. (41). At zero temperature (and in fact to a good approximation for $kT \ll \hbar\omega$) they are given respectively by the expressions ($x \equiv \hbar\omega/2\Delta$)

$$I_{\rm qp}(\omega) = \frac{2\omega}{R_N}\frac{|x|}{(|x|+1)}\left[1 - \frac{(|x|+1)^2}{x^2}K\left(\frac{|x|-1}{|x|+1}\right) + \left(\frac{|x|+1}{x}\right)^2 E\left(\frac{|x|-1}{|x|+1}\right)\right], \quad x \geqslant 1, \tag{11a}$$

$$I_c(\omega) = -\frac{4\,{\rm sgn}(\omega)}{\hbar(|x|+1)}\frac{\Delta}{R_N}K\left(\frac{|x|-1}{|x|+1}\right), \tag{11b}$$

$$I_{\rm qp}(\omega) = I_c(\omega) = 0, \quad x < 1. \tag{12}$$

Here we have denoted by x the ratio $\hbar\omega/2\Delta$, and by $K(x)$ and $E(x)$ the elliptic integrals of the first and second kind, respectively (a notation we shall follow throughout this paper). In the limit $\hbar\omega \ll kT \ll \Delta$ we have instead (for equal gaps)

$$I_{\rm qp}(\omega) = I_c(\omega) = \frac{2\omega}{R_N}\left[\ln\left(\frac{2kT}{\hbar|\omega|}\right) - \gamma\right]\frac{\Delta}{kT}\exp -\frac{\Delta}{kT}, \tag{13}$$

where γ is the Euler constant ($\cong 0.577...$). In the case of unequal gaps we get a similar formula which does not contain a logarithmic dependence on ω.

It is well known (see, e.g., Ref. 9, Sec. VI C) that *for any phenomenon in which only the low-frequency ($\omega \ll kT/\hbar, \Delta$) behavior is important* we can make the so-called adiabatic approximation and thereby reduce the deterministic equation of motion of the QPT model, Eq. (9), to that of a generalized RSJ model of the type discussed above, with ϕ-dependent resistance and capacitance. The relevant values of the parameters are[17]

$$\begin{aligned} I_c &\equiv -\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{I_c(\omega)}{\omega}, \\ R^{-1}(\phi) &\equiv R_0^{-1}(1+\epsilon\cos\phi), \\ C(\phi) &= C. \end{aligned} \tag{14}$$

The general form of the quantities R_0 and ϵ is complicated (see Ref. 9, Chap. 2), and in the case of equal gaps (only) R_0^{-1} actually depends logarithmically on ω [cf. Eq. (13)], but we can make the general statement that for $kT \ll \Delta$ the quantity R_0^{-1}, which is a measure of the number of excited quasiparticles, falls off as $e^{-\Delta k_B T}$, while ϵ tends to $+1$ (at least in theory!) in this limit. Note therefore that in the limit $T \to 0$ the quantity $R^{-1}(\phi)$ tends to zero for all ϕ. Thus, for any phenomenon for which dissipation is a zero-order effect it is inconsistent to use the "effective RSJ model" defined by (14), even though all characteristic frequencies involved may prima facie be small compared to Δ; as we shall see in Sec. III B, it is necessary to take into account higher harmonics of the motion, even though their amplitudes may be exponentially small. Despite this, it turns out, perhaps surprisingly at first sight, that even in the limit $T \to 0$ there are a number of features of the I-V characteristic which, since the effects of dissipation are a higher-order correction, can be adequately accounted for by the effective RSJ model.

Finally, it is possible in principle to discuss the case where the admittance shunting the junction is completely arbitrary, and in particular may have a substantial frequency dependence in the region of (the effective) ω_J (such a dependence could, for example, be caused by interaction with geometrical resonances of the junction). Some of the results derived below can in fact be generalized rather simply to this situation, and we shall indicate the generalizations at the appropriate points below.

It is sometimes convenient, for the purposes of order-of-magnitude estimates, etc., to define the dimensionless parameters β_J, μ, ρ [Eqs. (7)] also for general models. Whenever this is necessary, we shall use the definitions

$$\beta_J \equiv \frac{\pi}{4}\frac{I_r}{I_c}, \quad \mu \equiv \frac{\hbar\omega_J}{I_c\phi_0} \equiv \left(\frac{2\pi\hbar^2}{CI_c\phi_0^3}\right)^{1/2}, \quad \rho \equiv \beta_J^{-1}\mu, \tag{15}$$

where C and I_c are the values of these parameters at some convenient value of ϕ, e.g., $\phi = 0$. Thus, to an order of magnitude at least, β_J is still the relative energy loss per cycle and μ the ratio of the zero-point energy to the barrier height. At high temperature ($kT \gg \hbar\omega_J$ but $kT \ll I_c\phi_0, \Delta$, etc.) we again replace $\hbar\omega_J$ by kT in the definitions of μ and ρ.

III. THE DETERMINISTIC RETURN CHARACTERISTIC

A. Introduction

In this section we shall study the return characteristic close to the return to the zero-voltage state (more precisely, for $V \ll \Delta/e$ and $I - I_r \ll I_r$) in the approximation in which all effects of thermal and quantum fluctuations, and also any other quantum effects such as overbarrier reflection, are neglected. Thus we treat the quantity ϕ as a c number defined in the range $(-\infty, \infty)$, and moreover neglect the term $I_n(t)$ on the right-hand side of the equation of motion. The resulting problem is purely classical and deterministic. We shall first discuss two important special cases, that of the RSJ model with small β_J and that of the QPT model, then give numerically computed results for the latter and for the "mixed" model, and finally give the asymptotic form of the characteristic for the general case.

The general approach used throughout this section is the following: We consider the kinetic energy E with which the system reaches the top of the barrier of the washboard potential, and the energy W dissipated per cycle in Joule heating. In the approximation of this section, and in the steady state, both these quantities are independent of the cycle considered and are unique functions of the external current I. In fact the relation between W and I in the steady state is trivial: in our units we have simply

$$W = 2\pi I. \tag{16}$$

The dc voltage is, according to Eq. (2) rewritten in the reduced units, simply equal to the average rate of change of the phase, and hence is given in terms of the period T_0 of motion in the potential by

$$V = 2\pi T_0^{-1} \tag{17}$$

(or in conventional units as $V = \phi_0/T_0$). Our procedure, therefore consists in finding W and T_0 a functions of E, then inverting the latter relation and using (16) and (17) to find I as a function of V. This is similar to the method of Schlup.[18] An incidental advantage of doing the problem in this way is that the formula obtained for $W(E)$ can be used directly in the discussion of fluctuations in Sec. IV.

B. The RSJ model in the weak-damping limit

We consider Eq. (3), neglecting the noise term on the right-hand side,

$$C\ddot{\phi} + \dot{\phi}/R + I_c \sin\phi = I, \tag{18}$$

and specialize to the limit $\beta_J \ll 1$. It is easy to see (cf. Sec. II) that under this condition the energy dissipated per cycle is a small fraction of the characteristic energy $2I_c$, and consequently, from Eq. (16), that the return current I_r is small compared to I_c. We therefore write down the solution of (18) corresponding to the limit $R = \infty, I = 0$ (zero dissipation, zero bias) as a function of E: call this $\phi_u(t{:}E)$. Then we simply calculate the period T_0 corresponding to $\phi_u(t{:}E)$, and also substitute it in the expression for the energy dissipation W,

$$\int_0^{T_0} [\dot{\phi}^2(t)/R]\,dt.$$

This gives us $T_0(E)$ and $W(E)$ in the limit of zero dissipation and zero bias, and the $I(V)$ relation calculated from them should be valid to lowest order in β_J.

The procedure is straightforward to implement. It is convenient to introduce the dimensionless parameter v defined by

$$v^2 \equiv E/4E_0, \quad E_0 \equiv \tfrac{1}{2}C\omega_J^2 \;(= I_c\phi_0/4\pi). \tag{19}$$

Then in the limit $R \to \infty$, $I \to 0$ the first integral of Eq. (18) is

$$\dot{\phi}_u^2(t{:}v) = 4\omega_J^2\{\cos^2[\phi_u(t)/2] + v^2\}. \tag{20}$$

We note for future reference that the solution of this equation in the limit $v \to 0$ $(E \to 0)$ is

$$\phi_u(t{:}0) = 2\sin^{-1}(\tanh\omega_J t). \tag{21}$$

For finite v the quantities $W(v)$ and $T_0(v)$ can actually be obtained without an explicit solution for $\phi_u(t{:}v)$. In fact, using $K(x)$ and $E(x)$ as above to indicate the elliptic integrals of the first and second kind, respectively, we find

$$\begin{aligned} W(v) &= \frac{8\omega_J}{R}(1+v^2)^{1/2}E\left(\frac{1}{\sqrt{1+v^2}}\right) \\ &= \frac{8\omega_J}{R}\left(1 + \frac{v^2}{2}\ln\frac{4}{v} + \frac{v^2}{4} + O(v^4\ln v)\right), \end{aligned} \tag{22}$$

$$\begin{aligned} T_0(v) &= 2\omega_J^{-1}(1+v^2)^{-1/2}K\left(\frac{1}{\sqrt{1+v^2}}\right) \\ &= 2\omega_J^{-1}\left(\ln\frac{4}{v} + O(v^2\ln v)\right). \end{aligned} \tag{23}$$

Using (16) and (17) and restoring conventional units, we find for the return current I_r (i.e., the value of I for which $v^2 \to 0$) the well-known result

$$I_r = \frac{2}{\pi^2}\phi_0\omega_J R^{-1} \equiv \frac{4}{\pi}\beta_J I_c . \quad (24)$$

Furthermore, expanding (22) and (23) as indicated, we obtain for the behavior of the return current in the neighborhood of I_r the formula

$$\frac{I - I_r}{I_r} = 4\left(1 + \frac{V_0}{V}\right)\exp - \frac{V_0}{V} + O(e^{-2V_0/V}),$$
$$V_0 \equiv \omega_J \phi_0 . \quad (25)$$

Rather to our surprise, we have so far failed to locate a clear and explicit statement of the simple result (25) in the literature; and while it is certainly implicit in existing work,[19] it is not in our experience common knowledge among experimentalists wishing to analyze their data in this region. We shall see in Sec. V that the exp $- V_0/V$ behavior is actually much more general than the RSJ model. Formula (25) is valid to zero order in the dissipation, or equivalently in β_J; we will see later that corrections to it are of order β_J^2.

C. The QPT model

We now turn to the model defined by Eq. (9) (again neglecting the noise term on the right-hand side). We will assume that the dissipation per cycle is a small fraction of the potential scale $I_c\phi_0/\pi$, i.e., $\beta_J \ll 1$; this is almost always true for junctions well described by this model at low temperatures. In the limit $kT \gg \hbar\omega_J$ (actually, as it turns out, for $kT > \hbar\omega_J/2\pi$), the dissipation is predominantly due to the finite dc conductance; in this case it is clear from the considerations of Sec. II that apart from possible logarithmic corrections (which disappear in the unequal-gap case) the situation is very similar to that in the RSJ model discussed in the last subsection [cf. Eq. (33)].[20] We shall therefore concentrate here on the case $kT \ll \hbar\omega_J/2\pi$.

We proceed much as in the last subsection, with some special cautions. Multiplying Eq. (9) by $\dot\phi(t)$ and integrating over t, we obtain after some integration by parts the equation of energy balance:

$$\tfrac{1}{2}C\dot\phi^2(t) - I\phi(t) - I_c\cos\phi(t) + 8\int^t dt'\int^{t'} dt''\Big(A_+(t'-t'')\frac{d}{dt'}\{\sin[\phi(t')/2]\}\frac{d}{dt''}\{\sin[\phi(t'')/2]\}$$
$$+ A_-(t'-t'')\frac{d}{dt'}\{\cos[\phi(t')/2]\}\frac{d}{dt''}\cos[\phi(t'')/2]\Big) = 0 , \quad (26)$$

where we defined

$$A_\pm(t) \equiv \int^t dt'[\alpha_I(t') \pm \beta_I(t')]$$
$$\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{\cos\omega t}{1 - e^{-\beta\hbar\omega}}\left(\frac{I_{qp}(\omega) \pm I_c(\omega)}{\omega}\right), \quad (27)$$

and where I_c is defined in terms of the parameters of the model by Eq. (14a). Equation (26) is quite generally valid (not just in the limit $\beta_J \to 0$): it is clear that the last term represents the energy dissipated in Joule heating (cf. Ref. 13).

We now specialize to the case of steady-state motion. Let us define T_0 as the *average* period of $\phi(t)$, i.e., by the formula

$$T_0^{-1} \equiv \lim_{\tau\to\infty}\left(\frac{\phi(\tau) - \phi(0)}{2\pi\tau}\right), \quad (28)$$

so that the dc voltage is given as usual by ϕ_0/T_0. Because of the characteristic occurrence of half-angles in Eq. (26), the motion of $\phi(t)$ is actually periodic with period $2T_0$ rather than T_0. However, the departure from strict periodicity in T_0 is a consequence entirely of the dissipative term in Eq. (26), and hence is of relative order β_J; it is therefore consistent to neglect it in what follows.[21] In this approximation the Fourier transforms of the quantities $\sin[\phi(t)/2]$, $\cos[\phi(t)/2]$ (and hence of their derivatives) vanish for even-n values of $\omega_n \equiv 2\pi n/2T_0$, and for odd n are just twice the integral over a single period T_0. We therefore define

$$S_1(\omega_n) \equiv \int_{-T_0/2}^{T_0/2} dt\, e^{i\omega_n t}\frac{d}{dt}\{\sin[\phi(t)/2]\}, \quad (29a)$$

$$S_2(\omega_n) \equiv \int_{-T_0/2}^{T_0/2} dt\, e^{i\omega_n t}\frac{d}{dt}\{\cos[\phi(t)/2]\},$$
$$\omega_n \equiv [2\pi(n + 1/2)]/T_0 . \quad (29b)$$

The dissipation W per cycle is therefore given by the expression

$$W = (2T_0^{-1})\sum_{\omega_n}\left[\left(\frac{|S_1(\omega_n)|^2 + |S_2(\omega_n)|^2}{\omega_n}\right)I_{qp}(\omega_n) + \left(\frac{|S_1(\omega_n)|^2 - |S_2(\omega_n)|^2}{\omega_n}\right)I_c(\omega_n)\right]. \quad (30)$$

This formula is valid quite generally for weak dissipation. We note at once one general feature: If we substitute the zero-temperature forms of $I_{qp}(\omega)$ and $I_c(\omega)$ [Eqs. (11) in (30)], then because of the finite threshold 2Δ for dissipation the quantity W has a finite jump whenever $\omega_n = 2\Delta/\hbar$, i.e., whenever $eV = 2\Delta/(2n+1)$. This leads to the occurrence of the "odd-subharmonic" features of the I-V characteristic which have been extensively discussed in the literature (see, e.g., Ref. 9, Sec. VI), and which will be taken up again in the next subsection. For present purposes, however, it is sufficient to note that although the amplitude of these jumps falls off only as n^{-1}, their frequency on the voltage axis increases as n^2, and hence in the limit $V \to 0$ we may plausibly treat ω as a continuous variable and replace the sum in (30) by an integral.

Bearing this in mind, we now proceed exactly as in the last subsection: that is, we compute the trajectory for zero dissipation and zero bias current, as a function of E, and substitute this trajectory to find the dependent of T_0 and W on E. To find the return current we need the trajectory for $E=0$, which is given by Eq. (21): substituting this into (29), we find that for $E=0$ we have

$$S_1(\omega)=\frac{\pi\omega}{\omega_J \sinh(\pi\omega/2\omega_J)},$$

$$S_2(\omega)=\frac{i\pi\omega}{\omega_J \cosh(\pi\omega/2\omega_J)}. \quad (31)$$

We substitute these forms into (30), convert the sum into an integral (see above), substitute the appropriate forms of $I_{\mathrm{qp,c}}(\omega)$ [Eqs. (11) or (13)], and use Eq. (16). After some algebra we obtain in this way in the low-temperature limit the following expression for the return current I_r:

$$I_r=\frac{2}{\pi}\frac{\phi_0\omega_J}{R_N}\left(\frac{2\Delta}{\hbar\omega_J}\right)^2 \exp-\left(\frac{2\pi\Delta}{\hbar\omega_J}\right) \quad (kT\ll\hbar\omega_J\ll\Delta), \quad (32)$$

with corrections of relative order $\hbar\omega_J/\Delta$. In the high-temperature limit we obtain instead of (32) the result

$$I_r=\frac{8}{3\pi^2}\frac{\phi_0\omega_J}{R_N}\left[\ln\left(\frac{\pi kT}{\hbar\omega_J}\right)-a\right]\frac{\Delta}{kT}\exp-\frac{\Delta}{kT}$$
$$(\hbar\omega_J\ll kT\ll\Delta), \quad (33)$$

with corrections of relative order $\hbar\omega_J/kT$, kT/Δ, where the constant a is defined by

$$a\equiv\gamma+\frac{6}{\pi^2}\int_0^\infty dx\left(\frac{x^2\ln x}{\sinh^2 x}\right)=0.237$$
$$(\gamma=\text{Euler's constant}). \quad (34)$$

Apart from the difference between the numerical factors a and $[\gamma+\ln(\pi/2)]$, this is just what we should have got from the effective RSJ model of Eqs. (14) if we had set $\epsilon=+1$ and taken R_0 from (13) by the prescription $I_{\mathrm{qp}}(\omega)\equiv 2\omega R_0^{-1}$, with $\omega=\omega_J$ in the logarithmic term.

To find the limiting behavior for $I\to I_r$ we need to expand the trajectory $\phi_u(t{:}v)$ to lowest order in v^2. Although the general expression is an elliptic integral, it is well approximated, except in the regions $|\phi\pm\pi|\lesssim v$, by the expression

$$\sin[\phi(t)/2]=\tanh\tilde{\omega}_J t, \quad \tilde{\omega}_J\equiv\omega_J(1+v^2/2). \quad (35)$$

Now, in general we get a singular contribution to the dissipation from the region of ϕ very close to π (see Sec. II E). However, in the case of the QPT model at zero temperature this contribution vanishes, since in this region the system is moving very slowly and, according to Sec. II, will be described by an effective RSJ model with infinite resistance (and hence no dissipation). Consequently, to lowest order in v^2 the dissipation at finite v^2 is obtained from that at $v^2=0$ by the simple replacement $\omega_J\to\tilde{\omega}_J$. Using (16) we therefore find that in this limit ($kT\ll\hbar\omega_J\ll\Delta$) the I-V curve is given by

$$\frac{I-I_r}{I_r}=8\left(\frac{2\pi\Delta}{\hbar\omega_J}\right)\exp-\frac{V_0}{V}, \quad V_0\equiv\omega_J\phi_0, \quad (36)$$

with corrections of relative order $\hbar\omega_J/2\Delta$. We note that the leading (exponential) dependence is identical to that in the RSJ model but the prefactor is different.

We will not discuss here the asymptotic behavior of the curve in the limit $\hbar\omega_J\ll kT\ll\Delta$; it is clearly a special case of the general situation discussed in Sec. III E.

D. Numerical results

In this subsection we shall briefly present the numerical results we have obtained for the I-V characteristic at zero temperature, starting from Eq. (30). These results were obtained by substituting the *undamped* solution of (26) in (29) to obtain the quantities $S_{1,2}(\omega_n)$, and should therefore be a good approximation provided the relative energy loss per cycle is small. The results for the pure quasiparticle-tunneling model are shown in Fig. 4 for two values of the parameter $g\equiv 2\Delta/\hbar\omega_J$; they are qualitatively similar to those obtained in the existing literature, see, e.g., Ref. 22. In Fig. 5 we show the dependence of the height of the main peak (at $V=2\Delta/3e$) on g. We emphasize two points: (1) despite the spectacular appearance of the subharmonic peaks at $eV=2\Delta/(2n+1)$, the height of these peaks is, for $g\gtrsim 2$, several orders of magnitude less than the above-gap current; (2) the slope of the characteristic is *negative* in the regions between the peaks. The latter circumstance will be of vital importance when we consider the effect of fluctuations in Sec. IV.

It is interesting to consider also the characteristics which would appear in a "mixed" model, i.e., one in which the energy dissipation is described by the last term in (26) plus an additional RSJ-type term of the form

$$\int^t \dot{\phi}^2(t')/R_{\mathrm{RSJ}}\,dt'.$$

If the ratio $\delta\equiv R_{\mathrm{RSJ}}/R_N$ is large (where R_N is the normal-state resistance appropriate to the quasiparticle-tunneling model), we see that the extra term will have little effect on the above-gap characteristics but may modify the low-voltage behavior appreciably. In Fig. 6 we plot the characteristics for several (large) values of δ. We see that the main effect of the added term is to decrease considerably the size of the negative-slope regions; presumably, for given values of the parameters $\hbar\omega_J/\Delta$, etc., and small enough δ they will be

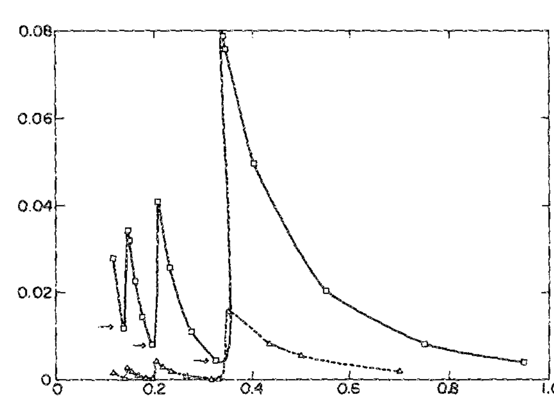

FIG. 4. Computed below-gap I-V characteristic for the QPT model at $T=0$. I and V are in units of $2\Delta/eR_N$ and $2\Delta/e$, respectively; the squares correspond to $g\equiv 2\Delta/\hbar\omega_J=2$ and the triangles to $g=3$. The curves are guides to the eye. Note the $1/n$ decrease in the height of successive peaks.

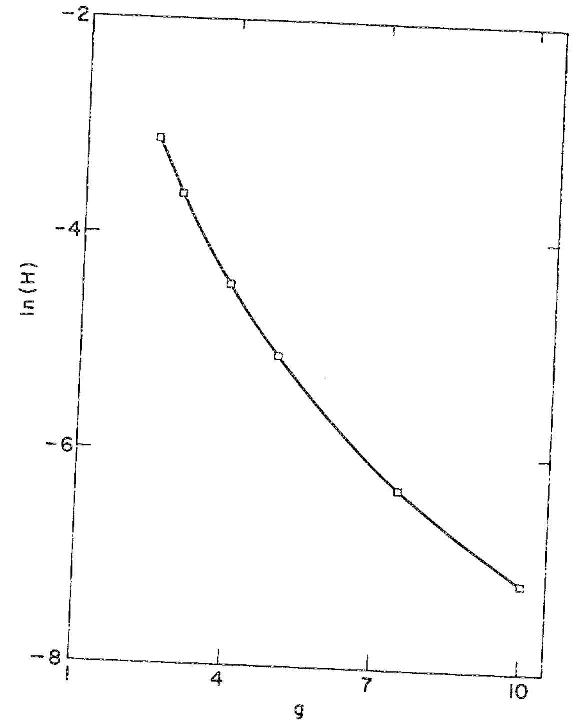

FIG. 5. Plot of ln H vs g for the pure QPT model at $T=0$, where H is the height of the first peak (at $V=2\Delta/3e$) in units of $2\Delta/eR_N$.

removed completely, but we have not investigated this point in detail.

E. Generalization to an arbitrary junction admittance

In this subsection we shall give a quite general discussion of the asymptotic form of the return I-V characteristic,

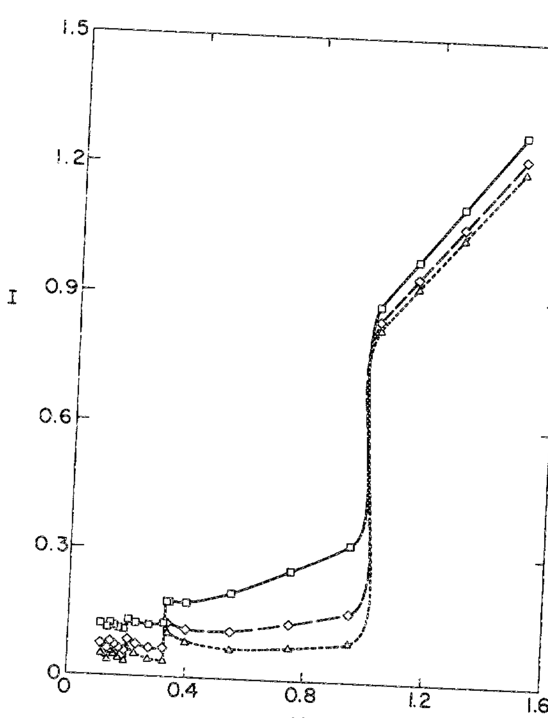

FIG. 6. Below-gap I-V characteristics for the "mixed" model, with $g=2$ and $\delta\equiv R_{\rm RSJ}/R_N$ equal to 2, 5, 10 for the squares, diamonds, and triangles, respectively. Units as in Fig. 4.

which will recover the results of Secs. III B and III C as special cases. In particular we shall show that the asymptotic dependence $(I-I_r)/I_r \propto \exp - V_0/V$ is quite universal, although in the general case V_0 need not be equal to $\omega_J\phi_0$. We do *not* restrict ourselves in this subsection to the case $\beta_J \ll 1$, or more generally to weak damping; the only assumption is that the damping is weak enough to make the characteristic appreciably hysteretic.

The discussion is based on the following simple observation: At the critical "return" current the system, when started from rest at the top of one barrier, reaches the top of the next with exactly zero kinetic energy, and takes an infinite time to do so. For values of I just slightly greater than I_r, it will take a time very long compared to any characteristic frequency of the problem (ω_J^{-1}, $\hbar/\Delta$, RC ...) to pass over the top of the barrier, and the period of motion (hence the voltage) will be overwhelmingly determined by this time. Moreover, near the top of the barrier the dynamics of the system, however complicated, can be described by an "effective" RSJ model with a (possibly ϕ-dependent) capacitance and resistance (cf. Sec. II and below). Thus, all the singular behavior of the asymptotic return characteristic will be a function only of the effective parameters in the region of the barrier top; the rest of the motion will enter only through nonsingular factors, which we shall not attempt to calculate in the general case.

Let us first consider the case of an explicit generalized RSJ model. We assume that when the current I is equal to the return value I_r the barrier top occurs at the point $\phi=\phi_{\rm max}$, and define as in Sec. II, effective parameters in the neighborhood of this point, putting a tilde on them to emphasize that they may in general be different from the parameters appropriate to (say) small oscillations. Explicitly, we define

$$\tilde{C}\equiv C(\phi_{\rm max}),\quad \tilde{R}\equiv R(\phi_{\rm max}),\quad \tilde{I}_c\equiv\left(\frac{\partial^2 U}{\partial\phi^2}\right)_{\phi=\phi_{\rm max}}. \quad (37)$$

Note that even for the simple RSJ model of Eq. (3), $\tilde{I}_c$ may differ from I_c because in a finite bias current the curvature of the maximum is decreased; clearly as $I_r\to I_c$, $(\beta_J\to 1)$, $\tilde{I}_c$ tends to zero. In a similar vein we define

$$\tilde{\omega}_J\equiv(\tilde{I}_c/\tilde{C})^{1/2},\quad \tilde{\gamma}\equiv\tfrac{1}{2}(\tilde{R}\tilde{C})^{-1},$$
$$\tilde{\beta}_J\equiv 2\tilde{\gamma}/\tilde{\omega}_J,\quad \tilde{E}_0=\tfrac{1}{2}\tilde{C}\tilde{\omega}_J^2\,. \quad (38)$$

Note that as β_J tends to 1, $\tilde{\beta}_J$ tends to infinity. For β_J not too close to 1, however, the order of magnitude of the tilde'd quantities in (38) is of the order of the original ones (ω_J, β_J, etc.).

Consider the steady state under external current bias I close to I_r, and let E, as above, be the kinetic energy with which the system reaches the top of the barrier. It is clear that the limit $E\to 0$ corresponds to $I\to I_r$; the value of I_r is determined, as in the specific examples considered earlier, by the condition that the energy W dissipated over a cycle for $E=0$ is equal to the energy $2\pi I_r$ gained from the external current. The resulting equation clearly depends in detail on the form of $C(\phi)$ and $R(\phi)$, as well as on β_J, and it is therefore not possible to give an analytic formula for I_r in the general case. However, we can remark that provided the functions $C(\phi)$ and $R(\phi)$ are not pathological, I_r will be of order $\beta_J I_c$. Now let I be just slightly greater than I_r. Denote by $\phi_{\max}$, as above, the position of the barrier top, and consider a small region $\phi_{\max} - \Delta\phi < \phi < \phi_{\max} + \Delta\phi$, where $\Delta\phi$ is small enough that the potential may be approximated by an inverted parabola and moreover $C(\phi)$ and $R(\phi)$ may be approximated by $\widetilde{C}$ and $\widetilde{R}$, respectively. Note that unless β_J is very close to 1 and/or $C(\phi)$, etc., is pathological, the quantity $\Delta\phi$ can still be of order unity. The motion in this region is of the form

$$\phi(t) - \phi_{\max} = (2E/\widetilde{C}\overline{\omega}_J^2)^{1/2} \sinh(\overline{\omega}_J t) e^{-\tilde{\gamma} t}, \quad (39)$$

where we have defined

$$\overline{\omega}_J \equiv (\widetilde{\omega}_J^2 + \tilde{\gamma}^2)^{1/2}, \quad (40)$$

and E as above is the kinetic energy at the barrier top. Note that for any finite value of $\widetilde{\omega}_J$ the system is not arrested at the barrier top. The time T taken to traverse the region in question is given in the limit $E \to 0$ by the formula

$$T = [\overline{\omega}_J/(\overline{\omega}_J^2 - \tilde{\gamma}^2)][\ln(\widetilde{E}_0/E) + \text{const}], \quad (41)$$

where $\widetilde{E}_0$ was defined in (38) and the constant is of order 1. Since the time taken over the rest of the cycle is small compared to T and is, except for $\tilde{\beta}_J$ very close to zero, of order $\overline{\omega}_J^{-1}$, the total period T_0 is given by a formula identical to (41) (with a different constant), and hence by (17) the dc voltage is given by the expression

$$V = 2\pi[(\overline{\omega}_J^2 - \tilde{\gamma}^2)/\overline{\omega}_J][\ln(\widetilde{E}_0/E) + C_0]^{-1}, \quad C_0 \sim 1. \quad (42)$$

Now consider the dependence of E on the external current I. As usual, this is determined by the requirement that the energy dissipation per cycle W is equal to $2\pi I$, or equivalently that the *extra* dissipation, over and above that corresponding to the return current I_r, is equal to $2\pi(I - I_r)$. We therefore have

$$I - I_r = \frac{1}{2\pi} \int_0^E \left(\frac{\partial W}{\partial E}\right) dE. \quad (43)$$

Now, the contribution to $\partial W/\partial E$ from the region far from the barrier top is a nonsingular function of E in the limit $E \to 0$ and may therefore be approximated in (43) by a constant, generally of order $\tilde{\beta}_J$. On the other hand, the contribution W_c from the small region near the barrier top (say within $\pm\Delta\phi$ of it) is logarithmically divergent; for the case $\widetilde{R} \to \infty$ we have explicitly

$$\frac{\partial W_c}{\partial E} = \frac{\partial}{\partial E} \int_{-\Delta\phi}^{\Delta\phi} \widetilde{R}^{-1} \dot{\phi}\, d\phi$$

$$\cong \widetilde{R}^{-1} \frac{\partial}{\partial E} \int_{-\Delta\phi}^{\Delta\phi} \widetilde{\omega}_J (\phi^2 + E/\widetilde{E}_0)^{1/2} d\phi$$

$$\cong \frac{\widetilde{\omega}_J}{2\widetilde{E}_0 \widetilde{R}} [\ln(\widetilde{E}_0/E) + \text{const}], \quad (44)$$

where the constant is of order 1 and the second approximate equality holds for $E/\widetilde{E}_0 \to 0$. For the general case we obtain a similar formula but with a more complicated prefactor, which, however, is still of order $\widetilde{\omega}_J/\widetilde{E}_0\widetilde{R}$, i.e., of order $\tilde{\beta}_J$. Substituting (44) in (43) and allowing also for the nonsingular contribution, we find

$$I - I_r = (1/2\pi)[A'E \ln(\widetilde{E}_0/E) + B'E], \quad A', B' \sim \tilde{\beta}_J. \quad (45)$$

Finally, inverting (42), substituting the result in (45), using the fact that I_r is itself of order $\tilde{\beta}_J \widetilde{E}_0$, and restoring conventional units, we find the asymptotic current-voltage relation to be

$$\frac{I - I_r}{I_r} = \left(\frac{AV_0}{V} + B\right) \exp - \frac{V_0}{V}, \quad (46)$$

$$V_0 \equiv \phi_0 \left(\frac{\overline{\omega}_J^2 - \tilde{\gamma}^2}{\overline{\omega}_J}\right) \equiv \phi_0 \left(\frac{2\pi \tilde{I}_c}{\widetilde{C}\phi_0}\right)^{1/2} \left(1 + \frac{\tilde{\beta}_J^2}{4}\right)^{-1/2}, \quad (47)$$

where the quantities A and B are generally of order unity and cannot be calculated in analytical form in general. We see, therefore, that provided the dynamics near the barrier top can be obtained from a generalized RSJ model the exp $- V_0/V$ dependence of the current-voltage characteristic near return is quite universal: we emphasize that this result is not restricted to the weak-damping limit. In the weak-damping RSJ case the result is particularly simple since in this case $\tilde{I}_c = I_c[1 + O(\beta_J^2)]$, $\tilde{\beta}_J \sim \beta_J$, and hence we have simply $V_0 = \phi_0 \omega_J +$ corrections of order β_J^2, as already obtained in Sec. III B.

The case of the quasiparticle-tunneling model at zero temperature may be regarded as a special case of formula (46) with $A = 0$; as noted in Sec. III C, in this case the singular contribution to $\partial W/\partial E$ vanishes, since it is associated with the very low-frequency behavior which in this case is entirely dissipation free. [At high temperatures ($kT \gg \hbar\omega_J$) and in the limit of weak damping this is still true to the order considered, because the dissipation coefficient is proportional to $(1 + \cos\phi)$ and hence is of order $(\phi - \pi)^2$ in the region of the barrier top.] The question of what "effective resistance" should be used in defining the $\tilde{\gamma}$ which appears in V_0 is quite delicate, and can be regarded as a special case of the more general problem considered in the next paragraph.

We finally turn to the case of a general model where the effective admittance $Y(\omega)$ shunting the junction is a quite arbitrary function of ω (and ϕ) subject only to the usual constraints imposed by causality, etc. We consider the form $\widetilde{Y}(\omega)$ appropriate to small deviations of ϕ near $\phi_{\max}$, and define (cf. Ref. 23) the quantity $K(\omega) \equiv i\omega \widetilde{Y}(\omega)$. The Fourier transform $\phi(\omega)$ of $\phi(t) - \phi_{\max}$ then satisfies the equation

$$[K(\omega) - \tilde{I}_c]\phi(\omega) = 0, \quad (48)$$

which must be supplemented by appropriate boundary conditions. The quantity $K(\omega)$ is analytic in the lower half plane (cf. Ref. 23). It is intuitively plausible that the behavior is dominated, in the region of the barrier top, by the poles of the functions $[K(\pm\omega) - \tilde{I}_c]^{-1}$ closest to the real axis in the lower half plane, and this is demonstrated—admittedly not with total rigor, but we believe adequately for our purposes—in Appendix A. Assuming this to be so, let us call the positions of these poles $-i\omega_+$, $-i\omega_-$ (where ω_+, ω_- are assumed real and positive, cf. Appendix B) and *define* the quantities $\overline{\omega}_J, \tilde{\gamma}$ by the relations

$$\overline{\omega}_J \equiv \tfrac{1}{2}(\omega_{\min}^{(+)} + \omega_{\min}^{(-)}), \quad \tilde{\gamma} \equiv \tfrac{1}{2}\text{Re}(\omega_{\min}^{(-)} - \omega_{\min}^{(+)}). \quad (49)$$

It is then clear that the whole analysis leading to Eqs. (46)

and (47) goes through exactly as above. Consequently, the $\exp - V_0/V$ behavior of the asymptotic return current-voltage characteristic is quite universal, although for complicated forms of $K(\omega)$ a direct interpretation of the quantity V_0 in terms of simple circuit parameters may not always be possible.

IV. EFFECT OF FLUCTUATIONS

A. General considerations

In this section we shall consider the effect of fluctuations on the return of the system to the zero-voltage state. If we regard the energy dissipation in the system as a stochastic process in which the energy associated with the principal degree of freedom [$\phi(t)$ in this case] is transferred by random collisions to the "environment," then it is clear that as we make the dissipation weaker and weaker, both the averaged effect of the collisions and that of the fluctuations around the average tend to zero, but the second effect tends to zero more slowly than the first, since it is proportional to $N^{1/2}$ rather than N itself, where N is some effective number of collisions per cycle. A somewhat more quantitative estimate (cf. below) shows that at zero temperature the relative fluctuation of the energy dissipated per cycle near the return to the zero-voltage state is of order[24] $\rho_0^{1/2}$, where $\rho_0 \equiv R/R_Q$ is the $T=0$ value of the dimensionless parameter defined in Eq. (7c); thus, for resistances much higher than the quantum unit of resistance $R_Q \equiv h/4e^2$ (~ 6.5 kΩ) the results calculated in Sec. III are prima facie meaningless, even at zero temperature. [At finite temperature we require for their validity the stronger condition $(R/R_Q) \lesssim (\hbar\omega_J/kT)$, cf. Sec. II.]

In the next few subsections we shall restrict ourselves explicitly to the case where the effective resistance R is much smaller than R_Q, or for high temperatures much smaller than $R_Q(\hbar\omega_J/kT)$, where R is *defined*, for the general case, by the statement that the classically calculated value of the ratio $\beta_J \equiv (\pi/4)(I_r/I_c)$ is $1/\omega_J RC$; we shall, however, return in Sec. IV F to the possible generalization of our results to the case $R \gtrsim R_Q$. We recall that throughout this paper we are also assuming that the fluctuation parameter μ [defined by Eq. 7(b)] is small compared to unity, and that β_J [Eq. 7(c)], though not necessarily small, is not too close to unity. It then follows, as remarked in Sec. II, that the relevant calculation is not of the strictly dc characteristic but of the first-passage-time statistics for the return to the zero-voltage state.

The effect of *classical* thermal fluctuations on the return of a Josephson junction to the zero-voltage state has been considered by a number of authors.[3–7] Unfortunately (from the point of view of the present paper) most of these papers have discussed regimes different from the one of most interest to us, namely the case $\beta_J \ll 1$. In fact, the only papers of which we are aware which explicitly discuss the first-passage-time problem for this case are those of Ben-Jacob *et al.*,[5] the validity of whose treatment appears to be restricted to the case $I \gg I_r$ which we do not discuss here, and the recent paper of Cristiano and Silvestrini[6]; the final result of this latter paper is a complicated integral expression [Eq. (11)] which is not evaluated explicitly to give the retrapping probability as a function of the bias current or voltage, so that it is not easy to compare their results with those we shall obtain below.

At very low temperatures ($kT \lesssim \hbar\omega_J$) we should expect quantum effects to be important. At first sight we can distinguish two different kinds of effect: First, even if the system has enough energy to surmount the barrier top in a classical calculation, it nevertheless has a finite probability of being reflected because of quantum-mechanical effects (and conversely, has a finite probability of transmission, by tunneling, even for energies below the barrier top). Second, one might think that if there are fluctuations in the energy dissipated per cycle at finite temperature owing to the effects of classical noise, then there should also even at zero temperature be fluctuations due to quantum effects (the so-called "quantum noise"). Actually, these two apparently disparate effects are nothing but different aspects of the basic phenomenon of de Broglie wave propagation in a multidimensional space (see below); however, since the first exists even for a totally undamped system, while the second vanishes (in absolute magnitude) as the dissipation tends to zero, it makes sense to distinguish them for a first consideration. In a recent paper[8] Mel'nikov and Sütö have discussed both these effects. However, while the method used in their work is similar to (though more complicated than) the one we shall apply below, there are a number of important differences in the emphasis. In the first place, they implicitly consider the more general case in which our parameter μ [Eq. 7(b)] is not very small compared to unity, and fluctuations into and out of the zero-voltage state can be simultaneously important; for this reason they focus their interest on the truly steady-state solutions of the problem, and do not explicitly calculate the first-passage-time statistics.[25] Second, they concentrate on the first quantum corrections, in the limit $kT \gg \hbar\omega_J$, to the classical formulas; we, by contrast, are interested in a quantitative analysis of the complete classical-quantum crossover behavior, which has probably been seen in recent experiments.[26] Third, they assume without discussion that once quantum effects in the barrier-transmission process itself are taken into account the phase can be treated as a purely classical variable—an assumption which is very probably correct in the regime which they consider, but which needs explicit attention in the one we shall discuss. Finally, they restrict themselves to the RSJ model, whereas we shall consider also the general case.

In the classical limit it is possible to give a complete account of fluctuations by using a Fokker–Planck equation (which in the underdamped case is most naturally formulated in energy space). As already noted by Mel'nikov and Sütö,[8] to discuss the general case a different approach is necessary. A useful general approach to the problem of the dynamics of a dissipative macroscopic system in which quantum effects may be important is to regard the environment of the system as composed of harmonic oscillators with a dense frequency spectrum and to couple it to the system by an interaction linear in the oscillator coordinates (but possible nonlinear in the system variables). Such an "oscillator-bath" model of dissipation is of course familiar in the context of laser physics (see, e.g., Ref. 27); arguments to justify it as a generic model of dissipation have been given in Ref. 28, Appendix C, and in Ref. 23, and in the case of a Josephson junction artificially shunted by an external ohmic resistance calculations based on it seem to give rather good quantitative agreement with recent MQT experiments.[29] In the specific case of an ideal tunnel-oxide junction described by the QPT model, an explicit discussion and justification has been given recently by Ambegaokar and Eckern.[30] Once the "oscillator-bath" model is accepted, the problem essentially reduces to the solution of the corresponding Schrödinger's equation, with the relevant boundary conditions, in the many-dimensional Hilbert space which is the direct product of the system and both spaces. One would hope that a consistent solution would not only give the low-temperature behavior correctly but also, where appropriate thermal boundary conditions are imposed, reproduce the relevant results of the classical Fokker–Planck equation in the limit $T \to \infty$ (cf. Ref. 31).

An exact solution of Schrödinger's equation is of course usually out of the question, and one is forced to make some approximations. Here we shall assume that the effects of the environment on the quantum motion are adequately represented by the "quasiclassical Langevin equation" (QCLE) as defined and discussed by Schmid,[32] that is, an equation which represents the system by a *classical* variable driven by a Gaussian random force whose correlations are given by the *quantum* Callen–Welton (Nyquist) formula. This assumption seems to be implicit also in the work of Mel'nikov and Sütö.[8] The general question of the criterion of applicability of the QCLE is a delicate one; it is discussed in Appendix B for the special case of ohmic dissipation (corresponding in our case to the RSJ model). The upshot is that for motion in the classically accessible regime a sufficient, though not in general necessary, condition for its applicability is that $\eta L_0^2 \pi \hbar \gg 1$, where η is the viscosity and L_0 a characteristic distance over which anharmonicity in the potential is noticeable. In our case this reduces to the criterion $R \ll R_Q$, which we have assumed fulfilled. Although strictly speaking this result is proved only for the RSJ model, it seems reasonable to hope that the predictions of the QCLE will not be qualitatively misleading in the general case, provided $R \ll R_Q$ where R is, as above, the "effective" resistance defined by the statement that the classically calculated ratio I_r/I_c is equal to $1/\omega_J RC$. One aspect of this hypothesis needs to be particularly noted: For $R \ll R_Q$ it is easy to see that the average energy dissipated per cycle is large compared to the typical "environment quantum" $\hbar\omega_J$, and hence the probability that the system undergoes a complete cycle without emitting even one quantum is negligibly small. Since emission of a quantum on one trajectory without a corresponding emission on another destroys the coherence between them, we conclude that for $R \ll R_Q$ it is unnecessary to keep track of the phase coherence between the parts of the wave packet respectively reflected and transmitted at the barrier top, or to worry about the vexed question[11] of the effective identity or otherwise of values of ϕ separated by $2n\pi$; in fact, we may safely apply the standard quantum measurement axioms as soon as the system is reflected from a barrier. (This conclusion may well continue to hold even in some circumstances where the QCLE breaks down for other reasons.) There is of course a separate question as to whether the QCLE adequately describes the process of quantum barrier reflection itself; fortunately, we do not need to decide this question since in the regime $R \ll R_Q$ we shall see that this is anyway a small effect relative to those of quantum noise.

In the next subsection we shall discuss our general procedure for handling the effects of fluctuations. In Sec. IV C we specialize to the RSJ model with $\beta_J \ll 1$ (but $R \ll R_Q$), concentrating on the statistics of the first passage time. In Sec. IV D we discuss the analogous problem for the QPT model, and moreover comment on the effects, at first sight rather surprising, of the fluctuations on the current-voltage characteristic at finite voltages before retrapping takes place. Both these discussions take into account only the "quantum noise" effect, not that of quantum reflection; we return to the latter in Sec. IV E and show that for $\mu \ll 1$ it is always small. Finally in Sec. V we generalize the principal features of our results to a quite general situation (with, however, still $\mu \ll 1$).

Our notation in this section is as follows: We denote by E the actual kinetic energy with which, on a given attempt, the system reaches the barrier top, and by W the actual energy dissipated over a particular cycle. $\overline{E}$ and $\overline{W}(E)$ are the values of E and $W(E)$, respectively, which would be calculated neglecting the noise terms; i.e., they are the quantities denoted as E and W in Sec. III.[33] ΔW is defined as $W - \overline{W}$; the quantity W_f is defined as the root-mean-square value of ΔW, i.e., the rms fluctuation in the energy loss per cycle, and E_f is the rms dispersion of E. I and V denote, as always, respectively the imposed dc external current and the dc voltage (averaged over many cycles while the system is in the *running* state), and T_0 denotes similarly the period as calculated in the absence of noise: for any given external current I this is a fixed quantity. Note that with this definition V is in general *not* equal to ϕ_0/T_0: see below.

B. General procedure

In the general discussion of this section it will be essential to recall that we are interested in the regime specified by the inequalities

$$\beta_J \ll 1, \quad \mu \ll 1, \quad \rho \equiv \beta_J^{-1}\mu \ll 1, \tag{50}$$

where the dimensionless parameters β_J, μ, ρ are defined for the RSJ model by Eqs. (7) at zero temperature, and at high temperature by the same formulas with kT replacing $\hbar\omega_J$. For more general models the parameters can be defined, at least to the required order of magnitude, as indicated at the end of Sec. II. We discuss in Sec. IV F possible generalizations to cases where one or more of the inequalities (50) fail.

Consider a system which starts from the top of a given barrier with kinetic energy E. The average energy it will lose on the next cycle is, by definition, $\overline{W}(E)$, and the fluctuations in energy loss are given by a Gaussian distribution with variance $W_f(E)$. As we shall verify in detail below, the order of magnitude of $W_f(E)$ is given by

$$W_f^2(E) \sim (\hbar\omega_J, kT) \times \overline{W}(E) . \tag{51}$$

Now we know that the order of magnitude of $\overline{W}(E)$ is given (cf. Sec. III) by

$$\overline{W}(E) \sim \beta_J E_0 [1 + \text{const } E/E_0 (\ln E/E_0 + \text{const})] \tag{52}$$

(where $E_0 \sim I_c \phi_0$ as above). At this point we will assume that for realistic ramping rates the retrapping will occur predominantly in the region $E/E_0 \ll 1$; we confirm subsequently that for $\mu \ll 1$ this is indeed so. It then follows that for an estimate of the fluctuations in energy loss we may approximate $W_f^2(E)$ in (51) by $W_f^2(0)$ (Ref. 34); thus we find

$$W_f \sim \beta_J^{1/2} [(\hbar\omega_J, kT) \times E_0]^{1/2} \sim (\beta_J \mu)^{1/2} E_0 . \tag{53}$$

Thus, the system effectively undergoes a random walk in energy space in which on each cycle it gains or loses an energy of order (53). On the other hand, if it strays too far from the energy $\overline{E}(I)$ appropriate to the imposed external current I, there will be a tendency to restore equilibrium over a period corresponding to of order β_J^{-1} cycles. Thus the random walk in effect consists of $\sim \beta_J^{-1}$ steps, and the distribution of E is therefore itself given to a first approximation by a Gaussian centered at $\overline{E}(I)$ with width

$$\sim (\beta_J \mu)^{1/2} E_0 \beta_J^{-1/2} \sim \mu^{1/2} E_0.$$

[This conclusion is strictly true only if $W(E)$ is a linear function of E; in the realistic situation there are some corrections—see below.]

Now it is easily verified that for $\rho \ll 1$ the system, once reflected from the top of a particular barrier, is very unlikely to be able to pass it on any subsequent attempt. (This is because on the next cycle it will lose an energy of order $2\overline{W}$ in friction while gaining nothing from the potential; since for $\rho \ll 1$ the quantity $\overline{W}$ is much greater than $\hbar\omega_J$ or kT, its chances of passing at the next attempt are negligible.) Thus, it is reasonable to take the rate of retrapping as simply equal to the average number of cycles per second (i.e., V/ϕ_0) times the fraction of the energy distribution which lies below zero. Clearly the above procedure makes sense only if this fraction is small, but this will automatically be so for the region where retrapping predominantly occurs, provided that the ramping rate is not unrealistically high (see below). We can thus express the retrapping probability as a function of $\overline{E}$ and hence of either the externally imposed curent I or the dc voltage V measured in the running state before the retrapping occurs, as preferred.

It is useful to note the following relations, which follow in the limit $E \ll E_0$ from Eqs. (19), (22), (23), and the definition of β_J:

$$\overline{W}(E) - \overline{W}(0) = \beta_J E [\ln(E_0/E) + \ln 64 + 1] , \tag{54}$$

$$\overline{E}/E_0 = 64 \exp - V_0/V . \tag{56}$$

Note also that the quantity $[W(\overline{E}) - W(0)]^2/W_f^2$ is of order β_J (up to logarithmic factors) and hence much less than unity. ($\overline{E}$ is assumed to be of order $\mu^{1/2}E_0$—cf. above.)

C. Fluctuations in the RSJ model

We proceed to implement the above procedure for the RSJ model. The relevant equation of motion is now the full Eq. (3), with the noise term kept:

$$C\ddot{\phi} + \dot{\phi}/R + I_c \sin\phi = I + I_n(t). \tag{56}$$

The statistics of the noise term $I_n(t)$ correspond to colored Gaussian noise, with

$$\langle I_n(t) I_n(t') \rangle \equiv \alpha(t - t'), \tag{57}$$

$$\alpha(\omega) \equiv \int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \alpha(t) = \frac{\hbar\omega}{R} \coth\left(\frac{\beta\hbar\omega}{2}\right). \tag{58}$$

As discussed above, it is sufficient to calculate the quantity $W_f^2(0) \equiv W_f^2$, the mean-square fluctuation in the energy dissipated on the separatrix cycle corresponding to $E = 0$. Since the energy ΔW dissipated due to the noise term is given by the expression

$$\Delta W = -\int_0^{T_0} V(t) I_n(t) dt = -\int_0^{T_0} \dot{\phi}(t) I_n(t) dt, \tag{59}$$

we find

$$W_f^2 = \int_0^{T_0} dt \int_0^{T_0} dt' \langle \dot{\phi}(t) \dot{\phi}(t') I_n(t) I_n(t') \rangle. \tag{60}$$

Under the assumption that the noise is a small perturbation on the original noise-free motion, which is true for $\mu \ll 1$, we can factorize the expectation value in (60) and use (57) to write

$$W_f^2 = \int_0^{T_0} dt \int_0^{T_0} dt'\, \dot{\phi}(t) \dot{\phi}(t') \alpha(t - t'). \tag{61}$$

Finally, in the limit $\beta_J \to 0$ we can use for $\phi(t)$ the undamped solution (21), thereby obtaining (after restoration of conventional units)

$$W_f^2 = \left(\frac{\phi_0^2}{2\pi R}\right) \int_{-\infty}^{\infty} \frac{\hbar\omega \coth(\beta\hbar\omega/2)}{\cosh^2(\pi\omega/2\omega_j)} d\omega. \tag{62}$$

It is clear that in the limit $\beta \to 0$ the quantity W_f^2 is simply equal to $2kT I_r \phi_0$, as of course it must be to satisfy the fluctuation-dissipation theorem, while for $\beta \to \infty$ it is of order

$$\hbar\omega_J I_r \phi_0 \sim [(\beta_J \mu)^{1/2} E_0]^2, \tag{63}$$

as stated above. It is convenient to express W_f^2 in terms of its value at zero temperature:

$$W_f^2(T) = W_f^2(T=0) f(T). \tag{64}$$

As we shall see, the quantity $f(T)$ completely determines the temperature dependence of the experimentally observed retrapping distribution; we return below to its evaluation.

In view of the Gaussian nature of the quantum noise spectrum, it follows that provided E/E_0 is small compared to unity the probability, starting with kinetic energy E, to lose an energy ΔW over and above the mean dissipated energy $\overline{W}(E)$ is given by the expression

$$P(\Delta W) = \frac{1}{(2\pi W_f^2)^{1/2}} \exp - (\Delta W)^2/2W_f^2. \tag{65}$$

Suppose now that the system starts from the top of the nth barrier with energy E_n. Over the next cycle it will on average dissipate in friction an energy $W(E_n)$; at the same time it will gain from the external current (potential) an energy $W(\overline{E})$, where $\overline{E}(I)$ is the value of E calculated in Sec. III for the given I. In addition, the system will lose an energy ΔW due to the noise term, with probability (65). Thus we have

$$E_{n+1} = E_n + \overline{W}(\overline{E}) - \overline{W}(E_n) - \Delta W. \tag{66}$$

Hence, if the distribution function for the energy at the nth barrier is $f(E_n)$, we have

$$f(E_{n+1}) = \int_0^\infty dE_n f(E_n) P\{E_{n+1} - E_n + [\overline{W}(E_n) - \overline{W}(\overline{E})]\}, \quad (67)$$

with the function $P(x) \equiv P(-x)$ given by (63). To obtain the steady-state distribution, we may extend the lower limit of integration to $-\infty$, since for the physically interesting regime $f(E)$ is small for $E<0$. Thus the steady-state distribution $f_s(E)$ satisfies the integral equation

$$f_s(E) = (2\pi W_f^2)^{-1/2} \int_{-\infty}^{\infty} dE' f_s(E') \times \exp - \frac{1}{2W_f^2} [E - E' + \overline{W}(E') - \overline{W}(\overline{E})]^2. \quad (68)$$

It is now an essential point in our argument that the quantity $\partial W/\partial E$ is of order $\beta_J \ln E_0/E$ [cf. Eq. (44)]. Since we shall see below that in the region of interest E/E_0 is of order $\alpha\mu^{1/2}$, where α is a constant which, though small compared to 1, is not exponentially small, we shall see that $\partial W/\partial E$ is of order $\beta_J |\ln \mu|$ and is hence small compared to 1 unless μ is exponentially small (a case we exclude). Under these conditions it is straightforward to show[35] that the approximate solution of (68) is

$$f_s(E) = \left[\left(\frac{\partial W}{\partial E}\right) \Big/ \pi W_f^2\right]_{E-\overline{E}}^{1/2} \times \exp - \left(\frac{2}{W_f^2} \int_E^{\overline{E}} [W(\overline{E}) - W(E')] dE'\right). \quad (69)$$

This approximation fails for $E \lesssim W_f$, but it will enable us to obtain an approximate expression for the retrapping rate. Before doing so, we note that since $W(E)$ is not a linear function of E, the distribution function (69) is not symmetric around $\overline{E}$ and therefore the quantity we have called $\overline{E}$ is not the average energy, but the peak in the distribution. Moreover, the dc voltage in the running state, which is proportional to the average of $T_0^{-1}(E)$, i.e., of $[\ln(E_0/E)]^{-1}$, is not equal to the value calculated by putting $E = \overline{E}$ as in Sec. III. However, it is easily seen that these effects are of the same relative order as the inverse logarithm of the retrapping rate per cycle, which we assume shall (cf. below) and can therefore be neglected. In other words, so long as the system remains in the running state its current-voltage characteristic should be given to an excellent approximation by the deterministic formula (25).

The exact expression (neglecting as always multiple crossing attempts) for the distribution function at negative energies follows from (65) and (67):

$$f(E) = (2\pi W_f^2)^{-1/2} \int_0^\infty f_s(E') \times \exp - \frac{1}{2W_f^2} [E - E' + \overline{W}(E') - W(\overline{E})]^2 dE'. \quad (70)$$

Expression (69) cannot be used for $f_s(E')$ for $E' \lesssim W_f$. However, since the second factor in the integrand of (70) is much faster varying than $f_s(E')$, it is adequate to set $f_s(E)$ equal to the value for $E=0$ as given by (69). Moreover, we may safely neglect the factor $W(E') - W(\overline{E})$ in the exponent of the second factor in (70), since this gives a correction of order $\exp - \beta_J^{-1}$. Thus we obtain

$$f(E)_{E<0} \cong (2\pi W_f^2)^{-1/2} f_s(0) \times \int_0^\infty \exp - [(E-E')^2/2W_f^2] dE', \quad (71)$$

and hence the total fraction of the distribution with $E<0$ is $(2\pi)^{-1/2} W_f f_s(0)$. Substituting from (69) and multiplying by the attempt frequency $T_0^{-1}(\overline{E})$, we finally get for the rate of retrapping P_r into the zero-voltage state:

$$P_r = T_0^{-1}(\overline{E}) \left[\frac{1}{2\pi^2}\left(\frac{\partial W}{\partial E}\right)\right]_{E=\overline{E}}^{1/2} \times \exp - \frac{2}{W_f^2} \int_0^{\overline{E}} [W(\overline{E}) - W(E')] dE'. \quad (72)$$

Note that to this point we have not used the specific form of $W(E)$, so that provided the general order-of-magnitude estimates made above are valid formula (72) is valid for an arbitrary behavior of $W(E)$. Equation (72) should be exact in the limit $\beta_J, \mu, \rho \to 0$ up to a numerical constant of order 1 in the prefactor.

For a linear dependence of $W(E)$ on E, the integral in the exponent of (72) would of course be exactly $\frac{1}{2}\overline{E}[W(\overline{E}) - W(0)]$. For the actual form (54), the error involved in identifying the two is of relative order $1/\ln(64E_0/E)$, which is in practical terms so small as to be negligible. Using (54) and (55) also in the prefactor, and also (41), we obtain

$$P_r = \left(\frac{\beta_J}{4\pi^2 \ln(E_0/\overline{E})}\right)^{1/2} \omega_J \exp - \left(\frac{\overline{E}[W(\overline{E}) - W(0)]}{W_f^2}\right). \quad (73)$$

Now from (25) and (55) we have to a good approximation [i.e., neglecting terms of the form $\ln(\ln x)$ and $(\ln x)^{-1}$]

$$\overline{E}/E_0 = 16\delta I/I_r (|\ln \delta I/I_r|)^{-1} \quad (\delta I \equiv I - I_r), \quad (74)$$

and hence, finally, to the same approximation [since $W(\overline{E}) - W(0) \equiv \delta I \phi_0$]

$$P_r \cong \omega_J \left(\frac{\beta_J}{4\pi^2 |\ln \delta I/I_r|}\right)^{1/2} \times \exp - \left[\left(\frac{\delta I}{I_r}\right)^2 \left(\frac{16 I_r \phi_0 E_0}{W_f^2}\right) \frac{1}{|\ln(\delta I/I_r)|}\right]. \quad (75)$$

Note that the factor $I_r \phi_0 E_0/W_f^2$ is at high temperature simply $E_0/2kT$, and generally is of the order of the inverse of the quantity we have called μ. Thus, up to logarithmic factors depending on the ramping rate (see below) the quantity $\delta I/I_r$, and hence $\overline{E}/E_0$ is of order $\mu^{1/2}$ when the retrapping takes place, as stated above. Equation (75) is the fundamental result of this section.

In a real-life experiment one normally sweeps the current I down towards I_r at a constant rate, and observes the distribution $P(I)$ of I values at which return to the zero-voltage state takes place. It is convenient to introduce the notation

$$x \equiv \delta I / I_r, \quad \omega_s \equiv I_r^{-1}\left(\frac{dI}{dt}\right).$$

It is also convenient, with a view to future generalizations, to write the quantity P_r in the generic form

$$P_r = \omega_0(x)\exp - Af(x). \tag{76}$$

In the present case we obviously have

$$A \equiv 16 I_r \phi_0 E_0 / W_f^2, \quad f(x) \equiv x^2/|\ln x|,$$

$$\omega_0(x) \equiv \omega_J (\beta_J / 4\pi^2 |\ln x|)^{1/2}. \tag{77}$$

The dependence of the prefactor $\omega_0(x)$ on x is so slow that we shall neglect it in what follows and simply write ω_0 as a constant which must eventually be determined self-consistently. With this approximation we have for the predicted distribution $P(x)$ of values x at which return occurs

$$P(x) = \left(\frac{\omega_0}{\omega_s}\right)\exp - Af(x) \exp - \int_x^\infty \left(\frac{\omega_0}{\omega_s}\right) e^{-Af(x')}\, dx'. \tag{78}$$

It is easy to verify that the maximum in the distribution is attained at the point x_0 given by the implicit equation

$$Af(x_0) = \ln[\omega_0/\omega_s A f'(x_0)]. \tag{79}$$

This information is not particularly useful, since to use it one would have to know the fluctuation-free return current to high accuracy, which is experimentally impractical. A more useful quantity is the full width of the distribution at half maximum. Apart from a factor which is close to unity and depends on the precise form of $f(x)$, this is the same as the quantity $K \equiv 2[P(x_0)/|P''(x_0)|]^{1/2}$, and we shall therefore give the latter, which after a little algebra is found to be

$$K = 2[(f'A)^2 + f''A]_{x=x_0}^{-1/2}. \tag{80}$$

Formulas (79) and (80) are exact and general except for the neglect of the variation of the prefactor in (76). We now substitute the RSJ forms of $f(x)$ and A [Eqs. (77)] and assume that A, which is of the order of μ^{-1}, is large. Neglecting iterated logarithms, we obtain

$$x_0 = A^{-1/2}[\ln A^{1/2} \ln(\omega_0/2\omega_s A^{1/2})]^{1/2} \tag{81}$$

(where we implicitly assume that $\omega_s \ll A^{-1/2}\omega_0$, which is almost certainly true in any real experiment). Further, neglecting terms of relative order $(\ln A)^{-1}$, we find

$$K = A^{-1/2}[\ln A^{1/2}/\ln(\omega_0/2\omega_s A^{1/2})]^{1/2}. \tag{82}$$

Note carefully that K is the width with respect to the reduced variable I/I_r (not I/I_c). It is clear that the temperature dependence of K is entirely determined by that of A, i.e., of the quantity $W_f(T)$; at high temperature ($\hbar\omega_J \ll kT \ll I_c\phi_0$) we have simply $K \propto T^{1/2}$, in contrast to the case of escape from the zero-voltage state where $K \propto T^{2/3}$. Note that in the present case, in contrast to the latter, the function $f(x)$ is the same for the classical and quantum regimes, and hence the *shape* of the curve is independent of temperature up to logarithmic factors, only the scale changing.

It remains only to find an expression for A as a function of T. It is convenient to write

$$A(T) \equiv 16 I_r \phi_0 E_0 / W_f^2(T) = [2\mu^{-1}/(\ln 2)][f(T)]^{-1}, \tag{83}$$

where as in (7b) we define

$$\mu \equiv \hbar\omega_J / I_c\phi_0 \equiv (2\pi\hbar^2/CI_c\phi_0^3)^{1/2}. \tag{84}$$

The quantity $f(T)$, which is normalized to unity at zero temperature, is given by the expression [cf. (62)]

$$f(T) \equiv (\ln 2)^{-1} \int_0^\infty x \operatorname{sech}^2 x \coth(\alpha x) dx,$$

$$\alpha \equiv \hbar\omega_J / \pi kT. \tag{85}$$

This expression is easily seen to tend to $(\pi/\ln 2)(kT/\hbar\omega_J)$ in the limit $T \to \infty$, and to be equal to $\pi^2/8 \ln 2$ at $kT = \hbar\omega_J/\pi$. The numerically computed expression for $f(T)$, which according to Eq. (82) should be approximately proportional to the square of the observed distribution width, is shown in Fig. 7.

It is interesting to compare the results for the "retrapping" distribution with those for the escape from the zero-voltage state for the same junction. Since the results of the present section have been derived explicitly for the case of weak damping we use for the escape rate the undamped WKB predictions, for which see, e.g., Ref. 36. If we define "crossover temperatures" $T_{\rm retr}$, $T_{\rm esc}$ for the retrapping and escape problems, respectively, as the temperature at which the extrapolated high-temperature curve gives a value equal to the $T = 0$ rate, then from Eq. (9) of Ref. 36 and the properties of $f(T)$ given above we see that we predict

$$T_{\rm retr}/T_{\rm esc} \cong (\ln 2/0.15\pi)(I_c\phi_0/\hbar\omega_J)^{0.2} \sim \mu^{-0.2}. \tag{86}$$

Thus for small μ the crossover should occur at a higher temperature for retrapping than for escape. The physical reason for this is that for the retraping problem the relevant characteristic frequency is the Josephson plasma frequency at zero bias, whereas for the escape problem it is the small-oscillation frequency in the much shallower well which corresponds to I close to I_c.

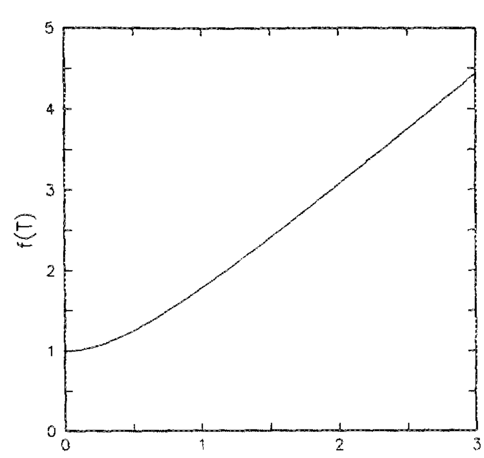

FIG. 7. Function $f(T)$, which is approximately proportional to the square of the predicted width of the retrapping distribution, plotted as a function of temperature. T is in units of $\hbar\omega_J/\pi k_B$.

D. Effect of fluctuations in the quasiparticle-tunneling model

In discussing the QPT model, it needs to be borne in mind that the spectrum of the resistive fluctuations has an explicit temperature dependence as well as that due to the $\coth(\beta\hbar\omega/2)$ factor. However, if as in the last section we express the experimentally observed results in terms of the *reduced* variable $x \equiv \delta I/I_r(T)$, this explicit dependence vanishes and the remaining temperature dependence is due entirely to the coth factor.

At first sight the QPT model can be discussed in a way similar to that of the last subsection, the main difference being that the quantity $\partial W/\partial E$ is now proportional, at $T=0$, to a constant rather than to $\ln E$ in the limit $E \to 0$. As shown by Ambegaokar and co-workers,[16] the correct procedure for describing fluctuations in the QPT model is to take the noise current $I_n(t)$ on the right-hand side of Eq. (9) to be given by the expression

$$I_n(t) = \xi_1(t)\cos[\phi(t)/2] + \xi_2(t)\sin[\phi(t)/2], \quad (87)$$

where the quantities $\xi_1(t), \xi_2(t)$ have independent Gaussian distributions with

$$\langle \xi_1(t) \rangle = \langle \xi_2(t) \rangle = 0, \quad (88a)$$

$$\langle \xi_1(t)\xi_2(t) \rangle = 0, \quad (88b)$$

$$\langle \xi_1(t)\xi_1(t) \rangle = \hbar[\alpha_R(t-t') + \beta_R(t-t')],$$

$$\langle \xi_2(t)\xi_2(t') \rangle = \hbar[\alpha_R(t-t') - \beta_R(t-t')], \quad (88c)$$

where $\alpha_R(t)$ and $\beta_R(t)$ are the real parts, respectively, of the quantities $\alpha(t), \beta(t)$ defined in Eqs. (10). The choice (87) and (88) assures that the fluctuation-dissipation theorem is satisfied in this model.

It is straightforward to calculate the quantity W_f^2 (the mean-square fluctuation in the energy dissipated on the "separatrix" cycle $E=0$). Quite generally we have [from Eq. (60)]

$$W_f^2 = \frac{4\hbar}{T_0}\sum_n \{[|S_1(\omega_n)|^2 + |S_2(\omega_n)|^2]\alpha_R(\omega_n) + [|S_1(\omega_n)|^2 - |S_2(\omega_n)|^2]\beta_R(\omega_n)\}$$

$$[\omega_n = 2\pi(n+1/2)/T_0], \quad (89)$$

where $S_1(\omega)$ and $S_2(\omega)$ are the quantities defined in Eqs. (29) and $\alpha_R(\omega), \beta_R(\omega)$ are, respectively, the Fourier transforms of $\alpha_R(t), \beta_R(t)$ and are explicitly given by

$$\alpha_R(\omega) = \tfrac{1}{2}\coth(\beta\hbar\omega/2)I_{qp}(\omega),$$

$$\beta_R(\omega) = \tfrac{1}{2}\coth(\beta\hbar\omega/2)I_c(\omega), \quad (90)$$

with $I_{qp}(\omega)$ and $I_c(\omega)$ the spectral densities defined in Eqs. (11). In the limit $E \to 0$, we can prima facie replace the sum in (89) by an integral, and take $S_1(\omega)$ and $S_2(\omega)$ to be given by the forms (valid for $\beta_J \ll 1$) (31). This gives

$$W_f^2(T) = \frac{8\hbar}{\pi}\int_0^\infty \left(\frac{\pi\omega}{\omega_J}\right)^2 \frac{\coth(\beta\hbar\omega/2)}{\sinh^2(\pi\omega/\omega_J)} \times [I_c(\omega{:}T) + I_{qp}(\omega{:}T)\cosh(\pi\omega/\omega_J)]\,d\omega. \quad (91)$$

It is clear that in the limit $\hbar\omega_J \ll kT \ll \Delta$ we have [cf. Eq. (13)]

$$W_f^2 = 2kTI_r\phi_0,$$

as of course we must to satisfy the fluctuation-dissipation theorem. The integral (91) can also be evaluated explicitly at zero temperature using Eqs. (11):

$$W_f^2(T=0) = \frac{2}{\pi}\frac{\phi_0^2\hbar\omega_J^2}{R_N}\left(\frac{2\Delta}{\hbar\omega_J}\right)^3 \exp -\left(\frac{2\pi\Delta}{\hbar\omega_J}\right) = 2\Delta I_r(T=0)\phi_0 \quad (92)$$

[cf. (32)] with corrections of relative order $\hbar\omega_J/2\Delta$. To evaluate $W_f^2(T)$ at intermediate temperatures we would need to substitute the appropriate forms of the spectral densities $I_c(\omega{:}T)$, $I_{qp}(\omega{:}T)$ [for which see, e.g., Ref. 16, Eq. (41)]. However, it is easy to draw qualitative conclusions without a detailed evaluation: see below.

We note that both the average energy dissipation $\overline{W}(E)$ and the mean-square fluctuation $W_f^2(T)$ are essentially a sum of two quite different contributions. One, arising from frequencies $\omega > 2\Delta$, has a weight which is proportional to $\exp -(2\pi\Delta/\hbar\omega_J)$ and is essentially temperature independent in the region of interest ($kT \ll \Delta$); as we saw in Sec. III, it has no singularity in the limit $E \to 0$. The other contribution comes from low frequencies, $\omega \lesssim \omega_J$: this has an intrinsic weight proportional to the number of thermally excited quasiparticles, i.e., to $\exp -\Delta/kT$, and also has no singularity in the limit $E \to 0$. In the limit of large $\Delta/\hbar\omega_J$ we may to a very good approximation neglect the second term for $kT < \hbar\omega_J/2\pi$ and the first for $kT > \hbar\omega_J/2\pi$, the transition region being of order $(\hbar\omega_J)^2/(2\pi)^2\Delta$ in width. We can then divide the temperature axis into three regions: (I) $kT < \hbar\omega_J/2\pi$, (II) $\hbar\omega_J/2\pi < kT \lesssim \hbar\omega_J$, and (III) $kT \gg \hbar\omega_J$. In region I everything is essentially temperature independent; in particular, there are no power-law corrections to the zero-temperature behavior as we expect in the RSJ case. In region III it follows from the fluctuation-dissipation theorem that the result for W_f^2, expressed in terms of the return current I_r, is identical to that of the RSJ model. In region II a more detailed calculation is probably necessary to get the exact behavior right, but it is clear that we must get a smooth interpolation between the formulas of regions I and III.

We can now proceed exactly as in the last subsection to obtain the retrapping statistics. In region I the main difference is that $\partial W/\partial E$ is proportional to a constant rather than to $\ln E$ in the limit $E \to 0$; Eq. (73) is in fact still valid except that the $(\ln E_0/\overline{E})^{-1/2}$ factor in the prefactor is replaced by $(\ln E_0/\overline{E})^{-1}$. The only difference in subsequent formulas is that $f(x)$ is of the simple form x^2 rather than $x^2|\ln x|$; this leads to a small correction to the shape of the retrapping histogram and to the omission of the factor $(\ln A)^{1/2}$ in (82). The same is true in region II, where in fact it is easy to see that (83) remains valid apart from the overall numerical factor.

The upshot of all this is that the qualitative behavior of the width σ of the retrapping distribution as a function of temperature for the QPT model will be very similar to that for the RSJ model; the principal differences are that (a) the "crossover temperature" as defined in Sec. IV C will be

$\hbar\omega_J/2\pi$ rather than $[(\ln 2)/\pi](\hbar\omega_J)$, and (b) the curve of σ^2 vs T will be much flatter at low temperature (reflecting the absence of appreciable dissipation at frequencies $\hbar\omega \sim kT$; compare the results for the MQT problem[37]).

There is one major complication to the above picture. We have implicitly assumed that we could take the quantity $W(E)$ to be given by its "smoothed" value, which is monotonic. But in fact at zero temperature the function $W(E)$ is composed, as seen in Fig. 4,[38] entirely of pieces with *negative* slope, separated by vertical drops; at finite temperature this remains true down to approximately the nth discontinuity, where $n \sim \hbar\omega_J/2\pi kT$, after which the slope becomes positive (but we still get discontinuities). We shall see that this phenomenon, while introducing a qualitative and surprising modification into the running-state I-V characteristics in the region where retrapping is negligible, has little effect on the retrapping statistics themselves.

Let us suppose that we are interested in a region of the characteristic where the root-mean-square fluctuation W_f in W, as calculated above, is small compared to the height of the discontinuities in $W(E)$. Then, once a fluctuation takes the system away from its deterministically calculated value of E for the given I, the effect of the external current will be to *amplify* the fluctuation rather than to damp it down, and this process will continue as long as $\partial W/\partial E$ remains negative. It is clear that the only stable states of the system will correspond to the points marked by arrows in Fig. 4: thus, we should expect that the voltage is *discretized*, at the values $V_n = 2\Delta/[e(2n+1)]$. If there are regions where $\partial W/\partial E$ is positive and others where it is negative (as happens for the "mixed" model, see Fig. 6, or for the simple QPT model at finite temperature) then we expect pieces of continuous curve in the I-V plot separated by forbidden voltage regions. Of course, in practice it may not be easy to distinguish the effects of fluctuations from those of a finite (and sufficiently small) lead impedance, which may lead to a qualitatively similar effect. Note that in any case we expect that at any finite temperature the gaps disappear sufficiently close to the voltage axis, since the slope always becomes positive in this limit (see above).

If, on the other hand, we are in the region where W_f is large compared to the step heights, then we should expect the effect to be washed out and the running-state I-V charcteristic to be essentially that calculated in Sec. III [e.g., Eq. (36)]. Since the height of the nth step decreases strongly with n, there will be only a finite number n_0 of forbidden voltage regions even at zero temperature. We may estimae this number crudely as follows: From Eq. (30) we find that the height of the nth step is of order $n^{-1}(\Delta/\hbar\omega_J)\beta_J E_0$. Moreover, [cf. Eq. (92)] the fluctuation W_f for the QPT model at $T=0$ is of order $(\Delta\beta_J E_0)^{1/2}$. Thus the number n_0 of forbidden voltage regions which survive the effects of quantum fluctuations is of order

$$n_0 \sim \frac{\Delta}{\hbar\omega_J}\left(\frac{E_0}{\Delta}\right)^{1/2}\beta_J^{1/2} \sim \left(\frac{\Delta}{\hbar\omega_J}\right)^{1/2}\left(\frac{R}{R_Q}\right)^{-1/2}. \qquad (93)$$

(The condition for forbidden regions to be seen at all is therefore the same, as regards orders of magnitude, as that for the whole treatment of this section to be valid: cf. Ref. 24. Now from (75) the order of magnitude of $\delta I/I_r$ in the retrapping regime is $W_f^2/I_r\phi_0 E_0 \sim (\Delta/\hbar\omega_J)\mu$, and the voltage is therefore of order $\omega_J\phi_0/\ln[(\Delta/\hbar\omega_J)\mu]$. The corresponding order of magnitude of n in the retrapping regime, namely Δ/eV, is therefore of order $(\Delta/\hbar\omega_J)\ln[(\Delta/\hbar\omega_J)\mu]$. The step structure will be negligible if this quantity is large compared to n_0, Eq. (93), i.e., if

$$(\beta_J E_0/\Delta) \ll \ln^2[(\Delta/\hbar\omega_J)\mu]. \qquad (94)$$

Now for the pure QPT model we have, at least to an order of magnitude, $E_0 \sim I_c \sim \Delta/R_n$, and hence the inequality (94) can be written

$$R/R_Q \gg \beta_J \ln^{-2}(R_N/R_Q), \qquad (95)$$

which is still compatible with the condition $R/R_Q \ll \hbar\omega_J/\Delta$ for the treatment of this section to be adequate. Under condition (95) we may legitimately neglect the complications due to the step structure of the I-V characteristic. We conclude that the results cited above for the widths, etc., of the retrapping distribution in the QPT model may be realistically compared with experiment.

E. Quantum reflection

In this section we shall consider an effect which was not explicitly accounted for in the QCLE analysis of the preceding sections, and which persists even in the limit $R \to \infty$, namely the phenomenon of quantum reflection of the system from a barrier which it has, classically, sufficient energy to cross. We will assume that quantum effects are important only near the barrier top, and that in this region the effects of dissipation are unimportant (this should certainly be true for $\beta_J \ll 1$), while over the rest of the cycles we describe the motion by the deterministic (but dissipative) equation of motion (18), or its analog (thus neglecting any interaction of the effects of fluctuations and barrier reflection). Furthermore, we will assume that it is unnecessary to keep track of the coherence between the parts of the wave packet which in the washboard-potential picture are in different wells. Thus, the effects of quantum mechanics are entirely encapsulated in the specification of the reflection coefficient from the barrier as a function of energy. This approximation should almost certainly be valid in the limit $\beta_J \ll 1, \rho \ll 1$; we return in the next section to the question of its validity in the more general case.

We consider the limit $I \to I_r \ll I_c$. Then the shape of the barrier top is an inverted parabola with curvature ω_J, and we have for the reflection coefficient the well-known expression (see, e.g., Ref. 39)

$$R(E) = [\exp(2\pi E/\hbar\omega_J) + 1]^{-1}, \qquad (96)$$

where E is, as usual, the kinetic energy with which the system reaches the barrier top in classical motion. It is not immediately clear that $R(E)$ is the retrapping probability per cycle, because of the possibility of repeat attempts. However, for $\rho \ll 1$ the system on average loses over the next back-and-forth cycle an energy large compared to $\hbar\omega_J$, and hence, if the probability for its initial reflection was appreciable, is very unlikely to pass on the next attempt. Thus in this case $R(E)$ is indeed to a good approximation the retrapping probability; note that it is not unity for $E<0$, because of the possibility of tunneling through the barrier. The case $\rho \gtrsim 1$ needs separate consideration, and will be discussed in the next subsection.

The crucial point now is that from the definition of the parameter μ [Eq. 7(b)] the quantity $\hbar\omega_J$ is of order μE_0, while according to the calculations of the last two subsections (which we shall see are self-consistent) E_f is of order $\mu^{1/2}E_0$ and W_f of order $(\beta_J\mu)^{1/2}E_0$, in the region where retrapping takes place. Thus, under the assumption $\mu \ll 1$ that is common to all the work of this paper, $\hbar\omega_J$ is *always* small compared to E_f; for $\rho \ll 1$ it is also small compared to W_f. Now in the limit $\hbar\omega_J \ll E_f$ it is clear that the principal effect of quantum reflection may be taken into account by multiplying the integrand of the right-hand side of Eq. (70) by $R(E')$ and extending the integration to $-\infty$; and moreover that for $\hbar\omega_J \ll W_f$ the effect of this substitution is negligible. Hence, we reach the conclusion that for $\rho \ll 1$ the effects of quantum reflection (and below-barrier transmission) are completely negligible relative to those of quantum noise and do not affect the results stated above.

F. Generalizations

The results given in this section so far were obtained under the assumption of either a RSJ or a QPT model with the inequalities $\beta_J \ll 1$, $\mu \ll 1$, $\rho \ll 1$ all well satisfied.[40] It is clear that they can be rather trivially generalized to the case where these inequalities, or the analogous ones, are still all well satisfied but the classical equation of motion is more complicated than the simple RSJ form. In fact, it is easy to see from the results of Sec. III E that provided only that the low-frequency dissipation is large enough that the logarithmic term in (44) dominates (a very weak condition, for any model but one of the extreme QPT type) then all the arguments leading to Eq. (75) go through verbatim with the quantities β_J, E_0, etc., replaced by the tilde'd quantities defined in Sec. III E. Indeed, in the limit $kT \gg \hbar\omega_J$ the only difference in the final result for the distribution $f(x)$ of reduced return currents $\delta I/I_r \equiv x$ [Eq. (78)] lies in the replacement of I_c by $\tilde{I}_c$, the quantity appropriate to the barrier top [and similar replacements in the prefactor ω_0 in (82)]. The detailed form of the crossover function $f(T)$ may be somewhat different from that in the simple RSJ model since the spectrum of the fluctuations is changed, but unless the behavior of the parameters as a function of ϕ is pathological the general behavior will not be much different from that shown in Fig. 7.

A less trivial question concerns the generalization to the case where one or more of the dimensionless parameters β_J, μ, ρ is not small. We will not consider in this paper the case $\mu \sim 1$: as noted in Sec. II, this is not realistic for most practical junctions in the limit $T \to 0$ (though at higher temperatures it may of course be of interest). Since we defined $\rho \equiv \beta_J^{-1}\mu$, the case $\beta_J \sim 1$, $\rho \gg 1$ is thereby excluded; and we have already considered the case $\beta_J \ll 1$, $\rho \ll 1$. Thus it remains to consider two further regimes: (a) $\beta_J \sim 1, \rho \ll 1$, and (b) $\beta_J \ll 1, \rho \gtrsim 1$. In case (a) we shall assume that $1-\beta_J$ is positive (otherwise no hysteresis occurs even in the absence of noise) and moreover is not too small compared to unity.

We consider first the regime $\beta_J \ll 1$, $\rho \gtrsim 1$. It is convenient to remind ourselves of a few characteristic energies, etc., as they appear at zero temperature and at temperatures $\gtrsim \hbar\omega_J$. The following is an estimate of orders of magnitude only:

	$T=0$	$kT \gtrsim \hbar\omega_J$
Average energy loss per cycle ($\overline{W}$)	$\beta_J E_0$	$\beta_J E_0$
Fluctuations in above (W_f)	$(\hbar\omega_J\beta_J E_0)^{1/2}$	$(kT\beta_J E_0)^{1/2}$
Width of "tunneling" region	$\hbar\omega_J$	$\hbar\omega_J$
Avg. number of quanta ($\sim\hbar\omega_J$) emitted *or* absorbed/cycle	$\beta_J E_0/\hbar\omega_J$	$(\beta_J E_0/\hbar\omega_J)(kT/\hbar\omega_J)$

We recall that we defined the parameter ρ to be $\beta_J^{-1}(\hbar\omega_J/E_0)$ at zero temperature and $\beta_J^{-1}(kT/E_0)$ at high temperatures, $kT \gtrsim \hbar\omega_J$. From the above we see that at zero temperature the regime $\rho \gtrsim 1$ $(R \gtrsim R_Q)$ differs from that corresponding to $\rho \ll 1$ in at least three respects:

(a) The fluctuations in the energy loss per cycle become comparable to or greater than the mean energy loss.

(b) The width of the region where tunneling (and quantum reflection) plays an important role becomes comparable to or greater than the mean fluctuation in energy loss per cycle.

(c) There is an appreciable probability of the system completing a cycle without emitting even one "typical" excitation $(\omega \sim \omega_J)$.

At high temperature, by contrast, the crossover to condition (a) still occurs at $\rho \sim 1$ $[R \sim (\hbar\omega_J/kT)R_Q]$, while the crossover to conditions (b) and (c) does not occur until $\rho \sim (kT/\hbar\omega_J)^2$ $[R \sim (kT/\hbar\omega_J)R_Q]$. Let us therefore first consider the case where (a) is relevant but (b) and (c) are not (high temperature, $(\hbar\omega_J/kT)R_Q \lesssim R \ll (kT/\hbar\omega_J)R_Q$).

The main difference between this regime and the one studied in Sec. IV C is simply that the mere fact that the system is reflected from a particular barrier is no guarantee that it will not pass it on a subsequent attempt. In principle, to discuss this regime completely quantitatively we would have to follow Mel'nikov and Sütö[8] and construct equations for the "backward-going" distribution as well as the "forward-going" one to which we have restricted ourselves up to now. However, it is easy to see that the ensuing considerations have an appreciable effect only on that part of the distribution which has $E \lesssim W_f$; for $E \gg W_f$, and in particular for $E \sim E_f$, the integral equation (68) is essentially unmodified and the solution is still given by (69). The trapping probability is still proportional to the value of (68) extrapolated to $E = 0$; only the coefficient is modified, in general being smaller than estimated in Sec. IV C. Thus the net upshot is to modify the prefactor in Eq. (73) but to leave the exponent unchanged. Thus, to the extent that we neglect logarithmic factors, the behavior of the observed widths of the retrapping distributions is identical to that found in Sec. IV C.

We finally consider the behavior in the regime where all of (a), (b), and (c) apply [$\rho \gtrsim 1$ at $T = 0, \rho \gtrsim (kT/\hbar\omega_J)$ for $kT > \hbar\omega_J$]. The effect of consideration (b) is to complicate further the process of reflection and transmission: in effect, W_f is now replaced, *as regards this process only*, by $\hbar\omega_J$. (The character of the distribution for $E \sim E_f \sim \mu^{-1/2}\hbar\omega_J \gg \hbar\omega_J$ is however still determined entirely by W_f.) Thus the effect is, again, simply a modification of the prefactor in (73) without change in the exponent. The question of the effect of consideration (c) is a delicate one and may depend critically on the behavior of the dissipation spectrum in the limit $\omega \to 0$. If we were to assume that the *only* relevant excitations are those with $\omega \sim \omega_J$ (e.g., because the spectrum has a gap at some point below but of order ω_J) then two related but different consequences would follow from (c): (1) the width of the probability distribution in ϕ space is not confined by dissipation to be small compared to 2π (see Appendix B) and (2) the coherence between the amplitudes reflected from and transmitted through a particular barrier is not necessarily destroyed by dissipation before they reach points differing by 2π (since there is an appreciable probability of completing a (half) cycle without emitting even a single excitation). The presence of a finite dissipation (more precisely, one at least of order ω) in the limit $\omega \to 0$ does not qualitatively alter conclusion (1) but knocks out (2): on this, see the argument of Ref. 41, which, somewhat counterintuitively, is apparently valid for any finite amount of "ohmic" dissipation, however small. Since at any finite temperature such dissipation exists even in the QPT mode, we may conclude that conclusion (2) never holds in any case of physical interest. Similar considerations would indicate that conclusion (1), while not necessarily false, is harmless, since parts of the probability distribution which differ by an amount of order 2π will be essentially incoherent with one another, so that the effect of "quantum uncertainty" is qualitatively no different from that of its classical counterpart.

The net upshot of this rather lengthy consideration, then, is simply that the transition from the regime $\rho \ll 1$ considered in Sec. III to the regime $\rho \gtrsim 1$ merely alters the prefactor of the expression for the retrapping probability without affecting the exponent. The main physical reason for this is that the major bottleneck in the retrapping process is simply the diffusion in energy space from the mean energy, which is of order (some fairly large number times) E_f, down to the much smaller energy ($\sim W_f$ or $\hbar\omega_f$) at which retrapping actually starts; this diffusion process is sensitive neither to quantum reflection nor to the details of the behavior over a single cycle, but is simply a random walk with a step length $\sim W_f$. Consequently, to the extent that in comparing experiment with theory we neglect logarithmic factors, then provided that both β_J and μ are small compared with unity the value of their ratio ρ is quite irrelevant, and the results of Sec. IV C apply irrespective of it.

Finally, we consider the case $\beta_J \sim 1, \rho \ll 1$. The deterministic dynamics of the system in this case was examined in Sec. III E and in particular the behavior of $W(E)$, and hence of the I-V curve in the absence of fluctuations, was seen to be of a structure essentially identical to that for $\beta_J \ll 1$, only the numerical coefficients A and B, and the characteristic voltage V_0 [cf. Eqs. (44)–(47)] being different. We note also that for $\rho \ll 1$ the fluctuations in the energy dissipated per cycle are small compared to the average value and moreover that it is still valid to replace $W_f^2(E)$ by $W_f^2(0)$ (cf. Sec. IV B). Hence the argument leading to Eq. (68) remains valid, with W_f^2 the mean-square fluctuation on the "separatrix" cycle, i.e., that corresponding to $E = 0$. However, the approximate solution (69), which relied on the fact that $\partial W/\partial E$ is small, is no longer valid. Were the quantity $\partial W/\partial E$ a constant, say α, rather than being logarithmically dependent on E, then the necessary change would simply consist in replacing the quantity 2α in the mean-square energy fluctuation $E_f^2 = W_f^2/2\alpha$ by $\alpha(2-\alpha)$. Since for the realistic case a "typical" value of the "effective" α is [cf. Eq. (44)] of order $\frac{1}{2}\beta_J|\ln \mu|$, it is plausible (though not rigorously demonstrated) that the results of Sec. IV C should apply up to a value of β_J of the order of $\frac{1}{4}|\ln \mu|^{-1}$. For larger values of β_J the main effect should be to multiply the quantity $(\delta I/I_r)^2/\ln(\delta I/I_r)$ in (75) by a function of β_J, of general order unity for β_J not too close to 1, which may also be logarithmically dependent on $\delta I/I_r$. To the extent, therefore, that we are prepared to ignore logarithmically small effects, we expect that the net result would be to multiply the overall widths of the observed retrapping distributions by a factor of order 1, but to leave the temperature dependence unchanged. (Here, of course, we are assuming that β_J is not itself a function of temperature.) However, it should be emphasized that this conclusion is plausible rather than rigorous.

V. CONCLUSION

In this paper we have considered the behavior of a Josephson junction with weak fluctuation effects ($kT, \hbar\omega_J \ll I_c\phi_0$) near the return to the zero-voltage state, with primary though not exclusive emphasis on the weakly damped case ($\beta_J \equiv 1/\omega_J RC \ll 1$). Our major results are the following:

(1) In the limit $I \to I_r$ the current-voltage characteristic in the running state is given quite generally by the formula

$$\frac{I - I_r}{I_r} = \left(\frac{AV_0}{V} + B\right)\exp - V_0/V, \qquad (97)$$

where the constants A and B are in general of order 1 (but A may be zero in special cases such as the QPT model at $T = 0$) and the quantity V_0 is defined by the effective junction parameters near the barrier top: in particular, in the weak-damping RSJ or QPT case V_0 is given up to correction terms of order β_J^2 by the simple formula

$$V_0 = \omega_J \phi_0. \tag{98}$$

(2) The *I-V* characteristic of a junction described by the QPT model (with $\hbar\omega_J \ll \Delta$) will in general exhibit a number of "forbidden voltage regions" before settling down, as $I \to I_r$, to a relatively smooth curve whose envelope is (97).

(3) For the RSJ model in the weak-damping limit the reduced width of the retrapping distribution observed in a standard ramping experiment should be given, up to logarithmically varying factors, by the expression

$$K = \text{const}\, \mu^{1/2} (2 \ln 2)^{1/2} g(T) , \tag{99}$$

where $\mu \equiv \hbar\omega_J / I_c \phi_0$ and $g(T) \equiv [f(T)]^{1/2}$, with the function $f(T)$ plotted in Fig. 7. For other models, or for stronger damping, the behavior should be qualitatively similar but the temperature dependence of K may be somewhat different from $g(T)$; in particular, for the QPT model it is considerably flatter at low temperatures.

The main significance of result (1) is that it shows that provided we know that β_J is small compared to unity (a conclusion we can draw directly from the ratio I_r/I_c) then an inspection of the running-state current voltage characteristic near return gives directly the value of the (pseudo) Josephson plasma resonance frequency $\tilde{\omega}_J \equiv (\tilde{I}_c/\tilde{C})^{1/2}$, *without* any need for knowledge of the effective resistance. For the simple RSJ or QPT model we can combine this with out knowledge of I_c to infer unambiguously the effective junction capacitance, while for more general (nonpathological) models we can derive at least its order of magnitude. This is of some importance, since the question of the effective junction capacitance at frequencies of the order of ω_J is one of the most hotly debated questions in the context of MQT and related phenomena. Knowing C, we can then obtain the "subgap" $(\omega \sim \omega_J)$ resistance quite unambiguously from $\beta_J [= (\pi/4) I_r/I_c]$.

The significance of result (2) is that it may help to explain the gaps in the current-voltage characteristic near return which seem to be a feature of many high-quality junctions examined in the laboratory.[42] The appearance of such gaps is a signature of an appreciable component of "QPT-like" behavior in the model. Of course, in most cases the voltage at which these start corresponds to $I \gg I_r$ and hence is too high to be quantitatively described by the theory of this paper (and the condition $\hbar\omega_J \ll \Delta$ may also not be satisfied) but the qualitative features should not be much changed. Needless to say, in practice, the finite (i.e., noninfinite) impedance of the leads may introduce complications: see below.

Finally the significance of result (3) needs no emphasis: experiments on the quantum retrapping behavior and in particular on the quantum-classical crossover should ideally complement the existing results on MQT and provide valuable further evidence for (or against!) the hypothesis of the applicability of the quantum formalism to the motion of a macroscopic variable.

It is clear that when combined with the well-developed theoretical predictions for MQT (for which see, e.g., Ref. 1), *assuming* the validity of the quantum-mechanical predictions in both cases, the results of the present paper can be used to put quite stringent consistency tests on any conjectured model for a given experimental junction. Suppose, for example, that we believe that a particular junction is well described, in the region $eV \ll \Delta$, by a simple effective RSJ model (cf. Sec. II) with weak damping. For measurements on the "outward" portion of the characteristic we can obtain an accurate value of the quantity I_c, and, from the MQT behavior, the "escape temperature" $T_{\rm esc}$ (see Sec. IV C). Next, by measurements of the running-state characteristic near the return to the zero-voltage state, we can obtain directly the quantity ω_J. Finally, we put this value into Eq. (86) to obtain a parameter-free prediction of the "retrapping temperature" $T_{\rm ret}$. Notice that in this argument we can check for consistency without ever having to invoke the absolute values of the reduced width (and hence know the absolute value of I_r), or even the absolute values of the temperatures involved (though predictions are of course made also for all these quantities). Thus we believe the results should be useful in determining in an unambiguous way the parameters of junctions considered, for example, for use in a "macroscopic quantum coherence" experiment.

One note of caution, however, needs to be added regarding the comparison of the theory of this paper with experiment. As is conventional, the theory is based on the assumption that the impedance of the junction leads is infinite; it will presumably break down when the *differential* dc resistance $\partial V/\partial I$ becomes comparable to the lead impedance. (Of course, it may be possible to take these into account as regards the running-state characteristics by the standard load-line construction, but it is not at all clear that the corrections to the retrapping statistics can be so simply handled.) Now with the estimates of Sec. IV C for the typical value of δI at which retrapping occurs, we find that in this region the differential conductance $\partial I/\partial V$ is given, to a crude order of magnitude, by $\mu^{1/2} I_r/V_0 \sim \beta_J \mu^{1/2} I_c/V_0$. If we take as typical the values $I_c \sim 1\,\mu$A, $V_0 \sim 1$ mV, $\beta_J \sim 10^{-3}$, $\mu \sim 10^{-4}$, we see that to avoid these complications we would need a lead impedance much greater than 100 MΩ. Thus the question of lead impedance cannot necessarily be neglected.

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APPENDIX A: BEHAVIOR NEAR THE BARRIER TOP FOR AN ARBITRARY SHUNT ADMITTANCE

The general equation of motion governing the dynamics of the phase $\phi(t)$ is of the form

$$\hat{K}[\phi(t)] = -\frac{\partial U}{\partial \phi}, \tag{A1}$$

where $U(\phi)$ is given by Eq. (5) and $\hat{K}$ is in the most general case a nonlinear integrodifferential operator. We consider, as in the main text, the motion in the region

$$\phi_{\max} - \Delta\phi < \phi(t) < \phi_{\max} + \Delta\phi,$$

where $\Delta\phi$ is small enough that any nonlinearity of $\hat{K}$ as a function of $(\phi - \phi_{\max})$ may be neglected. Then, since we are interested in "very slow" motion in which the system barely manages to pass the barrier, we assume that the motion in this region is well described by linearizing $\hat{K}[\phi(t)]$ around $\phi_{\max}$ (the implicit assumption here is that any effect of the nonlinear contributions to $\hat{K}$ will either be "forgotten" in a time of order ω_J^{-1} which is short compared to the passage time T, or can be incorporated in the boundary conditions at $\phi_{\max} \pm \Delta\phi$). If we could assume that the passage time is infinite, it would then follow that the Fourier transform $\phi(\omega)$ of $\phi(t) - \phi_{\max}$ would satisfy the equation (where we implicitly also linearize $\partial U/\partial\phi$)[43]

$$[K(\omega) - \tilde{I}_c]\phi(\omega) = 0, \quad K(\omega) \equiv i\omega Y(\omega), \tag{A2}$$

where $Y(\omega)$ is the linear admittance shunting the junction. Actually, for finite $T\phi(\omega)$ [defined as the Fourier transform of the quantity $\phi(t)$ between $-T/2$ and $T/2$] does *not* satisfy Eq. (A2), but a more complicated equation. However, we can use the following trick: We formally extend the range of the linearized equation from $-\infty$ to ∞, but impose appropriate boundary conditions at $t = \pm T/2$. Then the equation of motion is obtained by demanding that the quantity

$$\int_{-\infty}^{\infty} [K(\omega) - \tilde{I}_c]|\phi(\omega)|^2\, d\omega$$

shall be extremum as a function of $\phi^*(\omega)$ subject to the boundary conditions, which we assume can be specified in terms of linear functionals of $\phi(\omega)$ and can be incorporated via the usual Lagrange multiplier technique. As a result we obtain the solution

$$\phi(\omega) = f(\omega{:}T)/[K(\omega) - \tilde{I}_c], \tag{A3}$$

where $f(\omega{:}T)$ contains the Lagrange multipliers, and by taking the Fourier transform of $\phi(\omega)$ we obtain the actual motion $\phi(t)$. Finally, by using the fact that E is the value of $\frac{1}{2}C\dot{\phi}^2$ at $\phi = \phi_{\max}$, we can express the passage time T as a function of E, which is the object of the whole exercise.

For clarity of presentation we first carry out this procedure for the simple case $K(\omega) = -\tilde{C}\omega^2 + i\omega\tilde{R}^{-1}$, where the answer could of course be obtained much more simply by direct solution of the equation of motion in the time domain (see Sec. III). Since the equation of motion is second order in time, we need only two boundary conditions: let us specify those in the form

$$\phi(-T/2) = \phi_{\max} - \Delta\phi, \quad \phi(T/2) = \phi_{\max} + \Delta\phi. \tag{A4}$$

If we denote the corresponding Lagrange multipliers by ν and λ, respectively, the specific form of Eq. (A3) is

$$\phi(\omega) = \frac{\lambda e^{-i\omega T/2} - \nu e^{+i\omega T/2}}{\omega^2 - 2i\omega\tilde{\gamma} + \tilde{\omega}_J^2}, \tag{A5}$$

where $\tilde{\omega}_J$ and $\tilde{\gamma}$ are defined in Eq. (38) of the main text. In writing (A5) we have buried a minus sign and a factor of $\tilde{C}$ in the definitions of λ and ν. In the interval $-T/2 \leqslant t \leqslant T/2$ the form of $\phi(t)$ is therefore given by

$$\phi(t) - \phi_{\max} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{[\lambda \exp i\omega(t - T/2) - \nu \exp i\omega(t + T/2)]}{\omega^2 - 2i\omega\tilde{\gamma} + \tilde{\omega}_J^2}. \tag{A6}$$

We consider the two terms in the numerator of the integrand separately. The first must be integrated by closing the contour in the lower half plane. As to the latter, rather then closing the contour in the upper half plane, it is convenient with a view to the generalization below to make the change of variable $\omega \to -\omega$ and close again in the lower half plane. In this way we obtain an expression of the form

$$\phi(t) - \phi_{\max} = A \exp(\bar{\omega}_J - \gamma)t + B \exp - (\bar{\omega}_J + \gamma)t, \tag{A7}$$

where $\bar{\omega}_J \equiv +(\tilde{\omega}_J^2 + \gamma^2)^{1/2}$ as in (40). Needless to say, result (A7) is exactly what is obtained by elementary techniques: note that the first and second terms correspond respectively to the pole of $[K(\omega) - \tilde{I}_c]^{-1}$ and $[K(-\omega) - \tilde{I}_c]^{-1}$ in the lower half of the complex plane.

It is now clear how to generalize this result. Since the operator $\hat{K}(t)$ must be real, we have $K(-\omega) \equiv K^*(\omega)$, and hence, since $K(\omega)$ is analytic in the lower half plane, the only singularities occurring there of the quantities $[K(\pm\omega) - I_c]^{-1}$ are simple poles, whose positions we denote by $-i\omega_i^{(+)}$ and $-i\omega_i^{(-)}$, respectively. Thus we can immediately write down the result

$$\phi(t) - \phi_{\max} = \sum_i (A_i \exp \omega_i^{(+)}t + B_i \exp - \omega_i^{(-)}t). \tag{A8}$$

The prefactors A_i and B_i occurring in (A8) must be determined from the boundary conditions at $t = \pm T/2$, i.e., at $\phi(t) = \phi_{\max} \pm \Delta\phi$.[44] We now make the crucial assumption that these boundary conditions, whatever their detailed nature, do not depend explicitly on T, or at any rate not to lowest nontrivial order; this seems eminently reasonable since they arise as a result of fitting the motion near the barrier top to that in the rest of the well, which, at least as regards the second and higher derivatives,[45] should be quite insensitive to E and hence to T. It then follows immediately that the quantities A_i and B_i must be of order $\exp - \omega_i^{(\pm)}T/2$, and hence that over most of the range $-T/2 \leqslant t \leqslant T/2$ the motion is overwhelmingly determined by the terms corresponding to the smallest $\omega_i^{(+)}$ (for $t > 0$) and smallest $\omega_i^{(-)}$ (for $t < 0$). Consequently, the passage time T itself is also determined overwhelmingly by these terms. It is clear that if we define as in (49),

$$\bar{\omega}_J \equiv \tfrac{1}{2}(\omega_{\min}^{(+)} + \omega_{\min}^{(-)}), \quad \bar{\gamma} \equiv \tfrac{1}{2}(\omega_{\min}^{(-)} - \omega_{\min}^{(+)}),$$

then we recover all the results of the simple model discussed above, which is in fact a special case.

In the above argument it was implicitly assumed that the quantities $\omega_{\min}^{(+,\cdots)}$ (though not necessarily the higher $\omega_i^{(+,\cdots)}$) are real. Although at the time of writing we have not been able to exclude the opposite case rigorously, we know of no realistic example of it and it seems likely that any example would have to be rather pathological from a physical point of view.

APPENDIX B: THE c-NUMBER QUANTUM LANGEVIN EQUATION

It is known clasically that when in contact with a dissipative environment, a particle will be subject both to a random fluctuating force and to the associated irreversible energy dissipation. At high temperature, the correlation spectrum of the random noise is effectively "white," and the well-known Langevin equation has been very efficient for the study of the behavior of such dissipative systems. As the temperature decreases, the quantum nature of both the particle and the environment will manifest itself. One is thus forced to study the problem in an operator basis, treating the whole system quantum mechanically. In certain cases, it is found that the classical Langevin equation can be naturally generalized into a quantum one, provided we replace both the coordinate of the particle and the noise of the environment by their corresponding quantum operators (see, e.g., Ref. 46 and references therein). It is further found (cf. Refs. 32 and 47) that if the dissipation is strong, the quantum features of the motion of the particle will be heavily suppressed. This leads us to another slightly different approximation, the c-number quantum Langevin equation or, as it is called by Schmid,[32] the "quasiclassical Langevin equation," in which only the noise spectrum of the classical Langevin equation is modified by the quantum mechanical considerations. In this Appendix, we shall give a plausible path-integral derivation of the c-number Langevin equation and then investigate its conditions of validity. Our derivation is similar to that given by Schmid.[32]

The derivation

In order to formulate the problem quantum mechanically, one is generally required to take specific models of the environment. Here we consider a simple yet practically important case, namely, where the environment is taken as a bath of harmonic oscillators with the distribution of their frequencies being ohmic (see below). We start with the following Hamiltonian:

$$\hat{H} = \hat{H}_p + \hat{H}_{p-e} + \hat{H}_e\,, \tag{B1a}$$

where we denote $\hat{H}_p$, $\hat{H}_e$, and $\hat{H}_{p-e}$, respectively, the Hamiltonians for the particle, for the enviornment, and the coupling between them:

$$\hat{H}_p = (\hat{p}^2/2m) + V(\hat{x})\,, \tag{B1b}$$

$$\hat{H}_e = \frac{1}{2}\sum_j\left(\frac{\hat{p}^2}{2m_j} + m_j\omega_j^2\hat{x}_j^2\right), \tag{B1c}$$

$$\hat{H}_{p-e} = \hat{x}\sum_j C_j\hat{x}_j + \frac{1}{2}\hat{x}^2\sum_j\left(\frac{C_j^2}{m_j\omega_j^2}\right). \tag{B1d}$$

In $\hat{H}_{p-e}$ we have deliberately introduced a counterterm to cancel the implicit adiabatic potential shift due to the coupling (cf. Ref. 28, Sec. III, and Appendix A). It is convenient to introduce another quantity, the "spectrum of the bath,"

$$J(\omega) \equiv \sum_j \frac{\pi C_j^2}{2m_j\omega_j}[\delta(\omega_j - \omega) - \delta(\omega_j + \omega)]\,. \tag{B1f}$$

We shall see that $J(\omega)$ is of particular importance in the resultant formalism. Since we are only interested in the dynamics of the particle itself, we shall confine ourselves to the reduced density matrix defined by tracing out the bath degrees of freedom at the final time t_f. Nevertheless, one has to face an associated problem, namely, how to choose as natural as possible a form for the initial density matrix of the whole system. A popular choice is the so-called product state,

$$\hat{\rho}(t_i) = \hat{\tilde{\rho}}(t_i)\exp(-\beta H_e)\,, \tag{B2a}$$

with

$$\langle Q_i + r_i/2|\hat{\tilde{\rho}}(t_i)|Q_i - r_i/2\rangle \equiv \rho(Q_i,r_i,t_i) \tag{B2b}$$

being an arbitrary normalized function, where we have introduced, respectively, in the particle density matrix $\langle x|\rho|y\rangle$ the "center-of-mass" coordinate, $Q\equiv x + y/2$, and the relative coordinate, $r\equiv x - y$. We will take this choice. Note that it is not entirely obvious, as argued by some other people, that different choices will modify only the transient behavior of the dynamics. However, we will not address this question here (for further interest, see, e.g., Refs. 46 and 48). Write the evolution of the reduced density matrix in the following form:

$$\rho(Q_f,r_f,t_f) \equiv \int dQ_i\,dr_i\,J(Q_f,r_f,t_f;Q_i,r_i,t_i) \times\rho(Q_i,r_i,t_i)\,. \tag{B3a}$$

It is now straightforward to find that the kernel can be expressed in terms of a double path integral (see, e.g., Refs. 31 and 32):

$$J(Q_f,r_f,t_f;Q_i,r_i,t_i) = \int_{Q(t_i)=Q_i,r(t_i)=r_i}^{Q(t_f)=Q_f,r(t_f)=r_f} DQ(t)Dr(t)\exp[S(Q,r)]\,, \tag{B3b}$$

where

$$\begin{aligned} S(Q,r) \equiv{}& \frac{i}{\hbar}\int_{t_i}^{t_f} dt\,[m\dot{r}\dot{Q} + V(Q - r/2) - V(Q + r/2)] \\ &- \frac{i}{\hbar}\int_{t_i}^{t_f} dt\,r(t)\Big(\alpha_1(t)Q_i + \int_{t_i}^{t} dt'\,\alpha_1(t - t')\dot{Q}(t')\Big) \\ &- \frac{1}{\hbar^2}\int_{t_i}^{t_f} dt\int_{t_i}^{t} dt'\,r(t)\alpha_2(t - t')r(t')\,, \end{aligned} \tag{B3c}$$

with

$$\alpha_1(t) \equiv \int_{-\infty}^{+\infty} d\omega\,\frac{J(\omega)}{\pi\omega}\cos\omega t\,, \tag{B3d}$$

$$\alpha_2(t) \equiv \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\hbar J(\omega)\coth\left(\frac{\beta\hbar\omega}{2}\right)\cos\omega t\,. \tag{B3f}$$

It is of particular interest to consider the case of ohmic damping, i.e., take, in the limit of infinite number of environmental degrees of freedom, $J(\omega)\to\eta\omega$ where η will be identified as the viscosity.[44] Under this assumption, Eq. (B3c) can be reduced to

$$S(Q,r)\equiv-\frac{i}{\hbar}\eta Q_i r_i+\frac{i}{\hbar}\int_{t_i}^{t_f}dt\,[m\dot{r}\dot{Q}-\eta r\dot{Q}+V(Q-r/2)-V(Q+r/2)]-\frac{1}{\hbar^2}\int_{t_i}^{t_f}dt\int_{t_i}^{t}dt'\,r(t)\alpha_2(t-t')r(t')\,. \tag{B4}$$

It is important to notice that the last term in Eq. (B4) suppresses fluctuations in $r(t)$. Namely, if the viscosity η is large, only the portion of paths with small $r(t)$ will make important contributions to the path integral (B3b); we leave the detailed justification of this statement to the next section. Expanding

$$V\left(\frac{Q-r}{2}\right)-V\left(\frac{Q+r}{2}\right)=-V'(Q)r-\frac{V'''(Q)}{24}r^3-...,\tag{B5}$$

we would then be led to the suggestion that only the first-order term in the right-hand side will be significant. On the other hand, to complete the resulting path integral, we transform the real part of the exponent into the following form containing a random stochastic force

$$\exp\left(-\frac{1}{\hbar^2}\int_{t_i}^{t_f}dt\int_{t_i}^{t}dt'\,r(t)\alpha_2(t-t')r(t')\right)\equiv\left\langle\exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f}dt\,r(t)\xi(t)\right)\right\rangle \tag{B6a}$$

with the random Gaussian force satisfying

$$\langle\xi(t)\xi(t')\rangle=\alpha_2(t-t').\tag{B6b}$$

After these manipulations, we are ready to perform the path integral. We first integrate over $r(t)$, since the exponent is now linear with respect to it. Discretizing the time by $\epsilon=(t_f-t_i)/N$, one would find that the integration gives nothing but a δ function over the path space of $Q(t)$,

$$\prod_{k=1}^{N-1}\delta\left(-m\frac{Q_{k+1}-2Q_k+Q_{k-1}}{\epsilon}-\eta(Q_k-Q_{k-1})-\epsilon V'(Q_k)+\epsilon\xi_k\right).\tag{B7}$$

This clearly indicates that the possible paths for $Q(t)$ are restricted to the solution(s) of the Langevin equation

$$m\ddot{Q}+\eta\dot{Q}+V'(Q)=\xi(t),\quad Q(t_i)=Q_i,\quad Q(t_f)=Q_f\,.\tag{B8}$$

To integrate over $Q(t)$, we are led to determine the Jacobian of the transformation embedded in Eq. (B7). We do not intend to give the detailed algebra here, but it turns out, quite desirably, to be the density of paths around the solutions of Eq. (B8), i.e., it is proportional to $|\partial Q_f/\partial\dot{Q}_i|^{-1}$. Eventually, we obtain for a given $\xi(t)$ the following expression:

$$J(Q_f,r_f,t_f;Q_i,r_i;t_i:[\xi(t)])=\left[\left(\frac{2\pi\hbar}{m}\right)\left|\frac{\partial Q_f}{\partial\dot{Q}_i}\right|\right]^{-1}\times\exp\left(\frac{i}{\hbar}[m\dot{Q}_f r_f-(m\dot{Q}_i+\eta Q_i)r_i]\right),\tag{B9}$$

where $\dot{Q}_f$ and $\dot{Q}_i$ are obtained by solving Eq. (B8). It can be easily checked that the final density matrix is automatically normalized.

In order to see how the Langevin equation describes the motion of the particle, let us study, for example, the case of a wave packet moving according to (B9). Take

$$\rho(Q_i,r_i,t_i)=(2\pi\sigma^2)^{-1/2}\exp\left[\left(\frac{(Q_i-X_i)^2}{2\sigma^2}+\frac{r_i^2}{8\sigma^2}\right)+\frac{i}{\hbar}P_i r_i\right]\tag{B10}$$

and assume σ is small; we then have

$$\rho\{Q_f,r_f,t_f:[\xi(t)]\}\cong\left(\frac{2\sigma^2m^2}{\hbar^2}\right)^{1/2}\left(\left|\frac{\partial Q_f}{\partial\dot{Q}_i}\right|\right)^{-1}\times\exp\left[\frac{i}{\hbar}m\dot{Q}_f r_f-\frac{\sigma^2}{2}\left(\frac{m}{\hbar}\frac{\partial\dot{Q}_f}{\partial X_i}\right)^2r_f^2-\frac{2\sigma^2}{\hbar^2}(P_i-m\dot{Q}_i-\eta X_i)^2\right].\tag{B11}$$

The distribution of Q_f is implicitly through $\dot{Q}_i$. In fact, to determine the average final position and final velocity of the particle, we only need to weigh $Q_f(X_i,\dot{Q}_i,t_f-t_i)$ and $\dot{Q}_f(X_i,\dot{Q}_i,t_f-t_i)$ by the last exponent in Eq. (B11). In the presence of dissipation, the initial effects will, in general, quickly vanish after a certain time. Besides, in the semiclassical limit, the distribution of $\dot{Q}_i$ is usually sharp. In this way, one sees clearly how the c-number quantum Langevin equation (B8) determines the motion of the particle.

It is interesting to see that the most probable initial velocity is $(P_i-\eta X_i)/m$, rather than P_i/m. This is, in fact, due to our specific choice of the initial density matrix, which is equivalent to postulating that initially the environment is in what would be its equilibrium configuration if the particle were sitting at $X_i=0$. Thus, as soon as we switch on the coupling, the particle gains an extra momentum toward the origin at time t_i^+. (This "switch-on" of the coupling would also give a divergence in the mean square position, if the ohmic spectrum was taken seriously without high-frequency cutoff, cf. Ref. 46).

The validity criterion

Having given a plausible derivation of the c-number quantum Langevin equation, we now turn to a similar study of the validity criterion. Notice that the result in the last section would be exact if the potential of the particle were quadratic; it is an approximation only to the extent that the potential is anharmonic. Therefore, we try to introduce a typical length scale L_0 over which the anharmonic part of the potential would manifest itself. Clearly, for $|r(t)|/L_0\ll1$, the higher-order terms in $V(Q+r/2)-V(Q-r/2)$ are, in general, insignificant. Thus, a reasonable criterion would be that for any fluctuation of $r(t)$ with $|r(t)/L_0| \sim 1$, we have

$$\frac{1}{2\hbar^2}\int_{t_i}^{t_f} dt\,dt'\; r(t)\alpha_2(t-t')r(t') \gg 1 . \qquad \text{(B12)}$$

Notice that the function $\alpha_2(t)$ can actually be evaluated if we introduce a cutoff $\exp(-\delta|\omega|)$ at high frequency, where δ is much shorter than the typical time scale of the particle:

$$\alpha_2(t) = \frac{\eta\hbar}{\pi}\frac{d}{dt}\left[\frac{\pi}{\beta\hbar}\coth\left(\frac{\pi t}{\beta\hbar}\right) - \frac{1}{t} + \frac{t}{t^2+\delta^2}\right]. \qquad \text{(B13)}$$

In the low-temperature limit, the left-hand side of (B12) is then rather independent of the time interval $t_f - t_i$. Moreover, increasing temperature always helps the inequality. Thus, we obtain for the validity of the Langevin description a crude criterion,

$$\eta L_0^2/\pi\hbar \gg 1. \qquad \text{(B14)}$$

For the RSJ model of the current-biased Josephson junction discussed in this paper, for which $L_0 \sim 1$ and $\eta \sim \phi_0^2/R$, this criterion simply says

$$R \ll R_Q \equiv h/4e^2 \sim 6.5\ \text{k}\Omega, \qquad \text{(B15)}$$

which agrees with the result of Schmid.[32] Finally, it should be emphasized that the length scale L_0 varies for different problems. For example, in the case of macroscopic quantum tunneling in a Josephson junction, the relevant length scale L_0 of ϕ is usually much smaller than 1. Besides, in this case even if the criterion (B14) holds, it is still an open question whether it is legitimate to use the Langevin equation to compute the exponentially small escape probability (cf. Ref. 50). On the other hand, if one is interested in the global features of the dynamics of the particle, the range of validity of this Langevin approach might well be extended beyond (B14).

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[12]Note that throughout this paper we take as our two independent "fundamental constants" ϕ_0 and $\hbar$, not e and $\hbar$. This is essential for a clear identification of genuinely "quantum" fluctuations.

[13]W. Zwerger, Phys. Rev. B **35**, 4737 (1987).

[14]Unless otherwise stated we confine ourselves in this paper for simplicity to the case where the superconductors on the two sides of the junction are identical, so that the energy gaps Δ_1 and Δ_2 are equal.

[15]N. R. Werthamer, Phys. Rev. **147**, 255 (1966).

[16]U. Eckern, G. Schön, and V. Ambegaokar, Phys. Rev. B **30**, 6419 (1984).

[17]Note that it would be inconsistent in this context to keep the "capacitance-renormalization" term $\delta C(1 - 1/3\cos\phi)$ (see, e.g., Ref. 16), since $\delta C/C$ is of order $(\omega_J/\Delta)^2$, and in assuming that (14) describes the motion in the washboard potential we are already dropping other terms of this order.

[18]W. A. Schlup, J. Phys. C **7**, 736 (1974).

[19]See especially Ref. 7, Eq. (8). Schlup (Ref. 18) obtains a result (at the foot of p. 741) of the general form $\exp - V_0/V$, but his expression for V_0 appears to be quite different from ours.

[20]As we will see, however, there are some differences of detail due to the ϕ dependence of the effective conductance.

[21]It would be necessary to retain this term if we wished to discuss the "even-subharmonic" features of the current-voltage characteristic (i.e., these occurring at $eV = \Delta/n$, n integral), see, e.g., Ref. 22. However, in the weak-damping limit these are muchless spectacular than the "odd-subharmonic" features discussed below.

[22]D. G. McDonald, E. G. Johnson, and R. E. Harris, Phys. Rev. B **17**, 1028 (1976).

[23]A. J. Leggett, Phys. Rev. B **30**, 1208 (1984).

[24]Actually, for the QPT model at $T = 0$, if we define the "effective resistance" R by the statement that $\beta_J \equiv (\pi/4)(I_r/I_c) \equiv (\omega_J RC)^{-1}$ (cf. below), then it follows from Eqs. (32) and (92) that the ratio is of order $(2\Delta/\hbar\omega_J)(R/R_Q)$, and hence for this case the condition $R \ll R_Q$ should be replaced everywhere below by the stronger condition $R \ll (\hbar\omega_J/2\Delta)R_Q$. To avoid interrupting the flow of the argument we shall not explicitly insert this and related qualifications below (we note that for this case the quantity $\hbar\omega_J/2\Delta$ is only "weakly" small, $\sim \ln\beta_J$).

[25]Cf. also Ref. 11.

[26]D. J. Van Harlingen (private communication).

[27]R. P. Feynman and F. L. Vernon, Jr., Ann. Phys. **24**, 118 (1963).

[28]A. O. Caldeira and A. J. Leggett, Ann. Phys. (NY) **149**, 374 (1983).

[29]A. N. Cleland, J. M. Martinis, and J. Clark, Proceedings of the 18th International Conference on Low Temperature Physics [Jpn. J. Appl. Phys. Suppl. **26-3**, 1427 (1987)].

[30]V. Ambegaokar and U. Eckern, Z. Phys. B **69**, 399 (1987).

[31]A. O. Caldeira and A. J. Leggett, Physica **121A**, 587 (1983).

[32]A. Schmid, J. Low Temp. Phys. **49**, 602 (1982).

[33]However, W is now regarded as a function of E, which is *not* necessarily set equal to its steady-state value.

[34]Note for future reference that this replacement does *not* depend on the assumption $\beta_J \ll 1$.

[35]E.g., write $f_s(E) = \exp - \phi(E)$ and, on the right-hand side of (68) expand $\phi(E')$ and $W(E')$ around $E' = E$.

[36]A. J. Leggett, *Prodeedings of the 6th International Conference on Noise in Physical Systems,* edited by P. H. E. Meijer, R. D. Mountain, and R. J. Soulen, Jr., NBS Special Publication No. 614 (U. S. Govt. Printing Office, Washington, DC, 1981), p. 355.

[37]H. Grabert, U. Weiss, and P. Hanggi, Phys. Rev. Lett. **25**, 2193 (1984).

[38]Recall that I is proportional to W and V to $\ln E$.

[39]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, translated by J. B. Sykes and J. S. Bell (Pergamon, London, 1959), p. 178.

[40]But see the caveat in the footnote near the beginning of this section.

[41]W. Zwerger, A. T. Dorsey, and M. P. A. Fisher, Phys. Rev. B **34**, 6518 (1986).

[42]R. A. Webb (private communication).

[43]Conventions for Fourier transforms, etc., are as in Ref. 23.

[44]Here we have a course implicitly assumed that one pair of boundary condition remains the conditions that $\phi(\pm T/2) = \phi_{\max} \pm \Delta\phi$.

[45]As in the familiar Newtonian problem, the fact that we have specified $\phi(\pm T/2)$ absolves us from having to specify the first derivative.

[46]P. de Smedt, D. Durr, J. L. Lebowitz, and C. Lirerani (unpublished).

[47]R. Koch, D. J. Van Harlingen, and J. Clarke, Phys. Rev. Lett. **45**, 2132 (1980).

[48]V. Hakim and V. Ambegaokar, Phys. Rev. A **32**, 423 (1985).

[49]Note that this substitution removes the discrete nature of $J(\omega)$, thus producing the irreversibility of the dissipative dynamics. For more discussion, see Ref. 31.

[50]U. Eckern, Proceedings of the 18th International Conference on Low Temperature Physics [Jpn. J. Appl. Phys. Suppl. **26-3**, 1399 (1987)].