Exactly Soluble Model of Fractional Statistics

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Using notions of supersymmetry we present an exactly soluble model of anyons with both statistical and scalar interactions in 2+1 dimensions. We demonstrate that half-statistics particles with two spin flavors condense into a local singlet state which is both a charge superfluid and a "spin metal" in the sense that there is charge-pairing off-diagonal long-range order with gapless charge excitations but a gap in the collective spin-mode spectrum. The present results shed considerable light on the mean-field theory of fractional statistics.

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An anyon is a particle in 2+1 dimensions obeying fractional statistics[1] and may be viewed as a hard-core boson (or a fermion) to which a flux tube is attached. The statistical interaction is highly nontrivial and it has so far resisted full solution. Some progress can be made using a mean-field theory of statistics[2] in which the statistical flux is replaced by a uniform magnetic field B_{eff}. This idea[2] was first put to practical use in the theory of the fractional quantum Hall effect[3-5] (FQHE), where fermions in an external field B_{ext} can be mapped onto bosons in zero mean field ($B_{\text{eff}}+B_{\text{ext}}=0$). The ordering associated with the FQHE gap is revealed in the form of *algebraic* off-diagonal long-range order (ODLRO) for the composite particles (bosons plus flux tubes).[4] More recently this idea has been greatly expanded and extended by Laughlin[6,7] to the case of semions (particles with statistical angle $\theta/\pi=1/2$) for which the mean-field theory yields fermions filling the two lowest Landau levels. Laughlin pointed out the remarkable fact that pairs of semions can condense to form a Bose superfluid.

The mean-field approximation has several peculiarities[8-10] since the purely quantum Aharonov-Bohm phase is replaced by a classical Lorentz force and the particles see a preferred (quantum) length scale $l\equiv(\hbar c/eB_{\text{eff}})^{1/2}$. This turns out *not* to be a problem in the FQHE since the incompressible Laughlin state has a gap and its density is pinned at precisely this same length scale due to the (physical) external field.[4] However, mean-field theory for semions incorrectly suggests the existence of an excitation gap analogous to that in the integer Hall effect. Augmenting the mean-field theory with fluctuations at the RPA level[11] restores the linearly dispersing density mode expected in a superfluid. A local increase in density is accompanied by a compensating increase in the local B_{eff} such that the lowest two Landau levels can remain filled and there is no preferred length scale.[9,12]

The purpose of this paper is to present an exactly soluble model of anyons which sheds considerable light on all these points and which totally bypasses difficulties inherent in the mean-field solutions. The model differs in one (highly) nontrivial way from the usual one in that the particles have an *attractive* hard-core interaction. (The effect of the hard-core repulsion that is usually considered is to prevent intersection of the boson world lines and hence make the homotopy class of their braiding well defined.[1]) The present model was inspired by recent applications of supersymmetry and the Atiyah-Singer index theorem to particles in an arbitrary magnetic field.[13] We will prove below that the index theorem can be extended to the nontrivial many-body case where the flux is *not fixed in time but is carried on the particles themselves*; i.e., for anyons. Consider the pair of Hamiltonians

$$H^{\pm}=\sum_{j=1}^{N}\boldsymbol{\Pi}_j\cdot\boldsymbol{\Pi}_j\mp B_j\,,$$

with

$$B_j\equiv\nabla_j\times\mathbf{a}_j=\sum_{k\neq j}2\theta\delta^2(\mathbf{r}_j-\mathbf{r}_k)\,,$$

where $\boldsymbol{\Pi}_j\equiv-i\nabla_j+\mathbf{a}_j$, 2D pseudoscalar notation is used for the cross product, and without loss of generality we take $\theta>0$. We could interpret this as the Hamiltonian of a collection of spin-aligned particles which have a Zeeman energy corresponding to g factor 2, but prefer to view the system as a collection of spinless fractional-statistics particles with δ-function scalar interaction (attractive for H^+, repulsive for H^-). H can be factored into an $N=2$ supersymmetric form

$$H^{\pm}=\sum_j(Q_j^{\pm})^{\dagger}(Q_j^{\pm})\,,$$

where $Q_j^{\pm}\equiv\Pi_j^x\mp i\Pi_j^y$. Taking the vector potential to be divergenceless, we define $S[\mathbf{r}]$ by $a_j^{\mu}=\epsilon^{\mu\nu}\partial_j^{\nu}S$. We see that $\nabla_j^2S=-B_j$ and hence S is proportional to the Coulomb energy of an associated classical plasma

$$S=-\frac{\theta}{\pi}\sum_{k<j}\ln|\mathbf{r}_j-\mathbf{r}_k|\,.$$

It is straightforward to verify that the state defined by

$$\Psi^{+}=f[\bar{z}]e^{+S}=f[\bar{z}]\prod_{k>j}|z_j-z_k|^{-\theta/\pi},$$

where f is any entire function of the $\bar{z}_j$'s (with $\bar{z}_j \equiv x_j-iy_j$), is annihilated by Q_j^{+} for every j and hence[14] is an exact zero-energy eigenstate of H^{+}. The zero-energy eigenstates of H^{-} are

$$\Psi^{-}=f[z]e^{-S}=f[z]\prod_{k<j}|z_j-z_k|^{+\theta/\pi}.$$

These latter are unfortunately not normalizable except in the presence of a finite external field ($B_{\mathrm{ext}}<0$) which introduces the extra factor $\exp(+B_{\mathrm{ext}}\sum|z_j|^2/4)$ into the solution. These states were previously known.[12,15]

We turn now to the question of singlet superconductivity for half-statistics particles which carry either ordinary spin[16] or a flavor quantum number related to species doubling on a lattice.[17] (Note spin as used here has nothing to do with the spin or Zeeman energy used in the supersymmetry.) Consider the following state written in "Greek-Roman" notation[18] $\Phi=\mathcal{A}\Psi_{\mathrm{GR}}[z](\alpha,\alpha,\ldots,\alpha,\beta,\beta,\ldots,\beta)$, where $\mathcal{A}$ is the antisymmetrizer, and α and β are the up and down spinors, respectively. The spatial part of the wave function is given (in the fermion representation) by

$$\Psi_{\mathrm{GR}}=\prod_{i<j}(\bar{z}_i-\bar{z}_j)(\bar{z}_{[i]}-\bar{z}_{[j]})e^{S},$$

with

$$e^{S}\equiv\prod_{i<j}|z_i-z_j|^{-1/2}|z_{[i]}-z_{[j]}|^{-1/2}\times\prod_{k,l}|z_k-z_{[l]}|^{-1/2}. \tag{1}$$

In our notation $i=1,2,\ldots,N$ and $[j]=N+1,N+2,\ldots,2N$ refer to spin up and down, respectively. From the generalized index theorem, Φ is an exact, zero-energy eigenstate of the semion Hamiltonian with the appropriate δ-function attraction among all the particles (independent of flavor). The Hamiltonian is thus spin independent and Φ is a spin singlet since it manifestly obeys the Fock cyclic condition.[18] [Indeed, Φ has the same spin symmetry as, and is (as will be shown below) the unique adiabatic continuation of, the mean-field solution consisting of both spin states of the lowest Landau level being filled.]

Provided that we are interested in expectation values of operators which do not flip spins, we are allowed to deal only with Ψ_{GR} and ignore the antisymmetrization (since nontrivial permutations of the spinors yield orthogonal states[18]). It is a remarkable fact that

$$|\Psi_{\mathrm{GR}}|^2=\exp\left(-\beta\sum_{i<j}^{2N}(-q_iq_j)\ln|z_i-z_j|\right),$$

where the sum runs over all $2N$ particles, $\beta=1$, and $q_i=\pm 1$ for up and down spin, respectively. That is, the particle distribution is identical to that of the classical neutral Coulomb gas[19] with spin playing the role of Coulomb "charge." The coupling constant is $\Gamma\equiv\beta q^2=1$, which is well on the high-temperature side of the Kosterlitz-Thouless transition occurring[20] at $\Gamma=4$. Hence this state is a spin "metal." That is, opposite spins are *not* bound into real-space pairs (as they would be for $\Gamma>4$), but rather spin currents are free to flow and there is perfect, "metallic" screening of isolated spins with a screening wave vector given in the Debye approximation by[20] $\kappa^2=2\pi n_0\beta q^2$, where n_0 is the mean particle density. Thus this state is not merely a singlet but a *local singlet*[7] in the precise sense defined by Girvin.[21] In the single-mode approximation[9,22] (SMA) the spin-density collective excited state is $\Psi_{\mathbf{k}}=\rho_{\mathbf{k}}^{\sigma}\Phi$ with

$$\rho_{\mathbf{k}}^{\sigma}\equiv\sum_{j=1}^{2N}\sigma_j^z e^{i\mathbf{k}\cdot\mathbf{r}_j}.$$

The energy of the collective mode is $\Delta(k)=f(k)/s^{\sigma}(k)$, where $f(k)\equiv\hbar^2k^2/2m$ is the oscillator strength and the spin structure factor is given in the Debye approximation by

$$s^{\sigma}(k)\equiv N^{-1}\langle\Phi|\rho_{-\mathbf{k}}^{\sigma}\rho_{\mathbf{k}}^{\sigma}|\Phi\rangle=k^2/(k^2+\kappa^2).$$

Thus there is a finite "spin-plasmon" gap $\Delta(k)=\hbar^2(k^2+\kappa^2)/2m$, as is appropriate for a spin metal. Note that the corresponding $S^z=\pm 1$ spin waves are degenerate with the $S^z=0$ mode derived here. (The corresponding structure factors involve flipped spins but are the same as s^{σ} because of rotational invariance of the singlet state Φ.)

At the same time, this spin metal is also a charge superfluid. To see off-diagonal long-range order[23,24] in the two-body density matrix, it is necessary (as in the FQHE[4]) to make a singular gauge change[1]

$$\tilde{\Psi}_{\mathrm{GR}}=\Psi_{\mathrm{GR}}\prod_{kl}\frac{z_k-z_{[l]}}{|z_k-z_{[l]}|}.$$

The effect of this is to cause each particle to see $-\frac{1}{2}$ instead of $+\frac{1}{2}$ flux quantum on particles of the opposite spin. This in turn renders an up-down spin pair "gauge neutral" with respect to the statistical field. Now consider the two-body density matrix

$$\tilde{\rho}(z_1,z_{[1]};z_1',z_{[1]}')=Z^{-1}N(N-1)\prod_{k>1}\int d^2z_k\int d^2z_{[k]}\,\tilde{\Psi}_{\mathrm{GR}}^{*}(z_1',z_2,\ldots,z_N;z_{[1]}',z_{[2]},\ldots,z_{[N]})\times\tilde{\Psi}_{\mathrm{GR}}(z_1,z_2,\ldots,z_N;z_{[1]},z_{[2]},\ldots,z_{[N]}),$$

where Z is the norm of $\Psi_{\rm GR}$. Following the discussion in Ref. 4, this expression can be analyzed as a neutral plasma of $2(N-1)$ charges plus four "impurities" located at z_1, $z_{[1]}$, z_1', and $z_{[1]}'$. The state exhibits true two-body ODLRO since if $z_1 \approx z_{[1]}$ and $z_1' \approx z_{[1]}'$, the pairs form charge- and gauge-neutral objects which are invisible to the plasma. ("Charge" here means the fake Coulomb charge corresponding to spin.) The free energy of the plasma and hence $\tilde{\rho}$ become independent of $|z_1 - z_1'|$ at large distances. One can estimate the coherence length to be $\xi \sim n_0^{-1/2}$ (where n_0 is the mean density) since phase fluctuations in the integration will become severe once $|z_1 - z_{[1]}|$ exceeds the average particle spacing. This is similar to the RPA result for spinless semions.[11]

Associated with the ODLRO is a gapless Goldstone density-wave mode. For a system of N anyons with point flux tubes and a δ-function scalar interaction of strength g, simple dimensional analysis shows that the ground-state energy per particle is linear in the density n, $E = \lambda(\theta, g) N n$, so that the static susceptibility is precisely $\chi(0) = -1/2\lambda$. In the single-mode approximation,[9,22] the speed of the collective density mode is $v = [-n/m\chi(0)]^{1/2} = (2n\lambda/m)^{1/2}$, where m is the mass. Thus, knowing only the ground-state energy, we can compute the speed of the collective mode. The supersymmetric point is like the ideal Bose gas in that $\lambda = 0$ (however, any small perturbation will render λ finite). It also follows that Ψ must be scale invariant (i.e., homogeneous) at the supersymmetric point.

The present model contains both a scalar and a statistical gauge-field attraction.[7] Therefore we cannot prove that the gauge forces alone are sufficient to produce pairing,[7] although this appears to be well established.[7,9,11,23–25] In a physical system, we expect the "flux tubes" to result from an effective low-energy action for some bare theory and hence the flux tubes will have some finite size (i.e., there will generically be $F_{\mu\nu}F^{\mu\nu}$ and higher terms in the Chern-Simons action). Fortunately the index-theorem results still apply in this case. We need only replace Eq. (1) by the corresponding Boltzmann factor for a plasma of finite-size particles of the appropriate form factor. Using this result we can explicitly adiabatically continue the state from the mean-field limit (infinite diameter flux tubes) to the anyon limit (infinitesimal diameter flux tubes). However, a technically more convenient procedure is to follow the heuristic prescription of Greiter and Wilczek[26] which keeps the flux-tube diameter infinitesimal but gradually "gathers up" flux from the background and adds it to the flux tubes. Then we have at any stage in this process Eq. (1) replaced by

$$e^{S_\alpha} \equiv e^{(1-\alpha)S} \exp\left[-\frac{\alpha}{4l^2}\sum_{k=1}^{N}(|z_k|^2 + |z_{[k]}|^2)\right],$$

where $l^2 \equiv 1/2\theta n_0$, and S is the same as in Eq. (1). For any value of α between the mean-field limit ($\alpha = 1$) and the anyon limit ($\alpha = 0$) this is the exact solution. Within the SMA the collective spin excitation gap is independent of α, but because of the preferred length scale $l/\sqrt{\alpha}$ set by the mean field, there is a charge gap which varies as $\Delta_c = \alpha\hbar\omega_c$, where $\hbar\omega_c$ is the mean-field Landau-level spacing.

These arguments suggest that a good variational wave function for *spinless* semions is

$$\Psi = P_2[\bar{z}, z]e^S, \qquad (2)$$

where S is given by Eq. (1) and P_2 is the polynomial part of the mean-field solution (for two filled Landau levels). Because P_2 depends on both z and $\bar{z}$, Ψ does not fulfill the index-theorem conditions and so must have positive energy. However, the extra freedom permits P_2 to vanish linearly when any two particles approach each other, making this an allowed wave function even for the case of hard-core *repulsive* interactions. Numerical estimates of the variational energy of this state are under way.[27]

In the Laughlin plasma analogy[4,7] for the mean-field wave function, the Gaussian piece plays the role of a neutralizing background charge. Hence we can view the supersymmetric solution presented here as gathering up the background charge and placing it on the particles, thereby converting the problem from an incompressible one-component plasma to a compressible (scale-invariant) neutral gas. One can gain further insight into this by making a Hubbard-Stratonovic decoupling of the e^S term (ignoring various formal divergences[28]),

$$\Psi \sim \int D\phi \exp\left[-\frac{1}{4\theta}\int d^2r |\nabla\phi(\mathbf{r})|^2\right]\psi_\phi[z],$$

where

$$\psi_\phi[z] \equiv P_2[\bar{z}, z]\exp\left[-\int d^2r\, \rho(\mathbf{r})\phi(\mathbf{r})\right].$$

We can interpret Ψ as the RPA wave function for the problem in the sense that it is a linear superposition of (approximate) mean-field states ψ_ϕ corresponding to a fluctuating background field $[B(r) = -\nabla^2\phi = 2\theta\rho]$. Because of the plasma neutrality, these states tend to have the particles follow the background flux (charge) as it fluctuates, in the same spirit as the RPA.[11]

In this context it is useful to consider the test case[6] of ordinary bosons, treated as fermions plus flux tubes at $\theta/\pi = 1$. The mean-field theory gives one filled Landau level (and a large gap)

$$\Psi_{\rm MF} = \prod_{i<j}(\bar{z}_i - \bar{z}_j)\exp\left[-\frac{1}{4l^2}\sum_k |z_k|^2\right]. \qquad (3)$$

Moving the background charge onto the particles in the manner discussed above yields

$$\Psi = \prod_{i<j}(\bar{z}_i - \bar{z}_j)|\bar{z}_i - \bar{z}_j|^{-1}. \qquad (4)$$

Making a singular gauge transformation[1] to turn this back into a boson state gives precisely $\Psi = 1$, which is of course the exact ground state for free bosons and, in an important sense, a much better approximation even to the hard-core repulsion problem than the mean-field state in Eq. (3). In particular, this state explicitly exhibits the correct ODLRO, whereas the mean-field state (after singular gauge transformation back to the boson representation) only exhibits *algebraic* ODLRO since it is incompressible.[4] Thus this test case lends strong support to the thesis that Eq. (2) represents a good variational state for spinless, hard-core repulsive anyons.

Finally, we note that in analogy to the FQHE, quasiparticle excitations are charged vertices.[3,4,7,9,29] Jackiw and Pi[30] have recently considered soliton solutions to a nonlinear Schrödinger equation closely related to the present model. Johnson and Canright[31] have extended the supersymmetry idea to the study of excited states. It is also straightforward to generalize the solutions presented here to the case of pure charges interacting with pure flux tubes, and to spin-S anyons at $\theta/\pi = 1/S$.

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