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LOCALIZATION IN A MAGNETIC FIELD: TIGHT BINDING MODEL WITH ONE-HALF OF A FLUX QUANTUM PER PLAQUETTE

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The conduction properties of a two-dimensional tight-binding model with on-site disorder and an applied perpendicular magnetic field with precisely one-half of a magnetic flux quantum per plaquette are studied. A continuum hamiltonian is derived which enables the construction of a field theory for the diffusive modes. The field theory is shown to be in the universality class of the $O(2n,2n)/O(2n)\times O(2n)(n\to 0)$ non-linear σ-model implying that all the electronic states are localized. The system is shown to be related, via an analytic continuation, to a system of self-interacting fermions in 1 + 1 dimensions, in the $n\to 0$ limit.

1. Introduction

A good understanding of the localization problem originally proposed by Anderson [1] in 1958 was finally achieved several years ago [2]. It is now well accepted that in two dimensions a tight-binding model for non-interacting electrons with on-site disorder has all states localized. In the presence of a perpendicular magnetic field, however, this system is not so well understood. Hofstadter [3] and more recently Thouless et al. [4] have studied a pure $d = 2$ tight-binding model in a transverse magnetic field. Even in the absence of disorder this system has an exceedingly rich band structure and has served as a model for gaining an understanding of the integer quantum Hall effect. When both disorder and a magnetic field are present, however, an interesting question arises. Are all the states still localized? This question, although having received much attention recently [2, 5–9], has not been completely resolved.

We consider the conduction properties of a disordered non-interacting electronic system on a two-dimensional tight-binding square lattice immersed in a constant magnetic field B perpendicular to the system. The disorder is taken as a white-noise random potential on the sites of the lattice. In this work we consider the very special case in which the magnetic flux Φ per plaquette of the square lattice equals one-half of the quantum of flux Φ_0. A general magnetic field breaks time-reversal invariance. Thus two systems with flux Φ and $-\Phi$ are generally inequivalent. However if Φ is equal to $\frac{1}{2}\Phi_0$ there is no difference between Φ and $-\Phi$ since the physics of the

problem is periodic in Φ with period Φ_0. Thus time-reversal invariance is not truly broken. Consequently the Hall conductance σ_{xy} is identically zero for any filling fraction. This, however, does not mean that the magnetic field has no effect on the properties of the system.

In fact the magnetic field causes the system to split up into four sublattices. It is shown in sect. 2 that, for states near $E = 0$, the system effectively describes the propagation of massless relativistic particles, a property used by Kogut and Susskind [10] to write down a lattice version of the Dirac equation. This enables us to construct a continuum field theory for the diffusive modes when the Fermi energy is near the middle of the band. The resulting theory is shown to be in the universality class of the orthogonal $O(2n,2n)/O(2n)\times O(2n)$ $(n \to 0)$ non-linear σ-model [11], (this reference contains discussions of the orthogonal and unitary non-linear σ-models and their connection with Anderson localization) implying that all states are localized just as they are in the absence of the magnetic field [11–15]. This suggests then that if delocalized states do exist in the presence of a magnetic field it must be the direct breaking of time-reversal invariance, rather than some other property of the field, which is responsible.

In sect. 3 we show that there is a discrete symmetry in the system which, if unbroken, makes the density of states vanish at $E = 0$. However it breaks down spontaneously in the presence of disorder yielding a non-vanishing and smooth density of states near $E = 0$. This justifies the weak localization approach used in sect. 2.

We also point out in sect. 3 that the system studied here is related, via an analytic continuation, to a special one-dimensional relativistic interacting Fermi system in the $n \to 0$ limit. This system is close, but not identical, to some one-dimensional Fermi systems at finite n that have been solved with the Bethe ansatz [16].

Since in most experimental lattice systems it would take an astronomically large magnetic field ($B \sim 10^9$ G) to generate $\Phi = \frac{1}{2}\Phi_0$, this study is primarily of theoretical interest. However, there are some amorphous systems, such as granular aluminum, for which a magnetic field of the order of 10^5 G could on average satisfy this condition. Another system in which $\Phi = \frac{1}{2}\Phi_0$ could conceivably be achieved is a square array of tunnelling junctions.

2. The model and its critical behavior

We start with a tight binding model for a spinless electron on a two-dimensional square lattice with precisely one half of a magnetic flux quantum through each plaquette. In the Landau gauge the hamiltonian is

$$\hat{H} = -t\sum_{r}\left[e^{i\pi y/a}\hat{\phi}^{+}(r+\hat{n}_x)\hat{\phi}(r) + \hat{\phi}^{+}(r+\hat{n}_y)\hat{\phi}(r) + \text{h.c.}\right] + \sum_{r} V(r)\hat{\phi}^{+}(r)\hat{\phi}(r), \tag{2.1}$$

where $\hat{\phi}(\boldsymbol{r})$ is the electron field and a is the lattice spacing. The potential $V(\boldsymbol{r})$ will be taken as randomly distributed gaussian white noise centered about zero:

$$\langle V(\boldsymbol{r})V(\boldsymbol{r}')\rangle = \gamma\delta(\boldsymbol{r}-\boldsymbol{r}'), \qquad \langle V(\boldsymbol{r})\rangle = 0. \tag{2.2}$$

The pure system has an energy spectrum

$$E_{\boldsymbol{k}} = \pm 2t\sqrt{\cos^2(k_x a) + \cos^2(k_y a)}\,, \tag{2.3}$$

where $\boldsymbol{k}$ denotes the Bloch-wave momenta. When the Fermi energy is near zero, the Fermi surface consists of four circles centered about the points $\boldsymbol{k} = (\pm\pi/2a, \pm\pi/2a)$ (see fig. 1). Near these regions in k-space (2.3) describes a two-dimensional relativistic massless spectrum [10]

$$E = \pm 2ta|\boldsymbol{q}| + \mathrm{O}(q^2), \qquad \boldsymbol{q} = \boldsymbol{k} - \left(\pm\frac{\pi}{2a}, \pm\frac{\pi}{2a}\right). \tag{2.4}$$

By exploiting the simplicity of the Fermi surface we now obtain a continuum version of (2.1) appropriate for a system with $E_{\mathrm{F}} \simeq 0$.

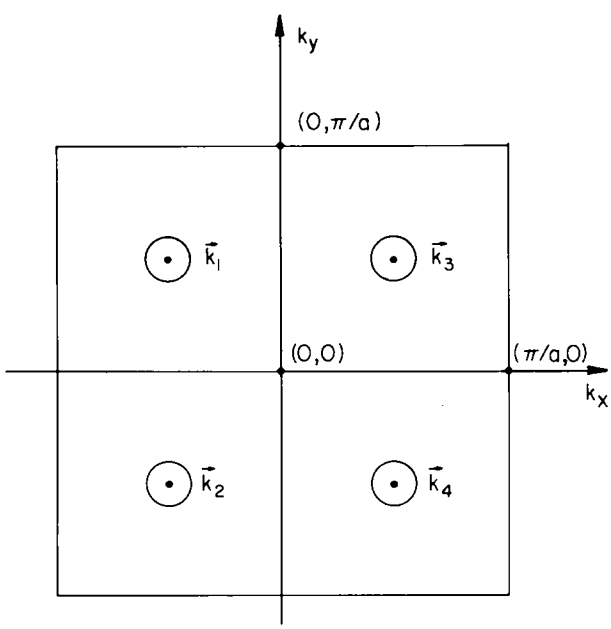

Fig. 1. With $E_{\mathrm{F}} \simeq 0$ the Fermi surface for the pure system consists of four circles in k-space.

From (2.1) we generate the equation of motion which upon fourier transformation is

$$i\frac{\mathrm{d}}{\mathrm{d}t}\phi(\boldsymbol{k},t) = -\frac{1}{a}\cos(k_y a)\phi(\boldsymbol{k}) - \frac{1}{a}\cos(k_x a)\phi\left(k_x, k_y - \frac{\pi}{a}\right)$$

$$+ \int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^2} V(\boldsymbol{k}-\boldsymbol{k}')\phi(\boldsymbol{k}'), \tag{2.5}$$

where for simplicity we have set $2ta = 1$. With the Fermi energy near zero one can, to a good approximation, restrict attention to the four regions in k-space near the light cones. This can be achieved by placing a cutoff Λ ($\Lambda < \pi/2a$) about the four points $\boldsymbol{k}_\alpha$. Then by defining four fields

$$\phi_\alpha(\boldsymbol{q}_\alpha) = \phi(\boldsymbol{k}), \qquad \boldsymbol{q}_\alpha = \boldsymbol{k} - \boldsymbol{k}_\alpha, \qquad \alpha = 1\ldots 4, \tag{2.6}$$

with $\boldsymbol{k}_\alpha$ shown in fig. 1, and four potential functions

$$V(\boldsymbol{k}) = \begin{cases} V_1(\boldsymbol{k}) \\ V_2\left(\boldsymbol{k} - \frac{\pi}{a}\hat{x} - \frac{\pi}{a}\hat{y}\right) \\ V_3\left(\boldsymbol{k} - \frac{\pi}{a}\hat{y}\right) \\ V_4\left(\boldsymbol{k} - \frac{\pi}{a}\hat{x}\right) \end{cases} \tag{2.7}$$

in the vicinity of each light cone, eq. (2.5) can be rewritten as a 4×4 matrix equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}(\boldsymbol{q},t) = \begin{bmatrix} q_y & -q_x & 0 & 0 \\ -q_x & -q_y & 0 & 0 \\ 0 & 0 & q_y & q_x \\ 0 & 0 & q_x & -q_y \end{bmatrix} \boldsymbol{\phi}(\boldsymbol{q})$$

$$+ \int_{q'}^{\Lambda} \begin{bmatrix} V_1 & V_3 & V_4 & V_2 \\ V_3 & V_1 & V_2 & V_4 \\ V_4 & V_2 & V_1 & V_3 \\ V_2 & V_4 & V_3 & V_1 \end{bmatrix} (\boldsymbol{q}-\boldsymbol{q}')\boldsymbol{\phi}(\boldsymbol{q}'). \tag{2.8}$$

In the kinetic energy in (2.8) we have expanded to linear order in qa. Notice that this term describes two decoupled Dirac fermions in $2+1$ dimensions [10]. In the presence of disorder, however, these are coupled through the potential energy.

After performing an appropriate orthogonal transformation to diagonalize the potential energy we re-express (2.8) back in real space. With an implicit cutoff Λ assumed this leads to the following hamiltonian:

$$H_{\alpha\beta}(\boldsymbol{r}) = i\boldsymbol{\alpha}_{\alpha\beta}\cdot\nabla_{\boldsymbol{r}} + \delta_{\alpha\beta}\tilde{V}_{\alpha}(\boldsymbol{r}), \tag{2.9}$$

where

$$(\alpha^{x})_{\alpha\beta} = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad (\alpha^{y})_{\alpha\beta} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \tag{2.10}$$

$$\tilde{V}_{\alpha} = \sum_{i=1}^{4} \Gamma_{\alpha}^{i} V_i, \tag{2.11}$$

with

$$\Gamma_{\alpha}^{1} = (1,1,1,1), \qquad \Gamma_{\alpha}^{2} = (1,1,-1,-1),$$

$$\Gamma_{\alpha}^{3} = (1,-1,1,-1), \qquad \Gamma_{\alpha}^{4} = (1,-1,-1,1). \tag{2.12}$$

Here σ_1 and σ_3 are 2×2 Pauli matrices. From (2.7) one deduces that the four potentials $\tilde{V}_{\alpha}(\boldsymbol{r})$ are real functions each with an independent gaussian white noise distribution

$$P\left[\{\tilde{V}_{\alpha}\}\right] = c\prod_{\alpha}\exp\left[-\frac{1}{8\gamma}\int \mathrm{d}^2\boldsymbol{r}\left[\tilde{V}_{\alpha}(\boldsymbol{r})\right]^2\right]. \tag{2.13}$$

Eq. (2.9) constitutes our continuum approximation to the hamiltonian in (2.1). Notice that the presence of the magnetic field is now hidden in the four-component nature of (2.9). We now proceed to analyze this continuum hamiltonian to see if the states are localized or extended.

The generating functional for the Green functions of the hamiltonian (2.9) is

$$Z = \int \prod_{\alpha} D\phi_{\alpha}\, D\phi_{\alpha}^{*}\, \mathrm{e}^{-S_0}, \tag{2.14}$$

with

$$S_0 = \int \mathrm{d}^2x\, \phi_{\alpha}^{*}(x)\left[(E \pm i\eta)\delta_{\alpha\beta} - H_{\alpha\beta}(x)\right]\phi_{\beta}(x). \tag{2.15}$$

From (2.15) it appears at first sight that the symmetry underlying the theory will be unitary*. However by introducing real fields

$$\phi_{\alpha} = \sqrt{\tfrac{1}{2}}\,(\phi_{\alpha,1} + i\phi_{\alpha,2}), \tag{2.16}$$

* In a previous unpublished version of this paper we incorrectly identified the symmetry as being unitary. We thank Dr. A. Pruisken for kindly pointing out to us that the actual symmetry is orthogonal. The argument presented below is actually his.

and performing an appropriate orthogonal transformation on the resulting 8×8 matrix, eq. (2.15) can be block diagonalized (into two 4×4 blocks) and in this new basis takes the form

$$S_0 = \tfrac{1}{2} \sum_{c=1,2} \int \mathrm{d}^2x \, \Phi_\alpha^c(x) \left[(E \pm i\eta)\delta_{\alpha\beta} - \tilde{H}_{\alpha\beta}(x) \right] \Phi_\beta^c(x), \tag{2.17}$$

where

$$\tilde{H}_{\alpha\beta} = \Gamma_{\alpha\beta} \cdot \nabla + \delta_{\alpha\beta} \tilde{V}_\alpha, \tag{2.18}$$

$$\Gamma^x = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad \Gamma^y = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \tag{2.19}$$

Here c labels the two 4×4 blocks which, as seen from (2.17), are identical to one another.

We now follow the standard procedure used to describe [11–15] the zero magnetic field localization problem in terms of an interacting matrix field theory. The square of the retarded single-particle Green function is obtained by squaring the functional integral. In order to average over the disorder the replica trick is used. The two 4×4 blocks in (2.17) are seen to simply double the number of replicas. After averaging over the disorder with the weighting (2.13), symmetric and real composite Q-fields are introduced, as first done by Wegner [12]. The resulting generating functional is

$$Z = \lim_{n_a, n_b \to 0} \int DQ \exp\left\{ - \int \mathrm{d}^2x \sum_{i,j,\alpha} \tfrac{1}{2} \left(Q_\alpha^{ij}(x) \right)^2 - \tfrac{1}{2} \operatorname{Tr} \log A \right\} \equiv \int DQ \, \mathrm{e}^{-S}, \tag{2.20}$$

where

$$A_{\alpha\beta}^{ij}(x, y) \equiv \delta(x - y) \left[\delta_{ij} \left\{ (E + is_j\eta)\delta_{\alpha\beta} - \Gamma_{\alpha\beta} \cdot \nabla_x \right\} - 2\sqrt{\gamma}\, \delta_{\alpha\beta} Q_{\alpha\beta}^{ij}(x) \right]. \tag{2.21}$$

The average of the absolute square of the retarded single-particle Green function is given by

$$\overline{|G_{\alpha\beta}(x, y, E + i\eta)|^2} = \gamma^{-1} \langle Q_\alpha^{ab}(x) Q_\beta^{ab}(y) \rangle, \tag{2.22}$$

where the brackets denote an average with respect to S in (2.20). In (2.20)–(2.22) i and j label the two sets of replica indices and run from $1 \ldots 2n_a + 2n_b$ where $n_{a(b)}$

denotes the number of replicas in the particle (hole) sector. The symbol s_j is equal to 1 (-1) when $j = 1 \ldots 2n_a$ $(2n_a + 1 \ldots 2n_a + 2n_b)$. The spinor components are labelled by the subscripts α and β.

The average single-particle Green function, on the other hand, can be obtained at the saddle-point (CPA) level from the following self-consistent equation:

$$\left[(E \pm i\eta)\delta_{\alpha\gamma} - i\Gamma_{\alpha\gamma}\cdot \boldsymbol{p} - 2\gamma\delta_{\alpha\gamma}\overline{G^0_{\alpha\alpha}(x, x, E \pm i\eta)}\right]\overline{G^0_{\gamma\beta}(\boldsymbol{p}, E \pm i\eta)} = \delta_{\alpha\beta}. \tag{2.23}$$

It is instructive to examine the CPA density of states $\rho_0(E)$ defined by

$$\rho_0(E) = -\frac{1}{\pi}\sum_{\alpha=1}^{4} \operatorname{Im}\, \overline{G^0_{\alpha\alpha}(x, x, E + i\eta)}\,. \tag{2.24}$$

By solving for the Green function in (2.23) one finds that the four diagonal components are all equal. At $E = 0$ this gives

$$\rho_0(E = 0) = \frac{4\Lambda}{2\pi\gamma} e^{-1/4\pi\gamma}\left[1 + O(e^{-1/4\pi\gamma})\right]. \tag{2.25}$$

The full functional form of $\rho_0(E)$ is shown in Fig. 2. Notice that as the disorder (γ) vanishes the linear density of states of the free 2 + 1 Dirac equation is recovered. However in the presence of disorder a non-zero density of states is introduced at $E = 0$. As shown in sect. 3 this non-zero density of states at $E = 0$ is in fact the result of a spontaneous breaking of a discrete symmetry in the original generating functional. It is important that the resulting density of states is smooth and finite

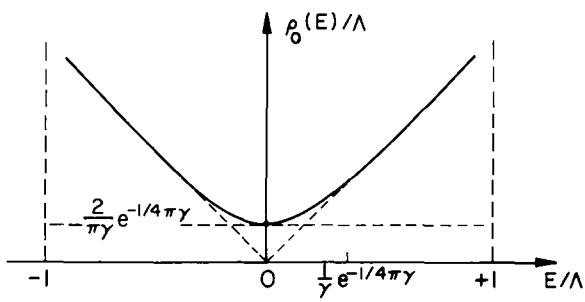

Fig. 2. The density of states at the CPA level is denoted $\rho_0(E)$. The presence of disorder $(\gamma \neq 0)$ has introduced a non-zero density of states at the center of the band. When $\gamma \to 0$ the linear density of states of the free 2 + 1 Dirac equation is recovered.

near the center of the band since otherwise the basic approach of weak localization theory would not apply.

We now proceed to analyze the action S defined in (2.20). The terms linear in Q can be eliminated by introducing the shift $Q_\alpha^{ij} = \langle Q_\alpha^{ij}\rangle_0 + \tilde{Q}_\alpha^{ij}$ with the choice

$$\langle Q_\alpha^{ij}\rangle_0 = \delta_{ij}\langle Q_\alpha^{ii}\rangle_0 = \sqrt{\gamma}\,\delta_{ij}\,\overline{G^0_{\alpha\alpha}(x, x, E + is_j\eta)}\,. \tag{2.26}$$

As mentioned above $\overline{G^0_{\alpha\alpha}}$ is independent of the spinor index α. Then by expanding about $\langle Q\rangle_0$ the action becomes a power series in $\tilde{Q}$. In particular at quadratic order S takes the form

$$\begin{aligned} S^{(2)} = \int \mathrm{d}^2x\,\mathrm{d}^2y \sum_{\alpha\beta}\Big\{ &\sum_{ab} \tilde{Q}_\alpha^{ab}(x) C_{\alpha\beta}^{+-}(x, y)\tilde{Q}_\beta^{ab}(y) \\ &+ \tfrac{1}{2}\sum_{a_1a_2} \tilde{Q}_\alpha^{a_1a_2}(x) C_{\alpha\beta}^{++}(x, y)\tilde{Q}_\beta^{a_1a_2}(y) \\ &+ \tfrac{1}{2}\sum_{b_1b_2} \tilde{Q}_\alpha^{b_1b_2}(x) C_{\alpha\beta}^{--}(x, y)\tilde{Q}_\beta^{b_1b_2}(y)\Big\}, \end{aligned} \tag{2.27}$$

with

$$C_{\alpha\beta}^{\pm\pm} = \delta_{\alpha\beta}\delta(x-y) - 2\gamma\,\overline{G^0_{\alpha\beta}(x, y, \pm)}\;\overline{G^0_{\beta\alpha}(y, x, \pm)}\,. \tag{2.28}$$

Putting $E = 0$ for simplicity enables a straightforward but tedious calculation of the C-matrices in (2.28) using $\overline{G^0}$ obtained from (2.23). One can then check for diffusive behavior at the tree level (CPA) by inverting the 4×4 matrix $C^{--}_{\alpha\beta}(\boldsymbol{p}, \eta)$. The result for small $\boldsymbol{p}$ is

$$(C^{--})^{-1}_{\alpha\beta}(\boldsymbol{p}, \eta) = \frac{2\pi\gamma\rho_0}{16}\left[\frac{1}{\eta + (2/\rho_0)\boldsymbol{p}^2}\right]. \tag{2.29}$$

As $\eta \to 0$ we pick up the familiar diffusion pole [13]. This pole is seen to be independent of the spinor indices. Eq. (2.29) together with Einstein's relation yields the CPA conductivity $\sigma_{\mathrm{CPA}} = (e^2/\hbar)\rho_0 D = 4e^2/\hbar$.

To determine the structure of the underlying field theory it is convenient to diagonalize the quadratic terms with respect to the spinor indices. This is achieved by performing an orthogonal rotation in α-space.

$$\tilde{Q}_\lambda = O_{\lambda\alpha}\tilde{Q}_\alpha\,, \tag{2.30}$$

where

$$O = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}. \tag{2.31}$$

The quadratic forms computed from (2.28) then take a diagonal form:

$$C^{\pm\pm}_{\lambda\lambda'}(\boldsymbol{p}) = \delta_{\lambda\lambda'} C^{\pm\pm}_{\lambda}(\boldsymbol{p}), \tag{2.32}$$

with $C^{\pm\pm}_{\lambda}(\boldsymbol{p})$ at $E = 0$ given by

$$C^{+-}_{\lambda}(\boldsymbol{p}) = \begin{bmatrix} (\sigma_{\mathrm{CPA}}/\pi\gamma\rho_0^2)\boldsymbol{p}^2 \\ 1 - 2\pi\gamma \\ 1 - 2\pi\gamma \\ 2(1 - 2\pi\gamma) \end{bmatrix}_{\lambda} + \mathrm{O}(\boldsymbol{p}^2), \qquad C^{\pm\pm}_{\lambda}(\boldsymbol{p}) = \begin{bmatrix} 4\pi\gamma \\ 1 + 2\pi\gamma \\ 1 + 2\pi\gamma \\ 2 \end{bmatrix}_{\lambda} + \mathrm{O}(\boldsymbol{p}^2). \tag{2.33}$$

In addition since $\langle Q^{ij}_{\alpha}\rangle_0$ in (2.26) is independent of the spinor index α we have in the new basis

$$\langle Q^{ij}_{\lambda}(x)\rangle_0 = \begin{cases} \delta_{ij} 2\sqrt{\gamma}\ \overline{G^0_{\alpha\alpha}(x, x, E + is_j\eta)} \\ 0 \end{cases}, \qquad \lambda = \begin{cases} 1 \\ 2,3,4. \end{cases} \tag{2.34}$$

Notice that in this representation only one mode ($\lambda = 1$) is broken and has massless (diffusive) excitations while the other three are unbroken and massive. We now discuss the symmetry of the effective field theory for the massless Q^{+-}_1 modes.

The saddle-point equation of the action S defined in (2.20) is obtained by setting $\delta S/\delta Q = 0$ giving

$$\langle Q^{ij}_{\alpha}(x)\rangle_0 = \sqrt{\gamma}\Big(\Big[\delta(x-y)\big\{\delta_{ij}\big[(E + is_j\eta)\delta_{\alpha\beta} - \Gamma_{\alpha\beta}\cdot\nabla\big] - 2\sqrt{\gamma}\,\delta_{\alpha\beta}\langle Q^{ij}_{\alpha}(x)\rangle_0\big\}\Big]^{-1}\Big)^{ij}_{\alpha\alpha}(x, x). \tag{2.35}$$

One solution to (2.35) is given in (2.26). However when $\eta = 0$ it is easy to verify that for any solution $\langle Q^{ij}_{\alpha}\rangle_0$ eq. (2.35) is also satisfied by $\langle Q^{ij}_{\alpha}\rangle'_0$ where

$$\langle Q^{ij}_{\alpha}\rangle'_0 = (U^{-1})^{ik}\langle Q^{kl}_{\alpha}\rangle_0 U^{lj}, \tag{2.36}$$

with U any invertible $4n \times 4n$ matrix. Since U is independent of the spinor

components α and the Q_λ are related to Q_α through a rotation in α-space, (2.36) also holds in the new basis. Thus from (2.34) the general solution to the saddle-point equation can be written

$$\langle Q_{\lambda=1}\rangle_0 = U^{-1}\begin{bmatrix} -i\frac{1}{2}\pi\sqrt{\gamma}\,\rho_0\hat{1} & 0 \\ 0 & i\frac{1}{2}\pi\sqrt{\gamma}\,\rho_0\hat{1} \end{bmatrix} U. \tag{2.37}$$

We now follow previous authors [13–15] and write an effective theory for the $Q^{+-}_{\lambda=1}$ sector in which the full configurations satisfy the saddle-point equations of the original theory. For a given $Q^{+-}_1 (\equiv Q)$ configuration we define a $4n \times 4n$ matrix field

$$h_0(x) = \begin{bmatrix} R_1(x) & Q(x) \\ Q'(x) & R_2(x) \end{bmatrix}, \tag{2.38}$$

where R_1 and R_2 will be chosen such that at each point x, h_0 is a saddle-point solution. From (2.37) this requires setting $\operatorname{tr} h_0 = 0$ and $h_0^2 = -\frac{1}{4}\pi^2\gamma\rho_0^2\hat{1}$. This determines R_1 and R_2 and gives

$$h_0(x) = \begin{bmatrix} -i\left(\frac{1}{4}\pi^2\gamma\rho_0^2\hat{1} + QQ'\right)^{1/2} & Q \\ Q' & i\left(\frac{1}{4}\pi^2\gamma\rho_0^2\hat{1} + Q'Q\right)^{1/2} \end{bmatrix}. \tag{2.39}$$

The effective theory is now chosen to include only the saddle-point configurations $h_0(x)$. Since in the full theory Q^{+-} enters to lowest order as $(\sigma_{\mathrm{CPA}}/(\pi\gamma\rho_0^2))\,\nabla Q^{ab}\,\nabla Q^{ab}$ it would be reasonable to suggest [12–14] that the effective theory for Q^{+-} is

$$H_{\mathrm{eff}} = \left(\sigma_{\mathrm{CPA}}/2\pi\gamma\rho_0^2\right)\int \mathrm{d}^2x\ \mathrm{tr}\ \nabla h_0 \cdot \nabla h_0. \tag{2.40}$$

Since h_0 in (2.39) can be written as [11–14]

$$h_0(x) = T(x)^{-1}\begin{bmatrix} -i\frac{1}{2}\pi\sqrt{\gamma}\,\rho_0\hat{1} & 0 \\ 0 & i\frac{1}{2}\pi\sqrt{\gamma}\,\rho_0\hat{1} \end{bmatrix} T(x), \tag{2.41}$$

with T belonging to $\mathrm{O}(2n,2n)$, the effective theory is an $\mathrm{O}(2n,2n)/\mathrm{O}(2n)\times\mathrm{O}(2n)$ non-linear σ-model. If we define $h = (2/\pi\sqrt{\gamma}\,\rho_0)h_0$ this becomes

$$H_{\mathrm{eff}} = \frac{1}{4t}\int \mathrm{d}^2x\ \mathrm{tr}\ \nabla h \cdot \nabla h, \tag{2.42}$$

with the inverse coupling constant proportional to the CPA conductivity

$$t^{-1} = \tfrac{1}{2}\pi\sigma_{\mathrm{CPA}}. \tag{2.43}$$

We conclude then that the tight-binding model with $\Phi = \frac{1}{2}\Phi_0$ is in the universality class of the orthogonal ensemble. The conductivity can be deduced from the β-function which with $n = 0$ and $d = 2$ is [11]

$$\beta(t) = \frac{\mathrm{d}t}{\mathrm{d}\log L} = 2t^2 + \mathrm{O}(t^5). \tag{2.44}$$

In the infrared t flows towards infinity (σ towards zero) implying zero d.c. conductivity in the infinite system limit. Thus near $E_{\mathrm{F}} = 0$ all states are localized. Far from $E_{\mathrm{F}} = 0$, where our particular continuum model is not a good approximation to the original tight-binding model, it is reasonable to assume that the orthogonal symmetry should still be apparent leading to localized states at all energies.

For arbitrary values of the flux Φ, terms which explicitly break time-reversal invariance are expected. The interplay between the localizing effects of the random potential and the additional time-reversal breaking terms is clearly essential in understanding fully localization in a magnetic field. A continuum theory has already been proposed by Levine, Libby and Pruisken [8, 9]. They show the existence of an additional term in the field theory which might be responsible for delocalization of the electrons. The coefficient of this extra term is the Hall conductance at the CPA level. Since in the model studied here ($\Phi = \frac{1}{2}\Phi_0$) σ_{xy} is identically zero it is consistent that this extra term is not found.

3. Discrete symmetry and one-dimensional Fermi systems

In this section we demonstrate the existence of a discrete symmetry which governs the behavior of the density of states near $E = 0$. In doing so we show that the system studied in sect. 2 is equivalent to a self-interacting Fermi field in the $n \to 0$ limit.

The Green functions which correspond to the quantum mechanical equations of motion (2.8) can be generated by means of a functional integral. In this section we will use the Grassmann representation, more transparent (although not necessary) for a connection with the $n \to 0$ Fermi system. Let $\Phi_\alpha(x)$ be a complex Fermi (Grassmann) field ($\alpha = 1, \ldots, 4$). The (one-particle) Green functions

$$G_{\alpha\beta}(x, y; E \pm i\eta) \equiv \langle x\alpha | \frac{1}{E \pm i\eta - H} | y\beta \rangle \tag{3.1}$$

have the representation

$$G_{\alpha\beta}(x, y; E) \equiv \langle \phi_\alpha(x) \phi^*_\beta(y) \rangle, \tag{3.2}$$

with

$$\langle 0 \rangle = \frac{\int D\phi_\alpha^*(x)\, D\phi_\alpha(x)\, O\, e^{-A}}{\int D\phi_\alpha^*\, D\phi_\alpha\, e^{-A}}. \tag{3.3}$$

In (3.1) H is the hamiltonian which governs the equations of motion in (2.8). The "action" A is given by

$$A = \int d^2x\, \phi_\alpha^*(x)(E \pm i\eta - H)_{\alpha\beta}\phi_\beta(x). \tag{3.4}$$

Of particular interest here is the density of states (DOS) which is given by

$$\rho(E) = -\frac{L^{-2}}{\pi} \operatorname{Im} \sum_{\alpha=1}^{4} \int d^2x\, G_{\alpha\alpha}(x, x; E + i\eta), \tag{3.5}$$

where L is a linear dimension of the system.

We now rewrite the four-component field ϕ_α in terms of a pair of spinor fields $\psi_a(x)$ $(a = 1, 2)$ defined by

$$\psi_1(x) = \sigma_3 \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix},$$

$$\psi_2(x) = \begin{pmatrix} \phi_3(x) \\ \phi_4(x) \end{pmatrix}. \tag{3.6}$$

The action A now reads

$$A = \int d^2x \sum_{a,b=1,2} \psi_a^+(x) M_{ab} \psi_b(x), \tag{3.7}$$

where

$$M_{11} = i\sigma_3\nabla_3 + i\sigma_1\nabla_1 - V_1 + \sigma_1 V_3 + (E + i\eta),$$

$$M_{22} = i\sigma_3\nabla_3 + i\sigma_1\nabla_1 - V_1 - \sigma_1 V_3 + (E + i\eta),$$

$$M_{12} = M_{21}^+ = -\sigma_3(V_4 + V_2\sigma_1). \tag{3.8}$$

In this new basis the Green functions are

$$G_{\substack{\alpha\beta \\ ab}}(x, y; E) = \langle xa\alpha | \frac{1}{M} | yb\beta \rangle$$

$$= \langle \psi_a^\alpha(x) \psi_b^{*\beta}(y) \rangle, \tag{3.9}$$

where $a = 1,2$ labels the spinors and $\alpha = 1,2$ labels the component of each spinor. It will be convenient to factor $i\sigma_1$ out of M in (3.9) giving

$$G_{\substack{\alpha\beta \\ ab}}(x, y; E) = -i\langle xa\alpha | \frac{1}{-i\sigma_1 IM}\sigma_1 I | y\beta b\rangle , \tag{3.10}$$

where I is the 2×2 identity matrix. The functional integral form of (3.10) involves a path integral whose lagrangian is

$$\begin{aligned} L = {} & \bar{\psi}_a \not{\nabla} \psi_a - iE\bar{\psi}_a\psi_a + iV_1(x)\bar{\psi}_a\psi_a \\ & - iV_2(x)\bar{\psi}_a\gamma_5(\tau_2)_{ab}\psi_b - iV_3(x)\bar{\psi}_a\gamma_1(\tau_3)_{ab}\psi_b + iV_4(x)\bar{\psi}_a\gamma_3(\tau_1)_{ab}\psi_b , \end{aligned} \tag{3.11}$$

where we have used the (euclidian) relativistic notation

$$\begin{gathered} \gamma_1 = \sigma_1 , \qquad \gamma_3 = \sigma_3 , \qquad \gamma_5 \equiv i\gamma_1\gamma_3 = \sigma_2 , \\ \bar{\psi} = \psi^\dagger\gamma_1 , \qquad \not{\nabla} = \nabla_1\gamma_1 + \nabla_3\gamma_3 , \\ \tau_i = \sigma_i \qquad (i = 1,2,3) . \end{gathered} \tag{3.12}$$

The lagrangian (3.11) represents two relativistic Dirac fields ψ_a in $1+1$ dimensions (in the euclidian metric), interacting with the random potentials $V_j(x)$ ($j = 1,\ldots,4$). The interaction terms represent the umklapp processes of the original lattice model. Making use of the replica trick to average over the disorder, one can reinterpret (3.11), after an analytic continuation to Minkowski space, as the lagrangian of a pair of self-interacting relativistic Dirac fields with a coupling constant proportional to the width of the distribution of the V_j. This type of lagrangian is close, but not identical, to the systems recently studied by means of the Bethe ansatz.

The Green functions in (3.10) can be expressed as

$$G_{\substack{\alpha\beta \\ ab}}(x, y; E) = -i\langle \psi_a^\alpha(x)\bar{\psi}_b^\beta(y)\rangle , \tag{3.13}$$

where the brackets denote an average with respect to the lagrangian in (3.11). Inserting (3.13) into (3.5) gives for the DOS

$$\rho(E) = -\frac{L^{-2}}{\pi}\mathrm{Re}\sum_{a=1,2}\int \mathrm{d}^2x\,\langle\bar{\psi}_a(x)\psi_a(x)\rangle . \tag{3.14}$$

At $E = 0$ the lagrangian (3.11) has the discrete (chiral) symmetry

$$\psi_a \to \gamma_5 \psi_a ,$$
$$\bar{\psi}_a \to -\bar{\psi}_a \gamma_5 , \tag{3.15}$$

provided that the probability distributions $P[V_j]$ are *even* (i.e. $P[V] = P[-V]$). This is not a symmetry for a system with a given configuration of the V but of *averaged* quantities. The operator $\bar{\psi}_a \psi_a$ is odd under (3.15). Hence, at $E = 0$, $\rho(0)$ must vanish *unless* the (chiral) symmetry is broken. For $E \neq 0$ the symmetry is *explicitly* broken and, of course, $\rho(E)$ is positive.

We would like to argue that this symmetry is spontaneously broken and that the *average* DOS is *non-zero* and *smooth near* $E = 0$. Indeed we have already confirmed this in sect. 2 (eq. (2.25)) where we showed that $\rho(0)$ was non-zero even at the CPA level. It is important to note that had $\rho(0)$ remained zero the weak scattering approach of sect. 2 would not have been applicable.

The lagrangian (3.11) is only useful for the computation of single-particle properties of the disordered system, such as the DOS and the mean-free-path. To get information about localization (i.e. conductivity and localization length) two-particle Green functions are needed. This necessitates doubling the number of Fermi fields in (3.11), with $E \to E + i\eta$ for the first half of the fields and $E \to E - i\eta$ for the remaining fields.

4. Conclusion

We have discussed the properties of an electron hopping in a disordered square lattice immersed in a magnetic field with one-half of quantum of flux per plaquette. We showed that, with time-reversal invariance not being broken, all states are still localized. The resulting localization problem falls in the universality class of the orthogonal non-linear sigma model $[O(2n, 2n)/O(2n) \times O(2n)]$ $(n \to 0)$. We established the presence of a symmetry in the problem which is spontaneously broken for arbitrary disorder rendering the density of states at $E = 0$ finite and the weak scattering non-linear sigma model approach applicable. We also pointed out an interesting connection between the problem studied here and a one-dimensional Fermi system in 1 + 1 space-time dimensions in the $n \to 0$ limit.

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