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ANYON SUPERCONDUCTIVITY AND CHARGE-VORTEX DUALITY

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We review recent work on a bosonic formulation of both anyon superconductivity and the fractional quantum Hall effect (FQHE). Central to this approach is the concept of charge-vortex duality in two-dimensional (2d) boson systems. A formal duality transformation which takes one from a particle to a vortex representation of a 2d boson model is described in detail. The duality transformation is employed to obtain detailed properties of both the FQHE hierarchy and a hierarchy of anyon superconducting phases.

1. Statistics Transmutation

The best known example in which electrons undergo "statistics transmutation" (ST) at low energies is Cooper pairing[1] in superconductors. Due to the existence of a phonon mediated attraction, electrons with opposite spin pair together to form composite particles of charge $2e$ (for the purpose of conceptual simplicity consider the case of real-space pairing). As a result, at length scales much greater than the size of the pair they behave like bosons. These composite bosons can subsequently acquire phase coherence and condense into a superconducting state.[2]

A new mechanism for statistical transmutation of electrons, which involves binding the electrons to magnetic fluxes or vortices, has recently been the focus of much attention.[3] In contrast to Cooper pairing, this mechanism is only possible in two space dimensions. In order to see how it works, imagine calculating the partition function for a collection of charge-vortex composite particles (each made up of charge q and vorticity Φ) via a Feynman path integral.[4] As usual these composite particles traverse world lines in the $2+1$-dimensional space-time which contribute to the partition function through the action. The action consists of a real and an imaginary part. For these composite particles, every time two world-lines braid around each other, the action picks up a pure imaginary contribution equal to $\pm i(\theta_i + q\Phi)$, the plus sign for counterclockwise and minus sign for clockwise exchanges. Here θ_i is the intrinsic statistics angle associated with the charge, i.e. $\theta_i = 0(\text{mod } 2\pi)$ for bosons and $\theta_i = \pi(\text{mod } 2\pi)$ for fermions. The other part, $q\Phi$, is an Aharonov-Bohm phase factor or Berry's phase[5] resulting from the adiabatic transport of charge relative to magnetic fluxes or vortices. Since the Berry's phase is added to the intrinsic phase angle θ_i the statistics of the particles is effectively changed. For example, if the charged particles are fermions and $q\Phi = \pi$, the composite particles have Bose statistics. In general, since $q\Phi$ need not be an integer multiple of π, the composite particles can have statistics lying between that of fermions and bosons. Such particles are called anyons.[6] Here it is important to notice that if $q\Phi \neq n\pi$, with n an integer, clockwise and counterclockwise exchanges give different phase factors, so that time reversal^T and parity^P symmetries are no longer respected. Strictly speaking, in order for the flux (or vortex) attachment not to introduce any singularities, the particles must have a hard-core repulsion among themselves. The reason that anyons can exist is due to the fact that a closed loop enclosing an impenetrable point cannot be contracted to a point in two space dimensions.

Unlike Cooper pairing, charge-vortex binding is favored by a short-range repulsion. This is because, due to the form of the kinetic energy operator, a vortex forces the wave function to vanish when two particles coincide. As a result, the short-range correlation energy is greatly reduced. In general, it is quite nontrivial to show that charge-vortex binding is energetically favorable. In fact the only well established case is in the quantum-Hall effect (QHE), where explicit wave functions exhibiting charge-vortex binding can be shown to be the exact ground state when the interaction between particles is sufficiently short range.[7] In this paper we will be less ambitious, in that we will assume that charge-vortex binding does occur, and ask what are the consequences.

In this paper we expand our recent work[8] on anyon superconductivity and more generally statistical transmutations in 2d electron systems such as that which arises in the FQHE. Our approach to anyon superconductivity and the FQHE is to reformulate both problems in terms of an underlying boson field theory with Chern-Simons term which attaches flux tubes to the bosons. In this way, with an appropriate choice of $q\Phi$, the boson plus flux-tube composite can either model an anyon or an ordinary fermion such as an electron. We emphasize that this approach differs from the vast majority of the literature, which employs a fermion representation of anyons.[3]

A bosonic formulation allows us to exploit the concept of charge-vortex duality.[8–10] Since vortices in a 2d boson system are point-like, there are two complementary representations of such a system: A conventional representation or a dual one in which the vortices are treated as particles and the particles as vortices. In Sec. 2 we describe in detail a formal duality transformation which allows one to pass between representations. In contrast to most previous works,[8–10] though, the duality transformation will be carried out in the spatial continuum, rather than on a lattice. The standard properties of a boson superfluid are extracted from the dual representation in this section.

Section 3, which comprises the bulk of this paper, is devoted to a discussion of the FQHE hierarchy in a bosonic representation applying the duality transformation. In Sec. 4 a boson representation of anyons is used to discuss the anyon superconducting phase. In both Secs. 3 and 4 an intuitive pictorial description is developed in parallel with the formal manipulations. Section 5 is devoted to a very brief discussion of the possible relevance of "semion" superconductivity to the high temperature copper-oxide superconductors.

2. Duality Transformation: Application to Boson Superfluidity

In this section we describe in detail a formal duality transformation[9] between particles and vortices for a 2d Bose system.[10] In simple terms duality means the following. For bosonic systems in two space dimensions, both the charge and vortices are point-like particles. There are two equivalent ways to represent the charge-vortex mixture. In the first representation, charges are viewed as charges and vortices are viewed as vortices. In doing the path integrals whenever a charge and a vortex braid around each other the action picks up a Berry phase. The alternative representation is to view the charges as vortices and the vortices as charges. These two representations are equivalent in the sense that the relative Berry phase is invariant under the exchange. The mathematical manipulation which leads from one representation to the other is the duality transformation.

In this section, we will illustrate the use of the duality transformation by considering a relatively simple problem, namely, the superfluidity of bosons. This will prepare us for the bosonic formulations of both anyon superconductivity and the FQHE in subsequent sections. Indeed, aside from a slight complication caused by the "Chern-Simons" term, which will be discussed in detail later, most of the boson duality results are directly applicable to anyon superconductivity.

Consider then a 2-D interacting boson Hamiltonian in second quantized form:

$$\mathrm{H} = \frac{1}{2m}\int d^2\mathbf{x}\left|\frac{\boldsymbol{\partial}}{i}\phi(\mathbf{x})\right|^2 + \frac{1}{2}\int d^2\mathbf{x}d^2\mathbf{x}'(\phi^+(\mathbf{x})\phi(\mathbf{x})-\bar{\rho})V(\mathbf{x}-\mathbf{x}')(\phi^+(\mathbf{x}')\phi(\mathbf{x}')-\bar{\rho})\,. \tag{2.1}$$

Here ϕ is the boson annihilation operator, m is the effective mass, and $\bar{\rho}$ is the average density. We are interested in the zero termperature state of this Hamiltonian. Following the standard steps leading from Hamiltonian formalism to the coherent state path-integral representation,[11] one can show that the partition function at zero temperature takes the form

$$Z = \int D[\phi,\bar{\phi}]e^{-\int d\tau d^2\mathbf{x}\,\mathcal{L}} \tag{2.2}$$

where

$$\mathcal{L} = i\bar{\phi}\frac{\partial_0}{i}\phi + \frac{1}{2m}\left|\frac{\boldsymbol{\partial}}{i}\phi\right|^2 + \frac{1}{2}(\bar{\phi}\phi-\bar{\rho})V(\bar{\phi}\phi-\bar{\rho})\,. \tag{2.3}$$

Here ϕ and $\bar{\phi}$ are c-number complex fields, and $\bar{\phi}$ is the complex conjugate of ϕ. The partial derivative, ∂_0, in (2.3) is with respect to imaginary time. Throughout the paper we take $h = c = e = 1$.

We now derive a dual representation for the partition function (2.2)–(2.3), which is expressed in terms of the vortices in the boson field ϕ. In Ref. 10a this duality transformation was carried out for a lattice model of bosons. Here we modify the formulation slightly and carry out the duality transformation in the continuum. Since the phase of the order parameter changes by 2π upon encircling a vortex, it is convenient to separate out the magnitude dependence of ϕ from its phase dependence by writing

$$\phi = \rho^{1/2}\chi \tag{2.4}$$

where ρ is positive definite, which gives the density of particles, and χ is a unimodular complex field: $\bar{\chi}\chi = 1$. The Lagrangian (2.3), when expressed in terms of ρ and χ by insertion of (2.4), takes the form,

$$\mathcal{L} = i\rho\bar{\chi}\frac{\partial_0}{i}\chi + \frac{\rho}{2m}\left|\bar{\chi}\frac{\boldsymbol{\partial}}{i}\chi\right|^2 + \frac{1}{2m}|\boldsymbol{\partial}\rho^{1/2}|^2 + \frac{1}{2}(\rho - \bar{\rho})V(\rho - \bar{\rho}) \,. \tag{2.5}$$

In the second term above we have added in an extra field $\bar{\chi}$, which leaves the term invariant since χ has unit modulus. We now decouple this term by introduction of a Hubbard-Stratanovich field, $\mathbf{J}$, which gives

$$\mathcal{L} = i\underline{J}\cdot\bar{\chi}\frac{\underline{\nabla}}{i}\chi + \frac{m}{2\rho}|\mathbf{J}|^2 + \frac{1}{2m}|\boldsymbol{\partial}\rho^{1/2}|^2 + \frac{1}{2}(\rho - \bar{\rho})V(\rho - \bar{\rho}) \,. \tag{2.6}$$

Here we have defined a 3-current, $\underline{J} \equiv (\rho, \mathbf{J})$. From now on we use the notation that an underbar denotes a 3-vector, i.e. $\underline{J}$ and a 2-vector is denoted $\mathbf{J}$. If the bosons are charged they couple to the external electromagnetic gauge field $\underline{A}$ via the usual minimal coupling: $\frac{\underline{\nabla}}{i} \to \frac{\underline{\nabla}}{i} - \underline{A}$ in Eq. (2.3). This amounts to adding $-i\underline{J}\cdot\underline{A}$ to Eq. (2.6). Thus $\underline{J}$ is identified as the physical 3-current of the bosons. To calculate the partition function one has to perform path integrals over $\underline{J}$ and χ.

We are now in the position to isolate the vortices in χ. In each time slice, $\chi(\mathbf{r},\tau)$ will in general be spatially smoothly varying except at a discrete set of singular points, the locations of vortices, at which ρ vanishes, and around which the phase of χ will wind by plus or minus 2π.[12] We thus decompose χ in each time slice as,

$$\chi = \chi_v e^{i\theta}, \tag{2.7}$$

where θ is a single valued real field, which describes the smooth "spin-wave" distortions of χ and

$$\chi_v(\mathbf{r},\tau) = \exp\left[i\sum_j q_j\Theta(\mathbf{r} - \mathbf{r}_j(\tau))\right] \,, \tag{2.8}$$

represents the vortices. Here the function $\Theta(\mathbf{r})$ gives the angle that $\mathbf{r}$ makes with the x-axis and $\mathbf{r}_j$ denote the positions of the vortices with "charge" $q_j = \pm 1$. Notice that as defined χ_v has unit modulus. Away from the vortex cores, $\bar{\chi}_v\boldsymbol{\partial}/i\chi_v$ is simply the gradient of the phase of χ_v, so that

$$\boldsymbol{\partial}x\left(\bar{\chi}_v\frac{\boldsymbol{\partial}}{i}\chi_v\right) = \rho_v \,, \tag{2.9}$$

where ρ_v is the vortex density:

$$\rho_v(\mathbf{r}) = 2\pi \sum_j q_j \delta^2[\mathbf{r} - \mathbf{r}_j(\tau)] \, . \tag{2.10}$$

We now insert (2.7) into the Lagrangian (2.6), and replace the functional integral over χ by a sum over all possible vortex positions and signs, in each time slice, and an integral over the real field θ. After an integration by parts, θ enters the Lagrangian only in the combination $i\theta\underline{\nabla}\cdot\underline{J}$. Performing the θ integral then leads to a constraint

$$\underline{\nabla}\cdot\underline{J} = 0 \, . \tag{2.11}$$

At this point the Lagrangian has the same form as in (2.6) except with χ_v replacing χ, and the integral over $\underline{J}$ subject to the constraint (2.11). Strictly speaking the functional integral over $\underline{J}$ depends on the positions of the vortices, since the boson density, J_0, must vanish at the vortex cores. Below we ignore this "coupling" and take the functional integral over $\underline{J}$ and the sum over the vortex world-lines as independent.

We solve the continuity constraint (2.11) on the boson particles 3-current by writing

$$\underline{J} = \underline{\nabla}\times\underline{b} \, , \tag{2.12}$$

where $\underline{b}$ is an unconstrained field. The functional integral over $\underline{J}$ is then replaced by a functional integral over the new field $\underline{b}$. Substituting (2.12) back into (2.6) then gives

$$\mathcal{L} = \frac{m}{2(\underline{\nabla}\times\underline{b})_0}|(\underline{\nabla}\times\underline{b})_\perp|^2 + \frac{1}{2m}|\partial(\underline{\nabla}\times\underline{b})_0^{1/2}|^2 + \frac{1}{2}[(\underline{\nabla}\times\underline{b})_0 - \bar{\rho}]V[(\underline{\nabla}\times\underline{b})_0 - \bar{\rho}] + i\underline{b}\cdot\underline{J}_v \, . \tag{2.13}$$

In (2.13) we have defined a vortex 3-current $\underline{J}_v$ as

$$\underline{J}_v \equiv \underline{\nabla}\mathrm{x}\bar{\chi}_v \frac{\underline{\nabla}}{i}\bar{\chi}_v \, . \tag{2.14}$$

From its very definition $\underline{J}_v$ satisfies a continuity equation $\underline{\nabla}\cdot\underline{J}_v = 0$. Moreover J_v^0 is precisely the vortex "charge" density as defined in (2.10). In (2.13) $(\underline{\nabla}\times\underline{b})_0$ and $(\underline{\nabla}\times\underline{b})_\perp$ denote the time and space components of the 3-vector $\underline{\nabla}\times\underline{b}$ respectively.

In its final representation the Lagrangian (2.13) represents a vortex 3-current coupled to a fluctuating gauge field. As (2.12) shows, the curl of this gauge field is the original boson particles 3-current. The partition function is expressed as a trace over the "gauge field" $\underline{b}$ and a sum over vortex world line configurations with vortex density (2.10) and a vortex current expressed in terms of the paths as

$$\mathbf{J}_v(\mathbf{r},\tau) = \sum_j q_j \partial_\tau \mathbf{r}_j \delta^2[\mathbf{r} - \mathbf{r}_j(\tau)] \, . \tag{2.15}$$

Thus, this final dual representation is in a sense "mixed", with the original particles represented as a second quantized "coarse-grained" 3-current (via (2.12)) and the vortices represented in a "first-quantized" world-line description.

In order to understand the final dual representation (2.13), it is useful to introduce a shifted "gauge"-field $\delta\underline{b}$ as,

$$\underline{b} = \delta\underline{b} + \langle \mathbf{b} \rangle \tag{2.16}$$

with $\boldsymbol{\partial}\mathbf{x}\langle\mathbf{b}\rangle = \bar{\rho}$. Upon substituting (2.16) into (2.13) we obtain

$$\mathcal{L} = \frac{m}{2\bar{\rho}}|(\underline{\nabla} \times \delta\underline{b})_\perp|^2 + \frac{1}{2}(\underline{\nabla} \times \delta\underline{b})_0 V (\underline{\nabla} \times \delta\underline{b})_0 + i\underline{b} \cdot \underline{J}_v \;, \tag{2.17}$$

where we have replaced $(\underline{\nabla} \times \underline{b})_0$ in the first term of (2.13) by $\bar{\rho}$ and ignored the second term, which should be valid at long wavelengths.

Let's consider first the properties of the Bose system in the absence of vortices. This will correspond to the superfluid or superconducting phase, as we shall see below. Setting $\underline{J}_v = 0$ then reduces (2.17) to a pure "gauge-field" Lagrangian,

$$\mathcal{L} = \frac{m}{2\bar{\rho}}|(\underline{\nabla} \times \delta\underline{b})_\perp|^2 + \frac{1}{2}(\underline{\nabla} \times \delta\underline{b})_0 V (\underline{\nabla} \times \delta\underline{b})_0 \;. \tag{2.18}$$

This is a simple quadratic Lagrangian and can easily be diagonalized to yield a normal mode dispersion

$$\omega_q^2 = \frac{\bar{\rho}}{m}|\mathbf{q}|^2 V(\mathbf{q}) \;. \tag{2.19}$$

If the two-body interaction is short range, $V(\mathbf{q}=0) = \text{const.}$, (2.19) gives the usual linear dispersion of the quantized sound mode.

If the bosons are charged, they couple to an external electromagnetic field $\delta\underline{A}$ via a term of the form $-i\underline{\nabla} \times \delta\underline{b} \cdot \delta\underline{A}$ in Eq. (2.18). Then, upon integration over $\delta\underline{b}$ an effective gauge field action quadratic in $\delta\underline{A}$ is obtained:

$$\mathcal{L}_A = -i\delta A_0 \bar{\rho} + \frac{\bar{\rho}}{2m}|(\delta\underline{A})_\perp|^2 + \frac{1}{2}\delta A_0 V^{-1} \delta A_0 \;. \tag{2.20}$$

Here we have chosen the Coulomb gauge, $\boldsymbol{\partial} \cdot \delta\mathbf{A} = 0$. Equation (2.20) is the famous Higgs gauge action of a superconductor.

In order to examine the role of vortices, a particularly convenient choice of gauge in (2.17) is the Coulomb gauge: $\boldsymbol{\partial} \cdot \mathbf{b} = 0$. The terms entering the Lagrangian which involve the time component of the gauge field then take the form:

$$\frac{m}{2\bar{\rho}}|\boldsymbol{\partial} b_0|^2 + i b_0 \rho_v \;. \tag{2.21}$$

It is clear from (2.21), that upon tracing out b_0 a logarithmic interaction between vortices will be generated. The full Lagrangian can then be written as $\mathcal{L} = \mathcal{L}_M + \mathcal{L}_v + \mathcal{L}_{in}$ with

$$\mathcal{L}_M = \frac{m}{2\bar{\rho}}|\partial_0 \delta\mathbf{b}|^2 + \frac{V}{2}(\boldsymbol{\partial} \times \delta\mathbf{b})^2, \tag{2.22a}$$

$$\int \mathcal{L}_v d\mathbf{r} = \frac{\bar{\rho}}{4\pi m}\int d\mathbf{r}d\mathbf{r}' \rho_v(\mathbf{r}) \ln(|\mathbf{r}-\mathbf{r}'|) \rho_v(\mathbf{r}') \;, \tag{2.22b}$$

and,

$$\mathcal{L}_{in} = i\mathbf{b} \cdot \mathbf{J}^v . \tag{2.22c}$$

As in (2.18) the matter Lagrangian (2.22a) represents the density fluctuations of the original boson particles, and describes the quantized sound mode. Equation (2.22b) describes the vortex excitations in the boson field. Just as in the classical treatment of superfluid (or superconducting) films, the vortices interact logarithmically with one another.[13] In the presence of an external magnetic field H_0 applied perpendicular to the 2d boson system, ρ_v in Eq. (2.22b) is replaced by $\rho_v - H_0$. Therefore, this magnetic field will induce a predominance of positive vortices, with one extra positive vortex per flux quanta of the applied magnetic field.

The third term in the above Lagrangian couples together the vortices and the boson density fluctuations. When the vortices move they "see" a fictitious magnetic field ($\boldsymbol{\partial} \times \mathbf{b}$) with strength proportional to the boson density. A given vortex, which feels a force from the other vortices as a gradient of their logarithmic interaction (in (2.22b)), is thus effectively moving in crossed electric and magnetic fields. It will thus undergo an $E \times B$ drift, perpendicular to both fields. It should be emphasized, though, that the vortices are quantum particles which themselves have Bose statistics. A full quantum treatment involves as usual a sum over all possible vortex paths weighted by the above Lagrangian. If the explicit forms for the vortex density (2.10) and current (2.15) operators are inserted into the above Lagrangian, a saddle point evaluation leads as usual to a classical equation of motion. The explicit form is

$$q_j \partial_t \mathbf{r}_j = \frac{1}{2\bar{\rho}} \hat{z} \mathbf{x}\, \boldsymbol{\partial}_j U \ , \tag{2.23}$$

where U is the potential of interaction between the vortices:

$$U = \frac{\bar{\rho}}{4\pi m} \sum_{i,j} q_i q_j \ln(|\mathbf{r}_i - \mathbf{r}_j|) \ . \tag{2.24}$$

As we saw above, the boson's superfluid or superconducting phase (with zero applied magnetic field) is essentially just a vortex vacuum — a phase with no vortices present. More correctly, there will only be virtual fluctuations of vortices, that is vortex–antivortex pairs which live for only a short time. Because there are no real vortex excitations (at $T = 0$), low energy properties of the superfluid phase can be correctly obtained from (2.18) and (2.20), wherein $\underline{J}_v$ has been set to zero.

At finite temperatures, at the Kosterlitz-Thouless transition,[13] the vortex anti-vortex pairs will thermally unbind and the 2d boson system become normal. Is it possible for the vortex pairs to unbind at zero temperature due to quantum fluctuations alone? Indeed it is! Once unbound, being at $T = 0$, the vortices will undergo a Bose condensation into a "vortex-superfluid" phase. Since the vortices "see" a fictitious magnetic field, the "magnetic" flux, $\boldsymbol{\partial} \times \mathrm{b}$, which corresponds to the original boson's density, will form an Abrikosov flux lattice. Since each flux quanta corresponds to one boson, this phase is none other than a boson crystal.[14] If the original bosons were moving in a random potential, as for example in a disordered superconducting film, the "vortex-superfluid" phase would correspond to a localized boson-insulator[15] (a Bose glass).

A comment is in order here about the notion of a vortex inertial mass. First notice the absence of a vortex mass term, $m_v|\mathbf{J}_v|^2/2$, in (2.22b). In general if the boson system is translationally invariant, as for example in a superfluid ^{4}He fluid, we expect that m_v is identically zero. Then the vortices cannot sustain a magnus force,[16] and at $T = 0$ must move with a velocity equal to that of the local superfluid flow[17] (i.e. Eq. (2.23)). But if the translational invariance is broken, either spontaneously as in the formation of a solid phase or by some external periodic or random potential, we expect that a vortex mass will be generated. Generally, however, this mass term will not effect the long wavelength physics, since it is less singular than the term quadratic in $\mathbf{J}_v$ that is generated upon integration over $\delta\mathbf{b}$ in (2.22). In the superfluid phase, on scales long compared to the vortices cyclotron radius, one still expects the guiding center to undergo an $E \times B$ drift as in (2.23).

In the rest of the paper we shall apply the above duality transformation to study problems where statistics transmutation plays an important role. Section 3 is devoted to the fractional quantum Hall effect (FQHE) and anyon superconductivity will be examined in Sec. 4. In each of these two sections we will present the technical and pictorial arguments in parallel.

3. The Fractional Quantum Hall Effect

a) *Technical discussion*

Our first example where statistics transmutation (ST) plays an essential role is the FQHE. Although the title of this paper emphasizes anyon superconductivity, we will discuss mostly the FQHE. This is because the FQHE involves a wide range of technical and conceptual aspects that are necessary for an understanding of anyon superconductivity.

In its simplest and best known form, the FQHE can be described as follows.[18] When a high mobility 2d electron gas is placed in a strong magnetic field, H_0, at low temperature, the Hall conductivity σ_{xy} develops plateaus at values e^2/kh when ν, the filling factor (defined as the total number of electrons divided by the total number of Dirac flux quanta passing through the sample), is near to $1/k$ where k is an odd integer. Moreover, the longitudinal conductivity σ_{xx} develops minima near the same filling factors. The values of σ_{xx} at these minima are found to vanish exponentially with temperature, $\sigma_{xx} \propto e^{-\Delta/k_B T}$.

An explanation of this effect, due to Laughlin,[19] involves the following wave function describing an incompressible quantum liquid at the filling factor $\nu = 1/k$,

$$|\Psi_k\rangle = \frac{1}{\mathcal{N}} \int dz_1 \ldots dz_N \Phi_k(\{z_i\}) \psi^+(z_1) \ldots \psi^+(z_N)|0\rangle \ , \tag{3.1a}$$

where

$$\Phi_k(\{z\}) = \prod_{i>j}(z_i - z_j)^k \exp\left[-(1/4)\sum_i |z_i|^2\right] . \tag{3.1b}$$

Here ψ^+ is an electron creation operator, $z = (x+iy)/\ell_H$ ($\ell_H = \sqrt{\hbar c/eH_0}$) is the dimensionless complex coordinate of the particles. The Laughlin wave functions have enjoyed tremendous success in explaining the ground state and the low-energy excited state properties of the quantum-Hall liquid. Rather than repeat these discussions, we will follow the seminal work of Girvin and MacDonald[20] and Read[21] and ask "Is there an order parameter characterizing the Laughlin state?". The task of finding an order parameter is to identify an operator whose auto-correlation function exhibits long-range order. In order to dramatize the effect of finding the right operator, let us start by considering the equal time single electron correlation function,

$$\Gamma_\psi(z - z') \equiv \langle \Psi_k | \psi^+(z)\psi(z') | \Psi_k \rangle . \tag{3.2}$$

It turns out that Γ_ψ can be computed exactly for Laughlin's wave functions (3.1) and the result is [20]

$$\Gamma_\psi(z - z') = \frac{1}{2\pi k}\exp[-|z - z'|^2/4]\exp[(\bar{z}z' - \overline{z'}z/4] . \tag{3.3}$$

Notice the gaussian decay of the single electron correlations, which is faster than the usual exponential decays in a localized electronic phase in the absence of magnetic field. In any case, as expected from Yang's theorem,[22] long-ranged order is not present in the single electron correlation function. How about multi-electron correlation functions? In a BCS superconductor, for example, while the single electron correlation function decays exponentially, the pair electron correlation function exhibits long-range order. Can the same thing happen here in the Laughlin state? Unfortunately, the multi-electron correlation functions can no longer be computed exactly. Nevertheless, there are numerical results[23] as well as physical arguments[24] which suggest that all finite number of electron correlation functions decay as gaussians. An operator that does exhibit (quasi-)long-ranged order, though, was found by Girvin and MacDonald[20] and Read.[21] Following Ref. 20 we write it as

$$\phi^+(z) \equiv \exp\left[ik\int dz' \Theta(z - z')\psi^+(z')\psi(z')\right]\psi^+(z) . \tag{3.4}$$

where $\Theta(z - z')$ is the angle subtended by the x-axis and a vector connecting the origin and the point $z - z'$. Notice that ϕ^+ as defined is an infinite-body operator, as can be seen upon expanding the exponential function. The correlation function between these ϕ-operators.

$$\Gamma_\phi(z - z') \equiv \langle \Psi_k | \phi^+(z)\phi(z') | \Psi_k \rangle \tag{3.5}$$

can be computed for Laughlin's wave function and it is found[20] that it decays algebraically with distance:

$$\Gamma_\phi(z - z') \propto |z - z'|^{-k/2} \tag{3.6}$$

exhibiting quasi-long-ranged order!

The operator ϕ satisfies $\phi^2(z) = 0$ and $[\phi(z), \phi(z')] = 0$ for $z \neq z'$. It is thus possible to represent the Hilbert space spanned by $\{\prod \phi^+|0\rangle\}$ as the finite-energy sub-Hilbert space of hard-core bosons. Despite its complicated appearance the operator ϕ in (3.4) has a simple physical interpretation. Specifically, the bosons created by ϕ^+ can be viewed as a composite particle made up of a fermion, created by ψ^+, and k attached vortices. As (3.6) shows, this boson has condensed and exhibits (quasi-)long-ranged order in Laughlin's state.

Since the boson in (3.4) is condensed in Laughlin's state, it can serve as an order parameter for the FQHE. This suggests an alternative formulation of the FQHE in which the original electron Hamiltonian is re-expressed in terms of these new composite boson operators.[25] To this end, it is useful to invert the transformation (3.4) between ϕ and ψ to give

$$\psi^+(z) = \exp\left[-ik\int dz'\Theta(z-z')\phi^+(z')\phi(z')\right]\phi^+(z)\,, \tag{3.7}$$

which expresses each electron as a boson plus k-attached antivortices. Then (3.7) can be substituted directly into the electron Hamiltonian appropriate for the FQHE,[25]

$$\begin{aligned} \mathrm{H} = &\frac{1}{2m}\int d^2\mathbf{x}|\mathbf{D}\psi(\mathbf{x})|^2 \\ &+\frac{1}{2}\int d^2\mathbf{x}d^2\mathbf{x}'(\psi(\mathbf{x})^+\psi(\mathbf{x})-\bar{\rho})V(\mathbf{x}-\mathbf{x}')(\psi(\mathbf{x}')^+\psi(\mathbf{x}')-\bar{\rho}) \end{aligned} \tag{3.8}$$

where $\mathbf{D} \equiv \frac{\boldsymbol{\partial}}{i} - \langle\mathbf{A}\rangle$ and $\boldsymbol{\partial}\mathbf{x}\langle\mathbf{A}\rangle = H_0$ with H_0 being the external magnetic field. This gives an effective boson Hamiltonian

$$\begin{aligned} \mathcal{H} = &\frac{1}{2m}\int d^2\mathbf{x}|(\mathbf{D}+\mathrm{a}(\mathbf{x}))\phi(\mathbf{x})|^2 \\ &+\frac{1}{2}\int d^2\mathbf{x}d^2\mathbf{x}'(\phi^+(\mathbf{x})\phi(\mathbf{x})-\bar{\rho})V(\mathbf{x}-\mathbf{x}')(\phi^+(\mathbf{x}')\phi(\mathbf{x}')-\bar{\rho}) \end{aligned} \tag{3.9}$$

where $\bar{\rho}$ is the average particle density and

$$\mathrm{a}(\mathbf{x}) = k\hat{z}\mathrm{x}\int d^2\mathbf{x}'\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2}\phi^+(\mathbf{x}')\phi(\mathbf{x}') \tag{3.10}$$

is the so-called "statistical vector potential".[25] In a coherent state path integral representation, the partition function associated with the boson Hamiltonian (3.9) can then be expressed as

$$Z = \int D[\phi, \bar{\phi}] D'[a_\mu] e^{-\int d\tau d^2\mathbf{x} \mathcal{L}} \tag{3.11}$$

where

$$\mathcal{L} = i\bar{\phi}\left(\frac{\partial_0}{i} + a_0\right)\phi + \frac{1}{2m}|(\mathbf{D} + \mathrm{a})\phi|^2 + \frac{1}{2}(\bar{\phi}\phi - \bar{\rho})V(\bar{\phi}\phi - \bar{\rho}) - \frac{i}{k}a_0 \boldsymbol{\partial} \times \mathbf{a}\ . \tag{3.12}$$

In (3.11) D' denotes that the path integral over the statistical gauge field a_μ is restricted to the Coulomb gauge, i.e. $\boldsymbol{\partial} \cdot \mathbf{a} = 0$. The last term in (3.12) is the so-called Chern-Simons term and it has the effect of attaching fluxes $\boldsymbol{\partial} \times \mathbf{a}$ to particles. More specifically, integration over a_0 gives a functional delta function which sets $\boldsymbol{\partial} \times \mathrm{a} = k\bar{\phi}\phi$, thereby attaching k fluxes in $\boldsymbol{\partial} \times \mathbf{a}$ to reach boson particle, turning them into fermions (cf. Eq. (3.7)). A gauge-invariant formulation is obtained via the replacement[25]

$$\frac{i}{k}a_0 \boldsymbol{\partial} \times \mathbf{a} \to \frac{i}{2k}\underline{a} \cdot \underline{\nabla} \times \underline{a} \tag{3.13}$$

in Eq. (3.12), and removal of the constraint on the path integrals over a_μ in (3.11). In this form the Lagrangian becomes

$$\mathcal{L} = i\bar{\phi}\left(\frac{\partial_0}{i} + a_0\right)\phi + \frac{1}{2m}|(\mathbf{D} + \mathrm{a})\phi|^2 + \frac{1}{2}(\bar{\phi}\phi - \bar{\rho})V(\bar{\phi}\phi - \bar{\rho}) - \frac{i}{2k}\underline{a} \cdot \underline{\nabla} \times \underline{a}\ . \tag{3.14}$$

Thus we have succeeded in re-expressing the original electron problem in terms of a boson field theory with Chern-Simons term. Morever, it is this boson which condenses in the FQHE phase. In Ref. 25, Zhang *et al.* directly exploited this fact and used (3.14) as a basis for a Ginzburg-Landau theory of the FQHE. Here we take a slightly different tack[8] and study the boson Hamiltonian (3.14) using the duality transformation of Sec. 2.

Here we outline the main steps in performing the duality transformation in the presence of the Chern-Simons term. As in Sec. 2 we first separate the modulus and phase degrees of freedom of the boson field using the decomposition (2.4). Upon substitution of (2.4) into (3.14) we obtain:

$$\begin{aligned} \mathcal{L} = i\rho\left(\bar{\chi}\frac{\partial_0}{i}\chi + a_0 - A_0\right) + \frac{\rho}{2m}\left|\bar{\chi}\frac{\boldsymbol{\partial}}{i}\chi + \mathbf{a} - \mathbf{A}\right|^2 \\ + \frac{1}{2m}|\boldsymbol{\partial}\rho^{1/2}|^2 + \frac{1}{2}(\rho - \bar{\rho})V(\rho - \bar{\rho}) - \frac{i}{2k}\underline{a} \cdot \underline{\nabla} \times \underline{a} \end{aligned} \tag{3.15}$$

where $\underline{A} \equiv \langle \mathbf{A} \rangle + \delta \underline{A}$. Here we have added in a source vector potential $\delta \underline{A}$ so as to be able to discuss the electromagnetic response functions. Once again we decouple the second term via a Hubbard-Stratonovich transformation to give

$$\mathcal{L} = i\underline{J} \cdot \left(\bar{\chi} \frac{\underline{\nabla}}{i} \chi + \underline{a} - \underline{A} \right) + \frac{m}{2\rho} |\mathbf{J}|^2 + \frac{1}{2m} |\partial \rho^{1/2}|^2 + \frac{1}{2} (\rho - \bar{\rho}) V (\rho - \bar{\rho}) - \frac{i}{2k} \underline{a} \cdot \underline{\nabla} \times \underline{a} \tag{3.16}$$

where $\underline{J} \equiv (\rho, \mathbf{J})$. The form of the coupling between $\underline{J}$ and $\underline{A}$ in the first term shows that $\underline{J}$ is the physical electron 3-current. To calculate the partition function one has to perform a path integral over $\underline{J}$, χ and a_μ.

We now follow identically the steps leading from (2.6) to (2.13): Decompose χ into a vortex contribution χ_v, and a spin wave phase θ, integrate over θ to give a continuity constraint $\underline{\nabla} \cdot \underline{J} = 0$, and solve this constraint by introducing a new field, $\underline{J} = \underline{\nabla} \times \underline{b}$. In addition, we now integrate over the statistical gauge field $\underline{a}$. In this way (3.16) is re-expressed as

$$\mathcal{L} = \frac{m}{2(\underline{\nabla} \times \underline{b})_0} |(\underline{\nabla} \times \underline{b})_\perp|^2 + \frac{1}{2m} |\partial (\underline{\nabla} \times \underline{b})_0^{1/2}|^2 + \frac{1}{2} [(\underline{\nabla} \times \underline{b})_0 - \bar{\rho}] V [(\underline{\nabla} \times \underline{b})_0 - \bar{\rho}] + i\underline{b} \cdot (\underline{J}_v - \underline{\nabla} \times \underline{A}) + i \frac{k}{2} \underline{b} \cdot \underline{\nabla} \times \underline{b} \tag{3.17}$$

where $\underline{J}_v \equiv \underline{\nabla} \times \bar{\chi}_v \frac{\underline{\nabla}}{i} \chi_v$ is once again the vortex 3-current. This is our final dual representation. Once again, as in (2.13), (3.17) describes a system of vortices, with 3-current $\underline{J}_v$, which "sees" a gauge field $\underline{b}$ whose curl is the original electrons 3-current; $\underline{J} = \underline{\nabla} \times \underline{b}$.

How can we describe the FQHE in this dual representation? As discussed above, in the QHE phase the boson field in (3.14) condenses. This implies an absence of vortices in the boson's wave function. But in general an applied magnetic field induces vortices (see (2.22b)). However, at the special fillings $\nu = 1/k$, the physical external magnetic field, H_0, is on average precisely cancelled by the statistical magnetic field carried with each boson. This can be made explicit by noting that the linear terms in b_0 in (3.17) are multiplied by $\rho_v - H_0 + k(\partial \times \mathbf{b})$. Moreover, in the Coulomb gauge, upon integration over b_0 a term of the form (2.22b) with $\rho_v \to \rho_v - H_0 + k\partial \times \mathbf{b}$ is generated which effectively sets $\langle \rho_v \rangle = H_0 - k \langle \partial \times \mathbf{b} \rangle = H_0 - k\bar{\rho}$. Thus at the special filling, $\nu = \bar{\rho}/H_0 = 1/k$, the mean number of vortices vanishes. The FQHE phase is thus a vortex vacuum.

In order to study this phase we define $\underline{b} = \delta \underline{b} + \langle \mathbf{b} \rangle$ where $\partial \times \langle \mathbf{b} \rangle = \bar{\rho}$, and at the filling factor $\nu = 1/k$ set $\langle \mathbf{A} \rangle = k \langle \mathbf{b} \rangle$ which gives

$$\mathcal{L} = \frac{m}{2(\underline{\nabla} \times \underline{b})_0} |(\underline{\nabla} \times \underline{b})_\perp|^2 + \frac{1}{2m} |\partial (\underline{\nabla} \times \underline{b})_0^{1/2}|^2 + \frac{1}{2} (\underline{\nabla} \times \delta \underline{b})_0 V (\underline{\nabla} \times \delta \underline{b})_0 + i\delta \underline{b} \cdot (\underline{J}_v - \underline{\nabla} \times \delta \underline{A}) + i \frac{k}{2} \delta \underline{b} \cdot \underline{\nabla} \times \delta \underline{b} + i \langle \mathbf{b} \rangle \cdot \mathbf{J}_v - i\delta A_0 \bar{\rho} \, . \tag{3.18}$$

Moreover, at long wavelengths, we replace $(\underline{\nabla} \times \delta \underline{b})_0$ in the first term in (3.18) by $\bar{\rho}$ and ignore the second term to obtain

$$\mathcal{L} = \frac{m}{2\bar{\rho}}|(\underline{\nabla} \times \delta\underline{b})_\perp|^2 + \frac{1}{2}(\underline{\nabla} \times \delta\underline{b})_0 V (\underline{\nabla} \times \delta\underline{b})_0 + i\delta\underline{b} \cdot (\underline{J}_v - \underline{\nabla} \times \delta\underline{A}) + i\frac{k}{2}\delta\underline{b} \cdot \underline{\nabla} \times \delta\underline{b} + i\langle \mathrm{b} \rangle \cdot \mathbf{J}_v - i\delta A_0 \bar{\rho} \ . \tag{3.19}$$

Finally we set the number of vortices, and the vortex current, to zero. The Lagrangian is now reduced to a purely quadratic form which in the absence of source terms, $\delta\underline{A} = 0$, is

$$\mathcal{L} = \frac{m}{2\bar{\rho}}|(\underline{\nabla} \times \delta\underline{b})_\perp|^2 + \frac{1}{2}(\underline{\nabla} \times \delta\underline{b})_0 V (\underline{\nabla} \times \delta\underline{b})_0 + i\frac{k}{2}\delta\underline{b} \cdot \underline{\nabla} \times \delta\underline{b} \ . \tag{3.20}$$

Equation (3.20) can be easily diagonalized to yield a normal mode dispersion

$$\omega_q^2 = \left(\frac{k\bar{\rho}}{m}\right)^2 + \frac{\bar{\rho}}{m}|\mathbf{q}|^2 V(\mathbf{q}) \ . \tag{3.21}$$

When V is short-ranged, Eq. (3.21) gives a normal mode frequency equal to $\omega_c \equiv k\frac{\bar{\rho}}{m}$ at $\mathbf{q} = 0$ and disperses quadratically away from $\mathbf{q} = 0$. Notice that this frequency is precisely equal to the electron cyclotron frequency, $\omega_c = H_0/m$. Indeed, this collective mode corresponds to the inter-Landau level particle-hole excitations, although it has been mis-identified[8,25] in the literature as the lowest Landau level collective mode discovered by Girvin, McDonald and Platzman.[26] (For details see Lee and Zhang.[27])

The response of the incompressible quantum-Hall liquid to an external electromagnetic field can be obtained by retaining $\delta\underline{A}$ in (3.19) and integrating out $\delta\underline{b}$ with $\underline{J}_v = 0$. This gives an effective gauge field action,[28]

$$\mathcal{L}_A = \frac{\bar{\rho}}{2mk^2}|(\underline{\nabla} \times \delta\underline{A})_\perp|^2 + \frac{1}{2k^2}(\underline{\nabla} \times \delta\underline{A})_0 V (\underline{\nabla} \times \delta\underline{A})_0 - \frac{i}{2k}\delta\underline{A} \cdot \underline{\nabla} \times \delta\underline{A} + \frac{i}{k}\langle \mathbf{A} \rangle \cdot (\underline{\nabla} \times \delta\underline{A})_\perp \tag{3.22}$$

which summarizes all electromagnetic response functions. For example, by calculating $\delta\mathcal{L}_A/\delta A_0$ in the limit $\delta\underline{A} = 0$ we obtain the average density $\bar{\rho} = \frac{1}{k}(\boldsymbol{\partial} x \langle \mathbf{A} \rangle)$. Taking the second derivative with respect to δA_0 gives the finite $\mathbf{q}$ compressibility, which vanishes in the $\mathbf{q} \to 0$ limit: $\mathbf{K}(\mathbf{q}, \omega = 0) = \frac{m}{\bar{\rho}k^2}|\mathbf{q}|^2$. By differentiating (3.22) with respect to $\delta\mathbf{A}$ we obtain the induced current. From the components proportional to $\mathbf{E} = \partial_0 \delta\mathbf{A} - \boldsymbol{\partial}\delta A_0$ and $\hat{z} \times \mathbf{E}$ follow the longitudinal and Hall conductivities, respectively. As expected we find $\sigma_{xx} = 0$ and $\sigma_{xy} = 1/k$ in the DC limit.

The correlation function between the ϕ operators in (3.5) can also be calculated from the effective gauge-field action (3.22). First note that $\delta\mathbf{A}$ enters into the original Lagrangian (3.14) in the combination $(\boldsymbol{\partial}\theta - \delta\mathbf{A})$, where θ is the phase of the ϕ-field. Provided we choose the Coulomb gauge, $\boldsymbol{\partial}\cdot\delta\mathbf{A} = 0$, the effective spin-wave Lagrangian for the θ-field describing the FQHE phase, can be obtained by simply replacing $\delta\mathbf{A}$ by $\boldsymbol{\partial}\theta$ in (3.22). (The Coulomb gauge insures that all cross terms of the form $\boldsymbol{\partial}\cdot\delta\mathbf{A}$ vanish.) Upon integration over δA_0 this gives

$$\mathcal{L}_\theta = \frac{\bar{\rho}}{2mk^2}|\boldsymbol{\partial}\partial_0\theta|^2 + \frac{m}{2\bar{\rho}}|\boldsymbol{\partial}\theta|^2 . \tag{3.23}$$

In Fourier space Eq. (3.23) is diagonal and gives a θ–θ correlation function equal to

$$\langle\theta(\mathbf{q},\omega)\theta(-\mathbf{q},-\omega)\rangle = \frac{m}{\bar{\rho}}\frac{1}{|\mathbf{q}|^2\left[\left(\frac{\omega}{\omega_c}\right)^2 + 1\right]} . \tag{3.24}$$

Using (3.24) it is straightforward to show that the $\phi \sim e^{i\theta}$ has an equal-time correlation function which indeed decays algebraically,

$$\Gamma_\phi(z - z') = \langle\bar{\phi}(z,t)\phi(z',t)\rangle \propto |z - z'|^{-k/2} . \tag{3.25}$$

The exponent $k/2$ agrees with Girvin and McDonald's result,[20] (3.6).

We can study the properties of the static vortex excitations by keeping a finite density of vortices in (3.19), i.e. $\rho_v(\mathbf{x}) \neq 0$, but setting the vortex current, $\mathbf{J}_v$, to zero. The energy associated with such a prescribed vortex density can then be obtained by integrating over the field $\delta\underline{b}$ in (3.19) and equating $E[\rho_v] = -(1/\beta)\ln Z$. This gives

$$E[\rho_v] = \frac{1}{2k^2}\int d^2\mathbf{x}d^2\mathbf{x}'\rho_v(\mathbf{x})V(\mathbf{x}-\mathbf{x}')\rho_v(\mathbf{x}') . \tag{3.26}$$

Notice that the quasiparticles described by (3.26) interact via the Coulomb potential $V(\mathbf{x})$, but with an effective charge of strength $1/k$. This is because each quasiparticle consists of one vortex and a screening cloud of $1/k$ electrons.[19] The screening cloud of electrons corresponds to an excess flux of $\boldsymbol{\partial}\times\mathbf{b}$, of strength $1/k$. Upon interchange of two vortices and their screening clouds,[8] this flux gives a Berry's phase of π/k. The quasiparticles thus have fractional $1/k$–statistics.[29,30]

Away from the special filling, $\nu \neq 1/k$, a non-zero density of vortices, $\langle\rho_v\rangle = H_0 - k\bar{\rho}$, is induced. (This can be inferred from (3.19) upon integration over b_0 in the Coulomb gauge, say.) Since the vortices see a fictitious magnetic field of average strength $\bar{\rho}$, they can form a Laughlin liquid themselves.[31] The only difference here is that the vortices are now being treated as (boson) particles. If $\langle\rho_v\rangle = \bar{\rho}/2p$ the vortices can condense into a boson fractional quantum-Hall liquid with $\sigma_{xy} = 1/2p$. By analogy to the original electron case, we can view this phase as a Bose condensate of composite particles, each made up of a vortex and $2p$ of their vortices. The appropriate Lagrangian for this phase can be obtained from (3.19) by attaching $2p$ flux tubes to each vortex:

$$\mathcal{L} = \frac{m}{2\bar{\rho}}|(\underline{\nabla} \times \delta\underline{b})_\perp|^2 + \frac{1}{2}(\underline{\nabla} \times \delta\underline{b})_0 V(\underline{\nabla} \times \delta\underline{b})_0 + i\underline{b} \cdot \underline{J}_v + i\frac{k}{2}\delta\underline{b} \cdot \underline{\nabla} \times \delta\underline{b} - i\underline{J}_v \cdot \underline{a} + \frac{i}{2}\frac{1}{2p}\underline{a} \cdot \underline{\nabla} \times \underline{a} - i\underline{b} \cdot \underline{\nabla} \times \langle\delta\mathbf{A}\rangle \,. \tag{3.27}$$

Here $\boldsymbol{\partial} x\langle\delta\mathbf{A}\rangle \equiv H_0 - k\bar{\rho}$, and $\underline{a}$ is the new statistical gauge field which attaches $2p$ fluxes to each vortex. When $\langle\rho_v\rangle = \frac{\bar{\rho}}{2p}$, which corresponds to filling

$$\nu = \frac{1}{k + \dfrac{1}{2p}} \tag{3.28}$$

the statistical magnetic field $\boldsymbol{\partial} \times \mathbf{a}$ seen by the vortices cancels on average the field $\boldsymbol{\partial} x\langle\mathbf{b}\rangle = \bar{\rho}$. This enables the vortices to Bose condense, since they now see zero average magnetic field. In this Bose condensed state the discrete particle nature of the vortices can be neglected and at long wavelengths $\underline{J}_v$ replaced by

$$\underline{J}_v \rightarrow \underline{\nabla} \times \underline{b}' \,. \tag{3.29}$$

Here $\underline{b}'$ is an unconstrained field. Equation (3.29) still insures that the vortex number density is conserved. Upon substituting Eq. (3.29) into (3.27) and integrating out $\underline{a}$ we obtain the following Lagrangian:

$$\mathcal{L} = \frac{m}{2\bar{\rho}}|(\underline{\nabla} \times \delta\underline{b})_\perp|^2 + \frac{1}{2}(\underline{\nabla} \times \delta\underline{b})_0 V(\underline{\nabla} \times \delta\underline{b})_0 + i\underline{b} \cdot \underline{\nabla} \times \underline{b}' + i\frac{k}{2}\delta\underline{b} \cdot \underline{\nabla} \times \delta\underline{b} - \frac{i}{2}2p\underline{b}' \cdot \underline{\nabla} \times \underline{b}' - i\underline{b} \cdot \underline{\nabla} \times \langle\delta\mathbf{A}\rangle \,. \tag{3.30}$$

Moreover, $\underline{b}'$ can also be integrated out to give:

$$\mathcal{L} = \frac{m}{2\bar{\rho}}|(\underline{\nabla} \times \delta\underline{b})_\perp|^2 + \frac{1}{2}(\underline{\nabla} \times \delta\underline{b})_0 V(\underline{\nabla} \times \delta\underline{b})_0 + \frac{i}{2}\left(k + \frac{1}{2p}\right)\delta\underline{b} \cdot \underline{\nabla} \times \delta\underline{b} \,. \tag{3.31}$$

Notice that this Lagrangian is identical in form to the first hierarchy Lagrangian (3.20) except that k is replaced by $k + (1/2p)$. The collective mode frequency can thus be obtained from (3.21) by replacing k by $k + \frac{1}{2p}$. The frequency at $\mathbf{q} = 0$ can be reexpressed in terms of the magnetic field and one finds it is equal to the cyclotron frequency, $\omega_c = H_0/m$, just as it was for the first hierarchy. Once again, the collective mode represents the inter-Landau level particle-hole excitations. The effective action for the external electromagnetic field can also be read off directly from (3.22) with the same replacement, thereby enabling calculation of conductivities and compressibilities.

If the filling factor does not satisfy the form given by (3.28), there will be a finite density of vortices in the *vortex* wave function. In that case we have to add $i\underline{J}'_v \cdot \underline{b}'$ to the Lagrangian in (3.30). Then following the steps between (3.27) to (3.31) will give an effective action as in (3.32), but with $k + \frac{1}{2p}$ replaced by $k + \frac{1}{2p+\frac{1}{2p'}}$. The filling fraction at which the new vortices condense is given by

$$\nu = \cfrac{1}{k + \cfrac{1}{2p + \cfrac{1}{2p'}}} . \tag{3.32}$$

By iterating the above procedure we can generate all filling fractions at which an incompressible fractional quantum-Hall liquid exists, namely[29,31]

$$\nu = \cfrac{1}{k + \cfrac{1}{2p_1 + \cdots + \cfrac{1}{2p_n}}} \tag{3.33}$$

where $k =$ odd integer and $p_1, \ldots, p_n$ are integers.

b) *Heuristic discussion*

Now we switch gears and develop a pictorial representation of the FQHE discussed so far. For simplicity, we focus on the case of $\nu = 1/3$, in which the external magnetic field induces, on average, three vortices per electron. In the FQHE phase, each electron simply binds with three vortices and converts itself into a boson (see Fig. 1a). We note that in order to transmute the statistics from Fermi to Bose, an odd integer number of vortices is required. The ground state is then a Bose condensate of these bosons which has algebraic long-range order. (The reason for algebraic instead of true long-ranged order will become clear below.) Consider now the outcome of a Hall measurement. Figure 1b shows a schematic representation of the Hall strip. The current flowing down the channel is carried by the electron-plus-vortex composite particles. Since each of them consists of a unit charge and three vortices, accompanying the charge current is a vortex current three times as big. To the particles in the system, uniform vortex current has the same effect as a moving uniform magnetic field. Therefore it generates a transverse electric field, which in turn produces a transverse Hall voltage V_H equals to h/e^2 times the vortex current. The Hall resistance, obtained by dividing V_H by the charge current, is thus equal to $3h/e^2$ where the "3" simply reflects the ratio of the vortex number to the charge number in the electron-vortex opposite.

How about the incompressibility? Based on the physics of 2D superfluids at $T \neq 0$, one would naively associate a linear dispersive sound mode with algebraic off-diagonal-long-range-order. However, we know that the Hall liquid is incompressible at the magic filling factors, at least in the absence of disorder, with a massive normal mode (3.21). What has gone wrong? The answer is that spatial fluctuations in the condensate-boson density interact logarithmically (see Fig. 1c). This is because when the boson density fluctuates it causes a local excess or deficit in vorticity, which in turn sets up a circulating current which costs a logarithmically divergent energy. This generates an effective logarithmic interaction between the composite bosons. This long range interaction is responsible for both opening up a gap for long wavelength density fluctuations, (3.21), as well as suppressing the true long range order of these bosons.[32] How about the vortex excitations? If we force in an extra static vortex, say by increasing the applied magnetic field by one flux quantum, the composite bosons will screen it by creating a deficit of one vortex in the vicinity of the extra vortex. Since each condensate boson carries three vortices, the screening cloud involves a net deficit of 1/3 bosons and hence a deficit of charge $1/3\ e$ (see Fig. 1d). This dressed vortex is the Laughlin quasihole. Since it consists of a vortex bound to one-third of a charge, as discussed after Eq. (3.26), it has fractional statistics, i.e. the Berry phase due to adiabatic exchange is $\pi/3$.[30]

With an additional concept–duality, the hierarchy[29,31] of the FQHE can also be understood pictorially. Imagine starting with N electrons moving under magnetic field at filling factor $\nu \neq 1/k$. Again, the field induces $N\nu^{-1}$ vortices. Each electron binds to k vortices leaving $N(\nu^{-1}-k)$ unbound ones (see Fig. 2a). Now we can exploit the concept of duality: If we regard each vortex as a "particle", then the composite bosons appear to these "particles" as vortices. This is because the composite boson carries charge, and when a vortex goes around a unit of charge it picks up a Berry's phase of 2π. (If we bring one vortex around another, there is no Berry's phase, so that the vortices behave like bosons among themselves.) Therefore, in the dual picture, we have $N(\nu^{-1}-k)$ new bosons and N new vortices (see Fig. 2b). A second level of charge-vortex binding can now occur. Since the new particles are bosons, they can bind with an even number, $2p_1$, of vortices (Fig. 2c) and still maintain their Bose nature. If such binding eliminates all the new vortices, then the new composite bosons can condense. The condition for this is $2p_1 N(\nu^{-1}-k) = N$, or $\nu = \frac{1}{k+1/2p_1}$-the special filling factors corresponding to the second hierarchy of the FQHE, (3.28). If we iterate the above procedure, a sequence of filling factors given by Eq. (3.33) is generated, at which an appropriate composite bosons can bose condense.

4. Anyon Superconductivity

a) *Technical discussion*

Finally we are in a position to consider anyon superconductivity.[3,8,33] Since the anyons most likely to have any relevance to the high T_c copper oxides are "semions", we will concentrate on this case. Semions have the property that when a pair of them are exchanged, counterclockwise say, they pick up a phase factor of $e^{i\pi/2}$. Since the ground state properties of both a collection of weakly interacting bosons, which

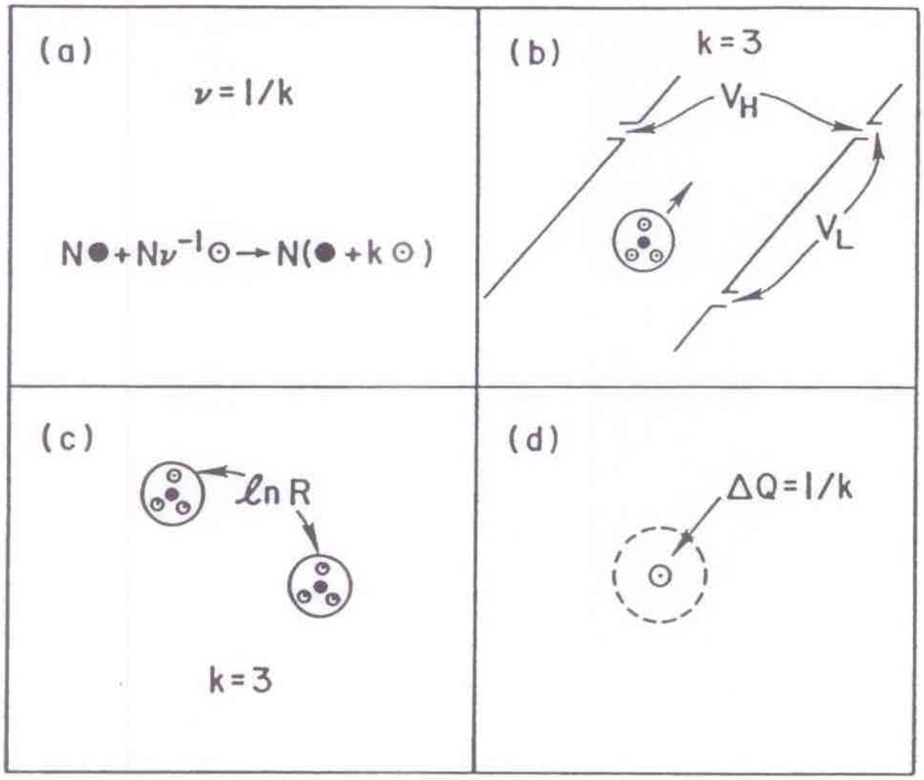

Fig. 1. a) Schematic representation of statistics transmutation (ST) in the FQHE at filling $\nu = 1/k$ with k an odd integer. Here the solid circle represents an electron, and the center-dotted circle represents a vortex. In the FQHE phase k vortices bind to each electron. b) A schematic representation of a Hall measurement. The charge current is carried by the bound electron-vortex composite bosons. Here V_L and V_H are the longitudinal and transverse voltage drops, respectively. c) Logarithmic repulsion between the composite bosons. d) Screening of an extra vortex by the composite bosons.

exhibit superfluidity and one-particle off-diagonal long-range order (ODLRO), and free fermions, which form a Fermi liquid, are well understood, it is extremely natural to inquire: "What is the ground state of a gas of non-interacting semions?". Since, as it turns out, a weak repulsive interaction does not qualitatively affect the zero temperature state of anyons, we will discuss repulsively interacting semions in the following. The Hamiltonian describing a collection of semions moving in zero magnetic field can be obtained from (3.9) and (3.10) by setting $\langle \mathbf{A} \rangle \to 0$ and putting $k = 1/2$. Similarly, the Lagrangian in a coherent-state path integral representation can be obtained from (3.14) with the same replacements. Then following the steps from (3.15) to (3.19) with $\langle \mathbf{A} \rangle \to 0$ we obtain

$$\mathcal{L} = \frac{m}{2\bar{\rho}} |(\underline{\nabla} \times \delta\underline{b})_\perp|^2 + \frac{1}{2}(\underline{\nabla} \times \delta\underline{b})_0 V (\underline{\nabla} \times \delta\underline{b})_0 + i\delta\underline{b} \cdot (\underline{J}_v - \underline{\nabla} \times \delta\underline{A})$$
$$- \frac{i}{4}\delta\underline{b} \cdot \underline{\nabla} \times \delta\underline{b} + i\langle \mathbf{b} \rangle \cdot \mathbf{J}_v - i\delta A_0 \bar{\rho} - \frac{i}{2}\delta b_0 \bar{\rho} \,. \tag{4.1}$$

Once again, the $\underline{\nabla} \times \underline{b}$ is the original semion 3-current and $\underline{J}_v$ is the vortex 3-current. Here $\delta\underline{A}$ is a source electromagnetic field used to generate correlation

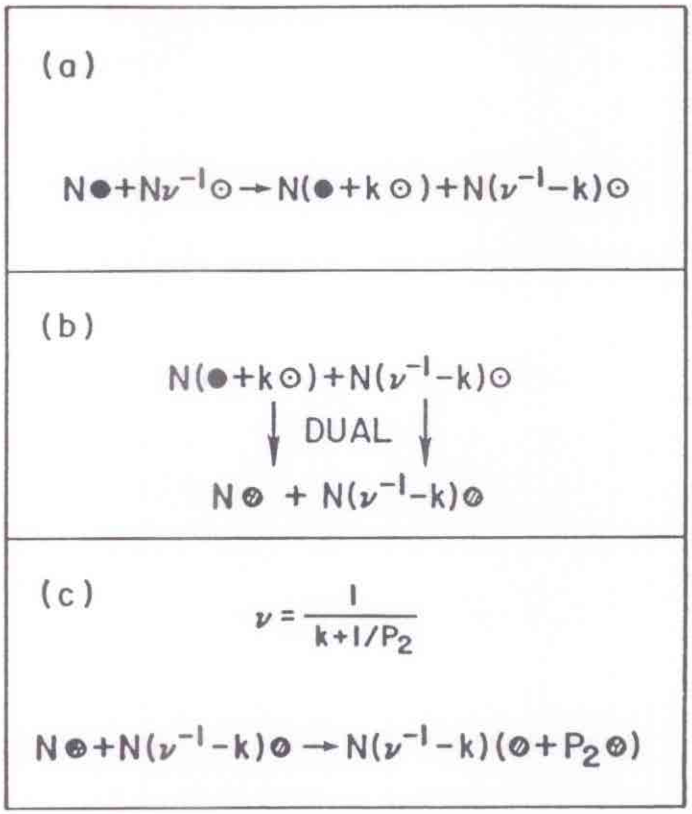

Fig. 2. a) Schematic representation of the first level in the FQHE hierarchical construction. Here k vortices bind to each electron, converting into a composite bosons. b) The dual representation of the composite boson-vortex mixture. Here the hatched circle represents an original vortex which behaves as a Bose particle in the dual picture, and the hatched and center-dotted circle represents the original composite boson which behaves like a vortex in the dual picture. c) Schematic representation of the second level in the hierarchical construction. Here p_2 new vortices bind to each new boson, with p_2 an even integer.

functions below. The last term in (4.1) effectively forces in a finite density ($=\bar{\rho}/2$) of vortices. This can be seen upon integration over b_0, in the Coulomb gauge, which generates a term of the form (2.22b) with $\rho_v \to \rho_v - \bar{\rho}/2$. As in Secs. 2 and 3, these vortices see an average magnetic field equalt to $\bar{\rho}$. As a result the effective filling factor for these vortices is $\nu = \frac{1}{2}$ which is a magic filling factor at which the vortices, being bosons, can themselves condense into a fractional quantum-Hall state. To implement this we follow (3.27) and introduce a statistical gauge field for the vortices to obtain:

$$\begin{aligned}\mathcal{L} = \frac{m}{2\bar{\rho}}|(\underline{\nabla} \times \delta\underline{b})_\perp|^2 + \frac{1}{2}(\underline{\nabla} \times \delta\underline{b})_0 V(\underline{\nabla} \times \delta\underline{b})_0 + i(\delta\underline{b} + \langle \mathbf{b} \rangle) \cdot \underline{J}_v \\ - \frac{i}{4}\delta\underline{b} \cdot \underline{\nabla} \times \delta\underline{b} - i\underline{J}_v \cdot \underline{a} + \frac{i}{4}\underline{a} \cdot \underline{\nabla} \times \delta\underline{a} - \frac{i}{2}\delta b_0 \bar{\rho} \,. \end{aligned} \tag{4.2}$$

Here we have momentarily dropped the source field $\delta\underline{A}$. We now follow precisely the steps between (3.29) and (3.31), namely, replace $\underline{J}_v \to \underline{\nabla} \times \underline{b}'$ and integrate out $\underline{a}$ and $\underline{b}'$. In this way we arrive at the final effective Lagrangian describing the anyon condensate.

$$\mathcal{L} = \frac{m}{2\bar{\rho}} |(\underline{\nabla} \times \delta\underline{b})_\perp|^2 + \frac{1}{2}(\underline{\nabla} \times \delta\underline{b})_0 V (\underline{\nabla} \times \delta\underline{b})_0 \ . \tag{4.3}$$

It is very important to notice that, in contrast to the FQHE condensate described by (3.31), the final effective anyon Lagrangian above has no Chern-Simons term. As a result, the collective excitation is gapless. The explicit dispersion relation follows by diagonalizing (4.3):

$$\omega_q^2 = \frac{\bar{\rho}}{m} |\mathbf{q}|^2 V(\mathbf{q}) \ . \tag{4.4}$$

For short range interactions this gives a linearly dispersing density or sound mode. If we turn the external electromagnetic source field $\delta\underline{A}$ back on, and integrate out $\delta\underline{b}$ we obtain an effective gauge field Lagrangian which in the Coulomb gauge ($\boldsymbol{\partial}\cdot\mathbf{A} = 0$) reads:

$$\mathcal{L}_A = -i\delta A_0 \bar{\rho} + \frac{\bar{\rho}}{2m} |(\delta\underline{A})_\perp|^2 + \frac{1}{2}\delta A_0 V^{-1} \delta A_0 \ . \tag{4.5}$$

Notice that Eq. (4.5) is identical to Eq. (2.20), i.e. it is precisely the Higgs gauge action of a conventional superconductor. Thus the anyon condensate will have identical electromagnetic response functions to a superconductor, exhibiting a Meissner effect, possessing an infinite d.c. conductivity, σ_{xx}, etc.

In general, if the original anyon model (3.14) had lacked translational symmetry, for example, being on a lattice or with a small random background potential, a vortex inertial mass would have been generated, being allowed by symmetry. This could be introduced phenomenologically into (4.1) by adding a term of the form $(1/2)m_v \underline{J}_v^2$, where m_v is the vortex mass. Carrying through the steps leading to (4.3) in the presence of this term leads to an additional term in (4.3) of the form,

$$i m_v \underline{\nabla} \times (\underline{\nabla} \times \delta\underline{b}) \cdot (\underline{\nabla} \times \delta\underline{b}) \ . \tag{4.6}$$

This is a Chern-Simons like term but involves $(\underline{\nabla} \times \delta\underline{b})$ rather than $\delta\underline{b}$ itself, and thus does not generate a mass in the dispersion relation (4.4). It does, however, modify the effective gauge-field Lagrangian (4.5): Upon integration over $\delta\underline{b}$ in (4.3) with inclusion of the additional term (4.6) gives,

$$\mathcal{L}_A = -i\delta A_0 \bar{\rho} + \frac{\bar{\rho}}{2m} |(\delta\underline{A})_\perp|^2 + \frac{1}{2}\delta A_0 V^{-1} \delta A_0 + i\alpha \delta\underline{A} \cdot \underline{\nabla} \times \delta\underline{A} \ . \tag{4.7}$$

Notice the additional Chern-Simons term with α a non-universal constant proportional to the square of the vortex mass. From this extra term follows a non-zero value for the Hall conductivity in the anyon superconducting phase, $\sigma_{xy} = \alpha$. (To see this recall that differentiation with respect to $\delta\mathbf{A}$ generates the current, and that the electric field is given by $\mathbf{E} = \partial_0 \delta\mathbf{A} - \boldsymbol{\partial}\delta A_0$ as usual.) In Ref. 8 we studied a lattice anyon model, and a non-zero value of σ_{xy} was indeed found. Using a Gallilean

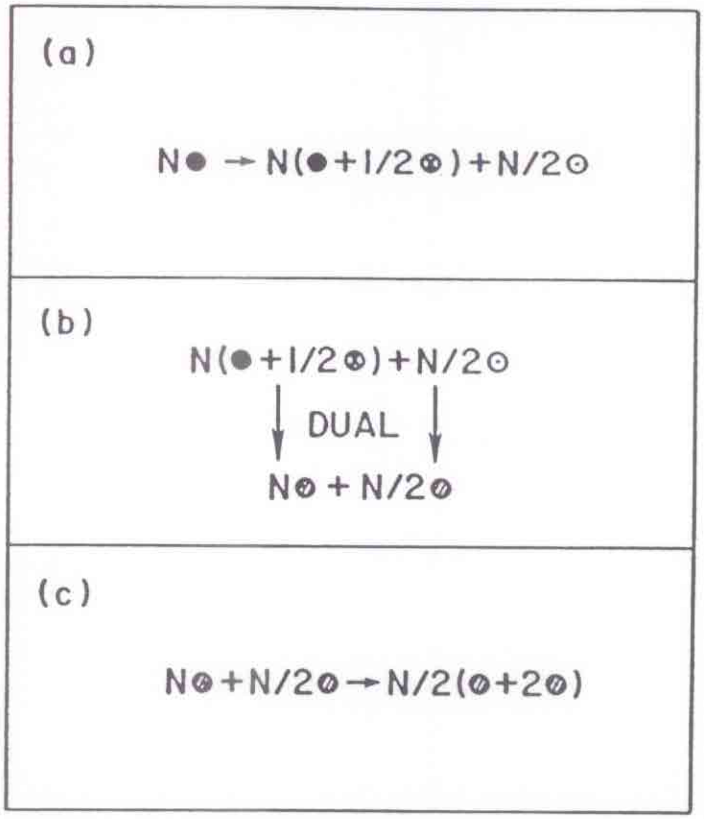

Fig. 3. a) Schematic representation of the first level in the hierarchical construction for a semion superconductor. Here the solid circle represents a semion, the center-crossed circle an antivortex and the dotted open circle a vortex. One-half of an antivortex binds to each anyon converting it into a composite boson. b) The dual representation of the composite boson-vortex mixture. Here the hatched circle represents an original vortex which behaves as a Bose particle in the dual picture, and the hatched and center-dotted circle represents the original semion-antivortex composite boson which behaves like a vortex in the dual picture. c) Schematic representation of the second level, in which each new boson binds to two new vortices and Bose condenses.

transformation, Chen *et al.*[33] have argued that for a translationally invariant system σ_{xy} should be identically zero, consistent with the above considerations.

b) *Heuristic discussion*

Let's consider the above technical discussion of anyon superconductivity in a pictorial representation. Ignoring for now any repulsive interaction between the anyons, the only energy scale is the kinetic energy, and the system will do whatever possible to minimize it. The state with the lowest possible kinetic energy is of course a condensate in which all the particles have zero momentum. Unfortunately, this state is forbidden by the statistics of the semions. From this point of view, the statistics of the semions frustrates minimization of the kinetic energy. In order to circumvent this frustration, the vacuum spontaneously nucleates $N/2$ vortices and $N/2$ antivortices, with N the number of semions. The antivortices bind to the semions and convert them into composite bosons (see Fig. 3a). In the dual picture the remaining $N/2$ free vortices behave like new bosons and the N composite bosons behave like new vortices (see Fig. 3b). Since the new vortex to new boson ratio is exactly 2:1, each new boson simply binds with two new vortices and Bose condenses (see Fig. 3c). By tracking back the duality transformation, depicted schematically in Fig. 3, one can easily see that the final boson which condenses consists of two charges and no original vortices.

Because the condensate boson contains no vortices, their density fluctuations do not have a long-range interaction. As a result, the density fluctuations, described by (4.4), are gapless in contrast to the FQHE. Moreover, since they possess no vorticity, the composite bosons cannot screen the interaction between vortices which are induced, say, by an external magnetic field. Thus, in contrast to the FQHE case, these vortices will interact via a bare logarithmic interaction. This fact is particularly important since it implies that thermally created vortex anti-vortex pairs can remain bound at low but non-zero temperatures. Anyon superconductivity will thus survive up to some Kosterlitz-Thouless transition temperature,[8,34] at which point the positive and negative vortices unbind. This should be contrasted to the FQHE effect which is a zero temperature phenomena: At any finite temperature the quantum Hall system will no longer exhibit long-ranged order, σ_{xx} will no longer be zero and the Hall conductivity will not be perfectly quantized.

It is possible[8] to generalize the hierarchical construction of the FQHE, which lead to the condition (3.33), to anyons with "statistics parameter" α_s not equal to 1/2 (as for semions). Specifically we can show that Bose condensation and anyon superconductivity is possible for any α_s which can be expressed as

$$\alpha_s = \cfrac{1}{2p_1 + \cfrac{1}{2p_2 + \cdots + \cfrac{1}{2p_n}}} \tag{4.8}$$

where all the p's are integers. If we define $\alpha_s = P/Q$ for the above fractions, in general Q anyons will bind and condense forming a superconductor.

5. On the Possible Relevance to Copper Oxides

How might semions be formed in the copper oxide planes of the high T_c superconductors? Here we outline one possible scenario.[35] The doped copper oxide planes consist of positively charged holes (i.e. empty sites) moving in a sea of short-range antiferromagnetically ordered spins.[36] As the hole hops around, the spins are transported. For example, if a hole hops around a closed loop, the final state of the spins along the loop differs from its initial state (see Fig. 4). The quantum amplitude for the hopping of the hole is proportional to the overlap between the initial and final spin states, which can be shown to be proportional to $e^{i\Omega/2}$, where Ω is the solid angle subtended by the three unit vectors $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$.[37] Therefore $\Omega/2$ simulates the loop integral of a fictitious gauge field. The idea is that due to quantum frustration in the spin degrees of freedom, there might be a condensation of these

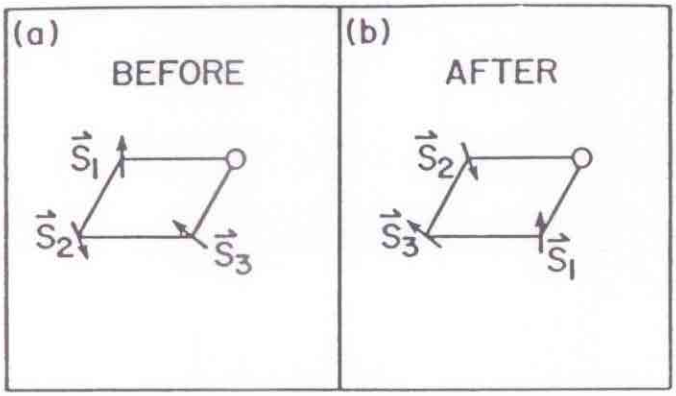

Fig. 4. The spin arrangements before (a) and after (b) a hole (open circle) hops counterclockwise around the square plaquette.

fictitious fluxes. Morever, it is argued that a half flux quantum of this fictitious gauge field will bind to a hole and transform it into a semion.

Is there any experimental evidence for semions in the high-T_c superconductors? Below we very briefly summarize the current status of the experimental searches for semion superconductivity in the copper oxides.

The first experiment is flux quantization. A semion superconductor, like ordinary superconductors, will exhibit charge $2e$ flux quantization. However, since time reversal symmetry must be broken just to have semions present in the first place, the value of the quantized flux can in general differ from $hc/2e$. However, the relevant theoretical issues here are not currently settled. The subtlety lies in the question "what is the appropriate boundary condition for anyons on a torus?".[38,39] In any case, the experimental results show no apparent shift[40] from $hc/2e$.

Since in real materials the translational symmetry is always broken, both by the crystal and by imperfections, an anyon superconductor will be described by (4.7) and have a nonzero Hall conductivity σ_{xy} in its superconducting phase. Hence, if one introduces a positively charged particle like a muon into an anyon superconductor, the electric field from the muon will induce a current which circles around the muon, out to the charge screening length. This current will generate a magnetic field which in turn will influence the precession of the muon spin. The influence of such internal fields on muon spin precession have not been observed in the copper oxides.[41]

Another effect expected in an anyon superconductor is photon polarization rotation. Provided both time reversal and parity (T-P) symmetries are indeed broken in the copper oxides, one expects[42] that photons propagating normal to the a-b plane will undergo rotation of their polarization axis at sufficiently low frequencies. The experimental result on this effect are currently controversial.[43]

The final effect is tunneling. If one makes a large area tunnel junction between a T-P violating superconductor and an ordinary superconductor, one might expect the absence of persistent current because the symmetries of the two superconductors are different. Here again the theoretical issues are not entirely settled.

Experimentally persistent current are observed between high T_c and conventional superconductors.[44]

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