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Hall effect at the magnetic-field-tuned superconductor–insulator transition

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The Hall effect for disordered superconducting films in magnetic field is discussed, with special focus on the vicinity of the magnetic field-tuned superconductor–insulator transition. Evidence is presented that upon ensemble averaging this transition possesses a "hidden" particle/hole symmetry. This symmetry implies that right at the transition the Hall resistance vanishes. Relevance to recent experiments is discussed briefly.

Hall effect studies in electronic systems had a renaissance in the 1980's with the discovery of the integer and fractional quantum Hall effects [1]. One of the most appealing theoretical descriptions of the quantum Hall effect involves a binding of two-dimensional vortices to the electrons, which effectively converts them into bosons [2,3]. Bose condensation then leads naturally to quantization and dissipationless flow [4,5], etc. In light of this, it is perhaps surprising that our understanding of the Hall effect in superconductors, one of the most extensively studied bosonic systems, is still quite rudimentary (for early theoretical work, see ref. [6]). This has become particularly apparent in the high temperature superconductors, which exhibit a mysterious sign reversal in the flux-flow Hall resistance [7].

Of all superconducting systems, thin disordered films in which the vortices are pointlike exhibit phenomena which are perhaps most closely related to the quantum Hall effect [8]. Indeed, in an applied perpendicular magnetic field, such thin films exhibit a field-tuned superconductor–insulator transition [9] which is intimately related to the transition between plateaus in the integer quantum Hall effect [8]. Right at this transition the films appear to exhibit metallic conduction, a finite resistance even in the $T \to 0$ limit. The value of the resistance, predicted to be universal [8], is found [9] to be clustered near 5 kΩ. The value of the Hall resistivity in this metallic state was also predicted to be universal [8], and the natural expectation was for a value comparable to the longitudinal resistance. Recent experiments [10], though, indicate a remarkably small Hall effect, with a Hall resistivity in the 10 Ω range, roughly a factor of 1000 smaller than the longitudinal resistance. Also, at this preliminary stage there appears to be a sample dependence and lack of universality to the Hall resistance.

This paper addresses the Hall effect in thin disordered superconducting films. Within a class of models which focus exclusively on the bosonic degrees of freedom, the magnetic-field-tuned superconductor–insulator transition is argued to possess a hidden "particle/hole" symmetry. This symmetry implies a vanishing of the fictitious magnetic field "seen" by the 2D vortices. Since this fictitious field is responsible for the Hall effect in superconducting films, its absence right at the transition implies a vanishing Hall resistance. Away from the transition a Hall effect should be present in general.

It is with greatest pleasure that I dedicate this article to my father, Michael E. Fisher, on the occasion of his 60th birthday. I feel this dedication is particularly appropriate since the conclusions in this work rely heavily on universality, a concept that Michael has taught us all so much about. On this special occasion I would like to wish Michael many more happy and productive years.

Consider first the forces on a vortex in a 2D superfluid. A vortex at rest in the presence of a transport current density $\boldsymbol{J}=\rho_s \boldsymbol{v}_s$, with ρ_s and $\boldsymbol{v}_s$ the superfluid areal number density and velocity respectively, experiences a Magnus force which is at right angles to the flow [11]: $\boldsymbol{F}_M=-q_v h \boldsymbol{J}\times\hat{\boldsymbol{z}}$. Here h is Planck's constant, $q_v=\pm 1$ is the vortex "charge" and $\hat{\boldsymbol{z}}$ is a unit vector orthogonal to the 2D system. When allowed to move in a translationally invariant superfluid such as ^{4}He, the vortex will not accelerate in the direction of the Magnus force as would a Newtonian particle [12], but rather will move downstream with the flow [13], at least at low temperatures where dissipation from the normal fluid can be ignored. The reason for this can be attributed to a fictitious or pseudo magnetic field, $\boldsymbol{b}$, that a 2D vortex feels when it moves [5,8] [#1]. The motion of the vortex downstream can then be understood as an $\boldsymbol{e}\times\boldsymbol{b}$ drift, resulting from the fictitious magnetic field crossed with the Magnus force, which can be viewed as a pseudo electric field, $\boldsymbol{e}$. Indeed, as we show below, in a translationally invariant fluid the fictitious field $\boldsymbol{b}$ has precisely the "right" strength to result in an $\boldsymbol{e}\times\boldsymbol{b}$ drift with velocity *equal* to the flow velocity v_s. Later we will argue that $\boldsymbol{b}$ might in fact be zero at the superconductor–insulator transition in disordered films, due to a hidden particle/hole symmetry.

The presence of the fictitious magnetic field $\boldsymbol{b}$ can most readily be inferred from the effective Euclidean action for a system of non-relativistic bosons [14b]:

$$S=\int \mathrm{d}\tau\,\mathrm{d}^2x\left(\psi^*\,\partial_\tau\psi+\frac{1}{2m}\,|\nabla\psi|^2-\mu|\psi|^2+u|\psi|^4\right), \tag{1}$$

with ψ the boson field. To this end decompose the field ψ in terms of the density and phase: $\psi=\sqrt{\rho}\,\exp(\mathrm{i}\phi)$. Insertion into the first term above reveals that it is pure imaginary and can be written

[#1] A vortex line in a 3D superfluid sees a 3D analog of the fictitious magnetic field. See e.g. ref. [14].

$$S_\tau = \mathrm{i}\int \mathrm{d}\tau\,\mathrm{d}^2x\,\rho\,\partial_\tau\phi = \mathrm{i}2\pi\int \mathrm{d}^2x\, m(x)\,\rho(x)\,, \tag{2}$$

where $m(x)$ is an integer winding number field which counts the number of times that the phase ϕ winds by 2π in moving from time 0 to $\beta\hbar$. The second equality above only follows strictly when it is legitimate to ignore the time dependence of the density.

Now imagine the motion of a single vortex through the superfluid at $T=0$. Consider specifically a vortex trajectory consisting of a single closed loop which is much larger than the vortex core radius. A little thought reveals that for this trajectory the winding number function $m(x)$ will be plus one for points x inside the loop and zero for those outside, except possibly for points lying within a core radius from the trajectory. For a vortex of the opposite circulation (or charge q_v) the winding number inside the loop will be minus one. Thus, precisely as for a charged particle moving in a perpendicular magnetic field, a 2D vortex picks up a complex factor $\exp(\mathrm{i}2\pi\rho A_{\mathrm{loop}})$ upon enclosing a loop with area A_{loop}. Notice that the phase factor is equal to 2π times the number of bosons encircled, so that the vortices see each boson effectively carrying one unit of fictitious magnetic flux. This feature plays a central role in recent charge-to-vortex duality transformations, in which the boson density is replaced by the curl of a fictitious fluctuating gauge field which is seen by the vortices [5,8,15].

Thus when a vortex moves through a superfluid, say with velocity $\boldsymbol{v}_v$, it experiences a pseudo-Lorentz force, F_L, from the fictitious magnetic field, with $\boldsymbol{F}_L = q_v h\rho\boldsymbol{v}_v\times\hat{\boldsymbol{z}}$. In a translationally invariant superfluid at $T=0$ in which $\rho_s=\rho$, this force precisely cancels the Magnus force provided the vortex moves with the flow: $\boldsymbol{v}_v=\boldsymbol{v}_s$. Put another way, the vortex $\boldsymbol{e}\times\boldsymbol{b}$ drift velocity in the presence of the Magnus force and the fictitious field $\boldsymbol{b}$ precisely equals the superflow velocity.

Vortex motion induces 2π slips in the phase of the boson order parameter and a resultant perpendicular chemical potential drop, given by $\nabla\mu = h\hat{\boldsymbol{z}}\times\boldsymbol{J}_v$, where $\boldsymbol{J}_v$ is the 2D vortex number current density. For a translationally invariant 2D superconducting film in magnetic field B, the vortex density is eB/h with e the Cooper pair charge, so that $\boldsymbol{J}_v = (eB/h)\boldsymbol{v}_v$. When all of the vortices move with the flow velocity, v_s, this implies a Hall resistance given by $R_{xy}=B/\rho e$.

As soon as the bosons are placed in an external potential, be it periodic or random, the above simplicity vanishes. No longer will a vortex move with the flow. Indeed, in superconductors, vortices typically move parallel to the Magnus force with only a small component downstream [7]. To see why this happens it is instructive to model the Cooper pairs in a superconductor as lattice bosons, rather than bosons moving in the continuum. To be concrete consider bosons hopping on a square lattice with a Hamiltonian $H=H_1+H_2$ where

$$H_1 = -\sum_{\langle ij\rangle}[\tilde{t}_{ij}\exp(\mathrm{i}A_{ij})\,\psi_i^\dagger\psi_j + \mathrm{h.c.}]\,, \tag{3a}$$

$$H_2 = \sum_i u(n_i + V_i - \bar{n})^2 . \tag{3b}$$

Here ψ_i is a boson destruction operator, the boson number operator is $n_i = \psi_i^\dagger \psi_i$, V_i is a random on-site potential and $\bar{n}$ sets the average density. An applied (physical) magnetic field has been included in the hopping term.

The partition function associated with this Hamiltonian can be expressed as an imaginary time path integral much as in (1), except with a discrete spatial lattice [16]. The complex piece of the action can still be expressed as in (2) but with a sum replacing the integral: $S_\tau = \mathrm{i}2\pi\sum_i m_i n_i$. Now notice that something very special happens when the boson density is chosen to be one boson per site. In this case, putting $n_i = \bar{n} = 1$, we see that $\exp(-S_\tau) = 1$, so that the vortices no longer see a fictitious magnetic field. More precisely, when a vortex moves through the lattice and encloses a loop it sees (on average) an integer number of fictitious flux quanta, one for each site enclosed. Since each contributes a factor of 2π it is as if the vortices are moving in zero fictitious field. In this case there is no reason why the vortices should move in any direction other than parallel with the Magnus force. The Hall resistance should thus vanish. For bosons with an average density of just less than one per site, the vortices will experience a small fictitious field of the "wrong" sign and will tend to move against the flow, changing the sign of the Hall effect.

Below we consider a model, closely related to (3) above, which has an exact "particle/hole" symmetry for integer boson densities per site. This symmetry enables us to show that the Hall resistance is identically zero. The model is a rotor model, popular in the context of Josephson junction arrays [17]. The hopping term in (3a) is replaced by

$$H_1 = -\sum_{\langle ij \rangle} t_{ij} \cos(\phi_i - \phi_j - A_{ij}) , \tag{4}$$

where ϕ_i is the phase of the boson field and $t_{ij} = \bar{n}\tilde{t}_{ij}$. The full Hamiltonian is a sum of H_1 in (4) above and H_2 in (3b), with the number operator n_i in (3b) taken to be conjugate to the phase: $[\phi_i, n_j] = \mathrm{i}\delta_{ij}$. In this rotor model the eigenvalues of n_i are all integers, including negative ones. Such negative integers will be costly in energy and should not affect the low energy physics.

For the special case of integer $\bar{n}$ in (3b), the rotor model has an exact particle/hole symmetry. In this case, $\bar{n}$ can be absorbed by the shift $n_i \to n_i + \bar{n}$. Then with $V_i = 0$, both the rotor Hamiltonian and the phase number commutation relations are invariant under the particle–hole transformation,

$$n \to -n , \qquad \phi \to -\phi , \qquad B \to -B . \tag{5}$$

Under this transformation the current operator, $J_{ij} = t_{ij} \sin(\phi_i - \phi_j - A_{ij})$, changes sign. Since the Hall conductivity, $\sigma_{xy}(B)$, can be expressed via a Kubo formula as a product of two current operators, the invariance in (5) implies $\sigma_{xy}(B) = \sigma_{xy}(-B)$. But since

σ_{xy} is odd in field B we conclude that the Hall conductivity (and hence resistivity) vanishes.

Although randomness in the hopping strength t_{ij} in (4) respects this particle/hole symmetry, a random potential V_i in (3b) in general does not. But suppose the distribution function $P(V)$ for the random potential, which is assumed independent from site to site, is *symmetric*: $P(V)=P(-V)$. In this case, although a given member of the ensemble will locally violate the symmetry (5), upon spatial averaging or equivalently ensemble averaging the symmetry will be restored. In the thermodynamic limit, the Hall resistance should thus still vanish.

A priori there appears to be no reason to expect that a real dirty superconducting film will have such a special particle/hole symmetry (even in the average sense). However, we will show below that in 1D and zero magnetic field, provided $V_i \neq 0$, the superconductor–insulator transition described by the rotor model is in the generic disordered boson universality class [16,18] even for a symmetric distribution of $P(V)$. Thus, at least in this 1D case, terms which break particle/hole symmetry in more realistic disordered boson Hamiltonians are apparently irrelevant in the critical regime of the $T=0$ phase transition. This 1D result suggests that the 2D field-tuned superconductor–insulator transition might also satisfy an ensemble-averaged particle/hole symmetry. If this is true, the Hall resistance right at the transition should vanish identically.

Consider then a 1D rotor model, (3b) and (4) with $A_{ij}=0$. To be concrete we assume that the random potential V_i is independent from site to site with a symmetric on-site distribution, and that the hopping $t_{ij}=t+\delta t_{ij}$, with a random piece δt_{ij} much smaller than the average t. The superconductor–insulator transition described by this model can be studied by expressing the partition function as a path integral and then performing a duality transformation [5,16,18]. Under duality the density $n_i \to \theta_{i+1}-\theta_i$, where θ_i is a real continuous field. The appropriate action takes the form

$$S=\int \mathrm{d}\tau \sum_i \left(\tfrac{1}{2}u(\theta_{i+1}-\theta_i+V_i)^2+\frac{1}{2t_{i,i-1}}(\partial_\tau\theta_i)^2-\lambda\cos(2\pi\theta_i)\right). \tag{6}$$

If θ is viewed as the height of a crystal interface, the boson (and boson–hole) world lines correspond to steps on the interface. The last term in (6) leads to discrete steps or equivalently discrete boson particles.

It is convenient at this stage to perform the change of variables $\theta_i \to \tilde{\theta}_i - W_i$ with $W_i=\sum_{j=1}^{i-1} V_i$, which shifts the random potential into the cosine term. Also, since the hopping strengths are assumed weakly random we can expand $1/t_{ij} \simeq 1/t+\delta t_{ij}/t^2$. In this way the action can be separated into two pieces, $S=S_1+S_2$ with S_1 independent of the randomness. After replacing the 1D lattice by a continuum we have

$$S_1=\frac{1}{2}\int_{x,\tau}\left(u(\partial_x\tilde{\theta})^2+\frac{1}{t}(\partial_\tau\tilde{\theta})^2\right), \tag{7a}$$

$$S_2 = \int_{x,\tau} \left(\frac{\delta t(x)}{t^2} (\partial_\tau \tilde{\theta})^2 - \lambda \cos(2\pi\tilde{\theta} - W_x) \right). \tag{7b}$$

Here $\delta t(x)$ can be taken as white noise with mean zero, and W_x has mean zero and $[W_x W_{x'}]_{\text{ens}} \sim \Delta |x-x'|$ where Δ is the variance of the on-site random potential V_i. This latter result follows even for symmetric distributions, $P(V) = P(-V)$.

The action S_1 in (7a) describes the sound mode of the superconducting phase, which has velocity $\sqrt{ut}$. As u and t are varied this term describes a fixed line of a renormalization group transformation which integrates out fast modes and rescales space and time isotropically [16,18]. The superconductor–insulator transition can be studied by studying the relevance of S_2 on this fixed line. To this end it is convenient to replicate the system and perform an ensemble average over the disorder fields, $\delta t(x)$ and W_x. Working perturbatively in λ the second term in (7b) generates, after an appropriate gradient expansion, a term of the form

$$\delta S \sim -\frac{\lambda^2}{\Delta} \sum_{\alpha,\beta} \int_{x,\tau,\tau'} \cos\{2\pi[\tilde{\theta}_\alpha(x,\tau) - \tilde{\theta}_\beta(x,\tau')]\}, \tag{8}$$

where α and β are replica indices running from 1 to m, with the $m \to 0$ limit understood as usual. This term has precisely the form obtained by Giamarchi and Schulz [16,18] in their original analysis and becomes a relevant perturbation to the fixed line for $\sqrt{ut} > 1/3\pi$. The point $\sqrt{ut} = 1/3\pi$ corresponds to the generic disordered boson superconductor–insulator transition. The first term in (7b) generates a term of the form

$$\delta S \sim \int_{x,\tau,\tau'} [\partial_\tau \tilde{\theta}_\alpha(x,\tau)\, \partial_{\tau'} \tilde{\theta}_\beta(x,\tau')]^2, \tag{9}$$

which is irrelevant (by power counting) at this transition.

Thus in 1D the rotor model describes a $T=0$ transition which is in the generic disordered boson universality class [16,18], even when the random potential distribution is symmetric so that upon ensemble averaging the model satisfies the symmetry (5). It is worth emphasizing that this result required the presence of the random on-site potential, which locally breaks the symmetry (5). Indeed were we to put $V=0$ in the action (6), it would describe a transition in a different universality class, namely that of a classical 2D xy model or a 1D Mott insulator to superconductor transition [16]. This follows since the term (9) generated from averaging the random piece of the second term in (6) is irrelevant at the pure system's 2D xy transition. Thus, as has been emphasized previously [16], a model with random hopping alone, which respects the particle/hole symmetry (5) for each member of the ensemble and locally, is not in the generic disordered boson universality class.

It seems most plausible that a similar ensemble-averaged particle/hole symmetry will also be present at the 2D superconductor–insulator transition. If this is the case, the Hall resistance right at the field-tuned transition in 2D should vanish. Since the symmetry will be manifest only right at the transition, away from the transition a Hall effect will in general be expected. Although preliminary measurements [10] at the field-tuned transition indicate a Hall resistivity much smaller than the longitudinal resistivity, a strictly zero value appears to be ruled out. Can this be reconciled with the above symmetry argument? It is conceivable that a non-zero Hall resistance could be due to the finite sample dimensions, which might be insufficient to adequately average over different regions with locally particle-like or hole-like Hall resistances. In this case, though, one would expect the measured Hall resistance to have a different sign from sample to sample, depending on whether the sample was predominately hole-like or particle-like. It is possible that the measured Hall resistance is due to residual electrons at the cores of the vortices [6], which have been neglected in the above theoretical considerations, which were based on the "boson-only" Hamiltonians (3) or (4). In ref. [19] it was argued that in the critical regime of the superconductor–insulator transition it should be valid to ignore unpaired electrons. This argument, while compelling in zero field, might perhaps be invalid at the field-tuned transition, due to electrons at the cores of the field-induced vortices. Finally, it is possible that for some unanticipated reason the particle/hole symmetry at the 1D transition breaks down in 2D. Further Hall effect measurements at the field-tuned transition should help clarify the current situation.

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