Nonlinear Transport and $1/f^{\alpha}$ Noise in Insulators

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We introduce and analyze a class of nonlinear Langevin equations that describe electrical transport in systems with electric-field thresholds, such as charge-density-wave systems biased just above threshold. For sufficiently large nonlinearities, the models are argued to exhibit scale-invariant phases wherein spatial and temporal correlations decay algebraically; in particular, current and voltage fluctuations show $1/f^{\alpha}$ noise with universal exponents α. The values of physical parameters required for the experimental observation of these phases are estimated.

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Fluctuations with a $1/f^{\alpha}$ power spectrum are a ubiquitous feature of electrical transport and many other nonequilibrium phenomena [1,2]. The power-law form and apparent universality of such fluctuations—$\alpha \simeq 1$ in many situations—is reminiscent of critical phenomena. It has nevertheless been argued [2] that, at least in condensed-matter systems, $1/f$ noise is not a consequence of collective behavior or spatial coherence that extends over long distances. Rather, it is usually ascribed to system-specific, short-distance physics, such as carrier trapping.

In recent years, however, it has been recognized that a class of stochastic, nonlinear, nonequilibrium systems exhibit nontrivial, universal $1/f^{\alpha}$ fluctuations as a consequence of either a conservation law or special symmetry that generates slowly (algebraically) decaying spatial and temporal correlations under generic conditions [3]. Prominent in this class are nonlinear growing interfaces [4], whose velocity noise spectrum has been explicitly shown [5] to behave like $1/f^{\alpha}$ with universal α. A second class of nonequilibrium systems, characterized by driving rates negligibly small compared to relaxation rates, have likewise been argued to display power-law spatial and temporal correlations, and hence $1/f^{\alpha}$ noise, without the tuning of external parameters, a behavior termed "self-organized criticality" [6].

Motivated by these recent theoretical developments, we ask whether there are circumstances in nonequilibrium electrical transport in which $1/f^{\alpha}$ fluctuations *do* arise from a universal source of long-length-scale fluctuations, and, if so, with what consequences. To attempt to answer this question, we introduce and study a new class of nonlinear Langevin equations appropriate for electrical transport in systems with electric-field thresholds, such as reverse-biased diodes, or charge-density-wave (CDW) systems biased just above threshold.

Our main result is the existence of a nontrivial, strong-coupling, scale-invariant phase in 2D systems, and in 3D systems with sufficiently strong nonlinearities. The scale invariance follows from gauge invariance (i.e., from the photon being massless), and results in current and voltage fluctuations that show $1/f^{\alpha}$ noise with universal exponents in these phases. We estimate the values of physical parameters required for the experimental observation of $1/f^{\alpha}$ noise from this mechanism.

In order to study fluctuations in the presence of a nonequilibrium transport current, consider a cylindrical sample with a current running parallel to its axis. A uniform current generates an azimuthal magnetic field which increases linearly with the radial distance from the axis, thereby destroying translational invariance and greatly complicating the analysis. In order to study nonequilibrium electrical transport in systems that are very nearly translationally invariant, we consider insulators with an electric-field threshold for conduction. Examples include CDW [7] systems, which allow very little current to pass below a threshold electric field E_T, or p-n junction diodes with reversed bias voltage, which admit negligible current flow below a breakdown voltage [8]. When biased just above threshold such systems, while clearly out of equilibrium, have a very small average transport current, so that inhomogeneities in the average background magnetic field are negligibly small. Time-dependent *fluctuations* of the magnetic field, which are present even in equilibrium, need *not* be small, however. The analysis below focuses exclusively on these fluctuations.

Traditionally, nonequilibrium transport is modeled by a Boltzmann equation that describes scattering in an electron gas. This approach typically ignores Coulomb interactions, effectively treating the electrons as neutral particles. In contrast, our approach focuses on the electromagnetic fluctuations, starting with Maxwell's equations for the electromagnetic fields in the presence of the electron charge (ρ) and electrical current (J) densities. These are supplemented by an empirical equation relating J to the electric and magnetic fields. For simplicity we consider first a very simple model system with a current which is zero below a threshold electric field E_T and is linear above it: $J=\sigma(E-E_T)$. In a gauge with zero scalar potential, both $\mathbf{E}$ and $\mathbf{J}$ can be expressed in terms of the vector potential, $\partial_t \mathbf{A} = -\mathbf{E}$, and from Ampere's law, $\mu_0 \mathbf{J} = \nabla \times \nabla \times \mathbf{A}$. In the latter we have dropped Maxwell's displacement current, $\partial \mathbf{E}/\partial t$, since it involves two time derivatives of $\mathbf{A}$ and hence is unimportant for the low-frequency fluctuations of interest to us. Within this model, the dynamical equation for $\mathbf{A}$, when the system is biased just above threshold with an electric field in the x direction, is

$$\partial_t \mathbf{A} = -E_T \hat{\mathbf{x}} - \nu \nabla \times (\nabla \times \mathbf{A}) + \boldsymbol{\eta}, \tag{1}$$

where $\nu = 1/\sigma\mu_0$; for the moment we ignore any anisotropy of the conductivity. Here we have added a fluctuating white-noise term $\boldsymbol{\eta}$, which is appropriate on length and time scales long compared to the correlation length and time of the transition at $E = E_T$. The electron charge density follows from (1) via $\rho = -\varepsilon_0 \nabla \cdot \partial_t \mathbf{A}$. Since Eq. (1) is linear, current and voltage noise spectra are easily shown to approach finite constants in the low-frequency limit [1].

In general, however, additional nonlinear terms should be added to (1). For example, the threshold field E_T must depend, even if weakly, on the magnetic field. This will produce on the right-hand side of (1) a term of the form λB_x^2, where $\lambda \sim (\partial^2 E_T / \partial B_x^2)_{B=0}$. Following the Langevin or fluctuating hydrodynamic approach often used to treat coarse-grained nonequilibrium systems [9], we consider *all* possible nonlinear terms allowed by the symmetries of the problem, and then restrict our attention to those which dominate at long lengths and times. As usual, this procedure can be systematized within a dynamical renormalization-group (RG) analysis [10].

The most restrictive symmetry is gauge invariance, which, by ensuring that the equations are unchanged by the transformation $\mathbf{A} \to \mathbf{A} + \mathbf{c}$ for any constant $\mathbf{c}$, is responsible for the scale invariance of the system. Even in the gauge with zero scalar potential, there is an additional gauge freedom, namely, invariance under $\mathbf{A} \to \mathbf{A} + \nabla \Lambda(r)$, for any time-independent function $\Lambda(r)$. We also assume the system is invariant under rotations about the x axis, the direction of the external electric field. Finally, combining parity and charge conjugation (change in sign of charge) shows that the equations must be invariant under $\mathbf{r} \to -\mathbf{r}$, even with the external field. Retaining only the most relevant terms which satisfy these symmetries yields

$$\partial_t A_x = -\nu \varepsilon_{\alpha\beta} \partial_\alpha B_\beta + \lambda_1 B_\alpha^2 + \lambda_2 B_x^2 + \eta_x, \tag{2a}$$

$$\partial_t A_\alpha = \varepsilon_{\alpha\beta} (\nu_1 \partial_x B_\beta - \nu_2 \partial_\beta B_x) + \lambda_3 B_x B_\alpha + \eta_\alpha, \tag{2b}$$

with $\alpha, \beta = y, z$ and $\mathbf{B} = \nabla \times \mathbf{A}$. The external white noise satisfies

$$\langle \eta_i(r,t) \eta_j(r',t') \rangle = \delta_{ij} D_i \delta^3(\mathbf{r} - \mathbf{r}') \delta(t - t') \tag{3}$$

with $D_x = D$ and $D_\alpha = D_\perp$. If the system were in thermal equilibrium one would have $D_j = 2k_B T/\sigma$, from detailed balance. Note that the symmetry allows (in fact, demands) three different ν values in Eq. (2), corresponding, e.g., to anisotropic conductivity in our simple model. Note too, however, that in deriving Eq. (2) purely from symmetry one no longer needs to assume that J is linear in E for $E > E_T$: Any arbitrary constitutive relation $\mathbf{J} = \mathbf{J}(\mathbf{E}, \mathbf{B})$ above threshold will result in the same equations.

The linear equations obtained by setting the three nonlinear couplings $\{\lambda_i\}$ to zero describe a fixed point of a simple RG transformation. Specifically, the equations in d space dimensions are invariant upon integration over modes with wave vectors in a shell between k/b and k, and all frequencies, and then a rescaling of all lengths by b, time by b^2, and the field A by $b^{(2-d)/2}$. Under this transformation, the $\{\lambda_i\}$ are multiplied by $b^{(2-d)/2}$, which implies that the nonlinearities are a marginal perturbation in 2D, and, to leading order, irrelevant in 3D. We now show, however, that for $d = 2$ they are marginally *relevant*, indicating the existence of a strong-coupling phase in 2D, and the probable existence of such a phase for sufficiently large values of the $\{\lambda_i\}$ in 3D.

Consider first the 2D situation, such as arises in a reverse-biased diode. Since essentially all of the voltage drop occurs across the thin depletion layer separating the two bulk electrodes, it seems reasonable to model the two electrodes as perfect metals, with zero internal electric fields [11]. In this case, one need only retain the x component of the vector potential, with $\partial_t A_x$ giving the electric field across the junction. Moreover, the spatial arguments of A_x can be restricted to the two coordinates, y and z, in the plane of the junction. Equations (2) then reduce simply to

$$\partial_t A_x = \nu \partial_\alpha^2 A_x + \lambda_1 (\partial_\alpha A_x)^2 + \eta_x, \tag{4}$$

which is precisely the Kardar-Parisi-Zhang (KPZ) equation [4] for a growing interface, with A_x playing the role of the interfacial "height." In terms of the dimensionless coupling constant $g = \lambda_1^2 D/\nu^3$, the RG flow equations to leading order are as found by KPZ: $\partial g/\partial l = g^2/4\pi$. Thus g grows under renormalization, and the system is driven into a strong-coupling phase.

Before discussing the noise spectra of this phase, we outline a perturbative RG analysis of the fully 3D equations (2). Although the nonlinear terms are irrelevant to leading order, a distinct strong-coupling phase is possible for larger values of λ_i. To see this, it is convenient to eliminate A_x from the linear terms on the right-hand side of (2b) by the change of variable: $A_\alpha = \partial_\alpha f + a_\alpha$, where $f(\mathbf{k},\omega) = \nu_1 k_x A_x(\mathbf{k},\omega)/(\nu_1 k_x^2 - i\omega)$. The perpendicular components of the magnetic field then become

$$B_\alpha = \varepsilon_{\alpha\beta} (\partial_\beta \phi - \partial_x a_\beta), \tag{5}$$

whereas $B_x = \varepsilon_{\alpha\beta} \partial_\alpha a_\beta$. Here we have introduced a new scalar field ϕ, defined by

$$(\partial_t - \nu_1 \partial_x^2) \phi(\mathbf{r},t) = \partial_t A_x(\mathbf{r},t). \tag{6}$$

In terms of ϕ and a_α the full equations (2) become

$$\partial_t \phi = \nu_1 \partial_x^2 \phi + \nu \partial_\alpha^2 \phi - \nu \partial_x \partial_\alpha a_\alpha + \lambda_1 B_\alpha^2 + \lambda_2 B_x^2 + \eta_x, \tag{7a}$$

$$\partial_t a_\alpha = \nu_1 \partial_x^2 a_\alpha - \varepsilon_{\alpha\beta} \nu_2 \partial_\beta B_x + \lambda_3 B_x B_\alpha + \eta_\alpha. \tag{7b}$$

To establish the existence of a strong-coupling phase, consider first the limit $a_\alpha = 0$. Physically this corresponds to having $B_x=0$, with magnetic field confined to lie perpendicular to the applied electric field. Formally it can be achieved by setting λ_3 and the transverse noise strength $D_\perp$ to zero, which allows (7b) to be solved by $a_\alpha=0$. In this limit the third and fifth terms on the right-hand side of Eq. (7a) can be dropped, reducing this equation to an anisotropic 3D KPZ-like equation for the scalar field ϕ, the nonlinear term having only two components: $\lambda_1(\partial_\alpha\phi)^2$. A dynamical RG analysis, perturbative in λ_1, can then be carried out [10,12]. One integrates over a shell in $k_\perp$ from $1/b$ to 1, and all k_x and ω. Rescaling as $\mathbf{r}_\perp=b\mathbf{r}'_\perp$, $x=b^\alpha x'$, $t=b^z t'$, and $\phi=b^\zeta\phi'$, and choosing the exponents α, z, and ζ to keep ν, ν_1, and D fixed, one computes the RG flows of the remaining coupling constant, λ_1. In terms of the dimensionless coupling $g=(\nu_1/\nu)^{1/2}D\lambda_1^2/\nu^3$, the resulting flow equation, analytically continued to $\varepsilon=d-2$ parallel and two perpendicular dimensions, is

$$\partial g/\partial l=(2-d)g+C_d g^2+O(g^3)\,, \tag{8}$$

where the coefficient C_d is positive for all $d\geq 2$. For $d=2+\varepsilon$ and ε small, there is an unstable critical point at $g_c=\varepsilon/C_2$ which separates weak- and strong-coupling phases. Assuming that this same flow structure persists up to $d=3$, one concludes that a strong-coupling phase occurs for all sufficiently large g in 3D as well [12,13].

Does this strong-coupling phase survive in the full equations (7), which include the fluctuations of the transverse field a_α? At the critical point found above all four couplings, ν, ν_1, D, and λ_1, are, of course, fixed: Treating the other four coupling constants ($D_\perp$, ν_2, λ_2, and λ_3) in (7), we show below that this critical fixed point is locally stable. This demonstrates that Eqs. (7) have (in $d=2+\varepsilon$) a perturbatively accessible critical fixed point with only one unstable direction [namely, g in (8)]. This critical point separates flows towards weak and strong coupling, providing powerful evidence for the existence of a strong-coupling phase in $d=3$ for the full equations.

It is straightforward to evaluate the dimensions of the four remaining coupling constants at the fixed point described by (8). We find that $D_\perp$ is irrelevant with dimension $-5g_c/64\pi$, λ_3 is dimensionless, and ν_2 and λ_2 are apparently relevant with positive dimensions, $g_c/32\pi$. However, whenever λ_2 or λ_3 enter diagramatically they contribute only in the combination $D_\perp\lambda_i\lambda_j/\nu_2$. This combination of couplings has a negative dimension, and so scales to zero at the fixed point. Thus in fact the contribution of λ_2 and ν_2 vanishes. The full equations thus have an (unstable) critical point with $g=g_c$, $D_\perp=1/\nu_2=1/\lambda_2=0$, and λ_3 finite and nonuniversal.

Having argued that strong electric fields drive certain insulators into a strong-coupling phase where nonlinearities are important, we now examine some of the scaling properties of that phase. Consider first the 3D model. The fundamental correlation function is $G(\mathbf{r},t)\equiv\langle\phi(\mathbf{r},t)\times\phi(\mathbf{0},0)\rangle_c$, where ϕ is defined in (6). The rescaling discussed before Eq. (8) implies the following RG equation for G:

$$G(\mathbf{r}_\perp,x,t)=b^{2\zeta}G(\mathbf{r}_\perp/b,x/b^\alpha,t/b^z)\,. \tag{9}$$

Voltage correlations of experimental interest can then be extracted from (9), since (6) relates the electric field to ϕ: $E_x=-\partial\phi/\partial t+\nu_1\partial_x^2\phi$. To be specific, consider the voltage drop across the sample in the x direction, averaged over y and z (where we assume the electrodes are as wide as the sample): $V(t)=[\int_0^{L_\perp}dy\,dz\int_0^{L_x}dx\,E_x(t)]/L_\perp^2$. The associated noise, $S_V(t)=\langle V(t)V(0)\rangle$, has three contributions, each of which can be related to G. For example, one contribution takes the form

$$S_V(t)\sim(L_x/L_\perp^2)\int dy\,dz\,d(x_1-x_2)\partial_{x_1}^2\partial_{x_2}^2 \times G(y,z,x_1-x_2,t)\,.$$

Using (9) and assuming the spatial integrals all converge at large distance yields a RG equation of the form $S_V(t)=b^{2\zeta-3\alpha+2}S_V(t/b^z)$, or equivalently, a power-law time dependence: $S_V(t)\sim(L_x/L_\perp^2)t^{(2\zeta-3\alpha+2+\theta)/z}$, with $\theta=0$. The other two contributions are also power laws, except with $\theta=2\alpha-z$ and $2(2\alpha-z)$, respectively. Clearly the sign of $2\alpha-z$ determines which contribution dominates at long times. Finally, Fourier transformation yields a voltage noise spectrum, $S_V(\omega)\sim(L_x/L_\perp^2)\omega^{-\kappa}$, where $\kappa\equiv 1+(2\zeta-3\alpha+2+\theta)/z$, valid down to a low-frequency cutoff, $\omega_L\sim L_\perp^{-z}$, set by the width of the sample (which is assumed smaller than its length L_x). This result is valid when the exponent κ is less than unity. For $\kappa>1$, the spatial integrals no longer converge, and one obtains the result $S_V(\omega)\sim L_xL_\perp^{2(2\zeta-3\alpha+\theta)}\omega^{-\kappa'}$, with $\kappa'\equiv 1-(2\zeta-3\alpha+\theta)/z$. *Spatial* dependences of voltage fluctuations can be obtained by similar applications of (9).

Current fluctuations can be extracted from the relation $\mathbf{J}\sim\nabla\times\nabla\times\mathbf{A}$ and Eq. (6). The most relevant experimental quantity is the noise spectrum of fluctuations of the total current in the x direction, i.e., the Fourier transform of the correlation function $S_I(t)\equiv\langle I_x(t)I_x(0)\rangle$, where $I_x\equiv\int d^2r_\perp J_x(t)$, and $J_x=-\partial_\alpha^2A_x+\partial_x\partial_\alpha A_\alpha$. Gauge invariance guarantees that the scaling of the two terms of this last expression is identical, so one need only consider the first of them. Using Eqs. (6) and (9) one again obtains three separate contributions of the form $S_I(t)\sim L_\perp^2 t^{(2\zeta-2-\theta)/z}$, where once again $\theta=0$, $2\alpha-z$, or $2(2\alpha-z)$. These yield a power spectrum dominated by the most singular contribution of the three: $S_I(\omega)\sim\omega^{-1-(2\zeta-2-\theta)/z}$.

The above results depend on the unknown exponents ζ, z, and α characterizing the strong-coupling fixed point. Recall that fluctuations in B_x were unimportant at the (unstable) 3D critical point analyzed above (i.e., the nonlinear terms with coupling constants λ_2 and λ_3 were irrelevant). It seems physically reasonable that this might also be true in the strong-coupling phase. If so, then one can argue that neither λ_1 nor ν_1 renormalizes, from which follow the relations $\zeta+z=2$ and $z=2\alpha$. In this case the current and voltage noise spectra respectively behave like $S_I(\omega)\sim\omega^{-2/z+1}$ and $S_V(\omega)\sim\omega^{-\kappa}$, where $\kappa=6/z-5/2$ if $z>12/7$, and $\kappa=9/2-4/z$ if $z<12/7$. If, for purposes of illustration, one uses the numerical estimate [14] $z\sim1.67$ for the isotropic 3D KPZ fixed point, one obtains $S_I(\omega)\sim\omega^{-0.20}$ and $S_V(\omega)\sim\omega^{-2.1}$. This should be compared with the results for the *linear* theory, Eq. (1), for which $z=2$, $\zeta=(2-d)/2$, and $\alpha=1$, so that both S_I and S_V approach constants in the low-frequency limit.

Similar scaling arguments can also be carried out in the case of a 2D reverse-biased junction, described by Eq. (4). For example, from the expression $J_x=-\partial_\alpha^2 A_x$ for the current density through the junction, one readily derives the noise spectrum for the total current: $S_I(\omega)\sim L^2\omega^{1-2/z}\sim\omega^{-0.25}$. Similarly, we find the power spectrum of fluctuations in the voltage averaged over the L^2 area of the junction: $S_V(\omega)\sim L^{-2}\omega^{-6/z+3}\sim\omega^{-0.75}$. Here we have used the exponent identity $\zeta+z=2$ for the 2D KPZ equation (4), and the numerical estimate [14] for the exponent z of 1.6.

It remains to argue under what conditions these various consequences of nonlinearity will be experimentally accessible. In 3D (e.g., in charge-density-wave materials), it is apparent from the flow equation (8) that one must find systems with sufficiently large values of the dimensionless coupling g to belong in the domain of attraction of the strong-coupling fixed point. From the simple model discussed above, where the source of the nonlinearity is the magnetic field dependence of the threshold electric field for depinning, one can write

$$g\sim2\Lambda k_B T\mu_0^3\sigma^2(\partial^2E_T/\partial B_\perp^2)^2\,, \tag{10}$$

where Λ^{-1} is a characteristic microscopic length (5 Å, say), and σ is the differential conductivity at the bias electric field above threshold. The requirement $g\gtrsim1$ can be cast in terms of the effective magnetic field $B_{\rm eff}$ needed to produce appreciable changes in the threshold electric field E_T through the definition $E_T/B_{\rm eff}^2\equiv\partial^2E_T/\partial B^2$. Numbers typical for CDW systems, $T\sim100$ K, $E_T\sim100$ V/m, and $\sigma\sim350/(\Omega\,{\rm m})$, imply that $B_{\rm eff}$ must be rather small—of order 0.1 G—in order for $g=1$. Although this does not seem particularly optimistic, there is considerable room for increasing g by choosing materials with larger values of σ and/or E_T.

Though in the 2D case the weak-coupling fixed point at $g=0$ is unstable to flows toward strong coupling for arbitrarily small g, g must still be of order unity for the nonlinear effects to be manifest on observable length and time scales. The *marginal* instability of the weak-coupling fixed point in 2D implies that for small g nonlinear effects first manifest themselves on exponentially long length scales: $\Lambda^{-1}e^{1/g}$. Similarly, the frequency below which the $1/f^\alpha$ noise predicted above is manifest decreases like $e^{-2/g}$, so $g\sim1$ is still imperative. Nevertheless, this requirement might be more easily obtained in 2D p-n junctions than in 3D CDW systems, due to the extremely large threshold fields in typical p-n junctions, $E_T\sim10^5$–10^7 V/m. With $E_T\sim10^7$ V/m, a conductivity of only $100/(\Omega\,{\rm m})$ yields $B_{\rm eff}$'s as high as 100 G.

Finally we note that as the electric field is increased further above threshold, all the above results eventually break down due to magnetic field inhomogeneities from the (now) non-negligible *average* current $\bar{J}_x$. Specifically, inhomogeneities become important when $\mu_0\bar{J}_xL_\perp\gtrsim B_{\rm eff}$, where $L_\perp$ is the sample dimension transverse to the applied current. For this reason, care should be taken to bias the system just above threshold.

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