LAUGHLIN STATES AT
THE EDGE

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Abstract

An effective wavefunction for the edge excitations in the Fractional quantum Hall effect can be found by dimensionally reducing the bulk wavefunction. Treated this way the Laughlin $\nu = 1/(2n+1)$ wavefunction yields a Luttinger model ground state. We identify the edge-electron field with a Luttinger hyper-fermion operator, and the edge electron itself with a non-backscattering Bogoliubov quasi-particle. The edge-electron propagator may be calculated directly from the effective wavefunction using the properties of a one-dimensional one-component plasma, provided a prescription is adopted which is sensitive to the extra flux attached to the electrons.

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1. Introduction

In introducing his chiral Luttinger liquid theory for the edge states in the fractional quantum Hall effect (FQHE), Wen [1] gave a persuasive, but indirect, argument for Luttinger-like behaviour of the the edge-electron Green functions. For droplets composed of electrons in the simplest FQHE phases with filling fraction $\nu = 1/m$ (m an odd integer) he concluded that there should be only one branch of edge excitations, and that the edge-electron propagator should decay with a power law

$$\langle droplet|T\{\psi^\dagger(x,t)\psi(0,0)\}|droplet\rangle \propto \frac{1}{(x-vt)^\xi}, \tag{1.1}$$

where v is the velocity of the unidirectional edge waves, and x the distance along the circumference. The exponent ξ turns out to be equal to m. The identification of the FQHE edge with a Luttinger liquid opens up a number of possibilities for confronting theory with experiment. For example, the problem of resonant tunneling between edge states can be mapped [2] onto previous work on one-dimensional electron gasses [3] and this gives results that agree quite well with experiment [4].

Despite this success there are still some aspects of the FQHE/Luttinger liquid correspondence that could be more transparent. As in the conventional Luttinger liquid [5] the exponent ξ is determined by a singularity in the single-particle occupation-number density at the Fermi surface. For a circular droplet the relation $\xi = m$ implies that the occupation of lowest Landau level orbitals with angular momentum N must go to zero as $(N_{max} - N)^{m-1}$. That this is true for the Laughlin wavefunction has been confirmed by numerical evaluation of the occupation numbers for small numbers of electrons [6], and by an analytic calculation of the density using the one-component plasma interpretation [7]. To the best of our knowledge however the Luttinger liquid behaviour of the edge states has not been directly connected to properties of the FQHE wavefunction. This paper is intended to provide such a link.

A relation between the edge correlator and the bulk FQHE wavefunction can be found by exploring the analogy between the conventional, non-chiral, Luttinger liquid and the $\nu = 1/m$ FQHE phases. This analogy goes further than the sharing of power law correlators. Both systems have Jastrow type wavefunctions composed of products of differences of particle coordinates. The product wavefunction proposed by Laughlin for the FQHE [8] has

a large overlap with the true ground state, and more importantly, precisely reproduces its long-distance part [9,10,11]. The ground-state wavefunction for the Luttinger (or Thirring) model is also known to be of Jastrow type [12]. It coincides with that of the Sutherland model [13] whose correlators were already known to be described by the same conformal field theory as the Luttinger-Thirring model [14].

Given the similarity of the correlators, can it be merely a coincidence that the wavefunctions of these systems also resemble one another? It is true that one is a product of coordinate differences in *one* dimension while the other is a product of differences in *two* dimensions, and that one system has only unidirectional (chiral) excitations while the other has modes propagating in both directions. Nonetheless, despite the difference of dimension, we expect the similarity of the wavefunctions to be more than an accident. The large magnetic field effectively halves the number of dimensions and converts the lowest Landau level into a phase-space with $x = $ 'q', $y = $ 'p'. This dimensional reduction leads to a very close analogy between the edge of a QHE droplet and a Fermi surface. The incompressible bulk of the QHE fluid corresponds to the Fermi sea, and heading inward from the edge to diving to deeper momentum.

For a *full* Landau level at least, we can make this dimensional reduction precise. By freezing the radial coordinates at the droplet radius, and treating the resulting wavefunction as one for free, one-dimensional electrons, we can recover most of the edge physics [15]. Unfortunately this simple recipe for creating an edge effective wavefunction does not extend straightforwardly to the fractional states. If we adjust the coupling constant in the Luttinger-Thirring model so as to give a Laughlin-like one-dimensional wavefunction with factors $(z_i - z_j)^m$, the Luttinger model fermion propagator does not have the desired $\xi = m$ exponent. Perhaps this is due to the very different dynamics of the Luttinger model? For non-vanishing coupling the left and right going Luttinger particles mix with each other, so there may be a profound difference between the chiral and non chiral systems. It turns out that this fear is groundless and the recipe can be made to work.

What has gone wrong is that we have misidentified the one-dimensional operator corresponding to the edge electron. In the Luttinger-Thirring model there is a family of operators which create charge e particles with fermi-like statistics, but which in some sense acquire a phase $e^{i\pi m}$, (m odd) when interchanged [16]. The first member of this

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family is the fundamental Fermi field in the relativistic Thirring model. In non-relativistic systems the operators with $m > 1$ are responsible for the appearance of subdominant terms in the two-point functions arising from the discreteness of the underlying charges. We will see that the FQHE edge-electron operator must be identified with one of these $m\pi$ statistics fields. This hyper-fermion is not simply related to the fundamental Luttinger fermions but can be written as an exponential of one of the non-backscattering Bogoliubov quasi-particles. Since the edge electrons do not backscatter in the absence of inter-edge tunneling, this shows that the identification can be used for dynamical properties, and not just for ground-state expectation values.

In section two we will give more details of the edge correlators and their connection to one-dimensional systems. In section three we will review some of the properties of the Luttinger-Thirring model and identify the edge fermion with an appropriate Haldane hyper-fermion operator. In section four we will use a one-component plasma method due to Hellberg and Mele [17] to calculate the edge correlator directly from the Laughlin wavefunction. This calculation will show how the two dimensionality of the Hall system requires us to distinguish between $e^{i\pi}$ and $e^{3i\pi}$. For completeness, the appendix contains a derivation of the ground state wavefunction for the Luttinger-Thirring model.

2. Edge correlators

There is an obvious similarity between the expression for the wavefunction of a droplet of N lowest Landau level electrons at filling fraction $\nu = 1$

$$\Psi_{\nu=1}(z_1, z_2, \ldots, z_N) = \prod_{i<j}(z_i - z_j)e^{-\frac{1}{4}\sum z_i^2} \tag{2.1}$$

and the wavefunction of a set of N one-dimensional, non-relativistic, fermions filling the N lowest energy plane wave states on a ring of circumference L

$$\Psi(x_1, x_2, \ldots, x_N) = e^{-ik_f \sum x_i} \prod_{i<j}(e^{2\pi ix_i/L} - e^{2\pi ix_j/L}). \tag{2.2}$$

($k_f = \pi(N-1)/L$ if N is odd). In particular, apart from trivial normalization and Fermi-momentum factors, the wavefunctions coincide at the boundary of the droplet where $|z| = R = \sqrt{2N}$, $z = Re^{i\theta}$, and $x = R\theta$ is the distance along the circumference.

This similarity extends to the correlators. The field operator for electrons in the lowest Landau level is

$$\psi(z) = \sum_{n=0}^{\infty} \hat{a}_n \frac{1}{\sqrt{2\pi 2^n n!}} z^n e^{-\frac{1}{4}|z|^2}, \tag{2.3}$$

where the $\hat{a}_n$ are fermionic annihilation operators obeying $\{\hat{a}_n^{\dagger}, \hat{a}_{n'}\} = \delta_{nn'}$. We can evaluate its equal-time two-point function — $i.e$ the one-particle density matrix

$$\langle \nu = 1 | \psi^{\dagger}(z)\psi(z') | \nu = 1 \rangle = G(z, z'). \tag{2.4}$$

Away from the boundary of the droplet $G(z, z')$ decays as a gaussian with a range of the magnetic length

$$|G(z, z')| = \frac{1}{2\pi} e^{-\frac{1}{4}|z-z'|^2}, \tag{2.5}$$

but when both z, z' approach the boundary, $G(z, z')$ becomes long ranged. Explicitly

$$G(x, x') = e^{-iN(x-x')/R} \int_{-\infty}^{\infty} d\xi \sqrt{\frac{B}{2\pi}} e^{-B\xi^2/2} \frac{1}{(x - x') - \xi + i\epsilon}. \tag{2.6}$$

Once again x denotes distance along the boundary (assumed small compared to the circumference), and we have temporarily restored the magnetic field B to make it manifest that (2.6) is a one-dimensional free-fermion correlator convoluted with a factor which serves merely to smooth it on scales shorter than a magnetic length. Actually we find only part of the usual $\sin k_f x$ prefactor, but if we were to consider an annulus rather than a disc we would find a contribution from the other fermi point on letting z, z' approach the inner edge.

Can this dimensional reduction continue to work when we replace the $\nu = 1$ droplet with a Laughlin $\nu = 1/m$ state

$$\Psi_{\nu=1/m}(z_1, z_2, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^m e^{-\frac{1}{4}\sum z_i^2} \quad ? \tag{2.7}$$

A one-dimensional wavefunction that resembles (2.7) is the Luttinger-Thirring model ground state. This wavefunction is (see ref. [12] and the appendix)

$$\Psi_{\{\lambda\}}(x_1, x_2, \ldots, x_N) = \prod_{i<j} \sin\frac{\pi}{L}(x_i - x_j) \left|\sin\frac{\pi}{L}(x_i - x_j)\right|^{\lambda-1}, \tag{2.8}$$

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where λ depends on the interaction. Perhaps this wavefunction is not immediately obviously a dimensionally reduced Laughlin state. For $\lambda = m$ an odd integer, however, a seemingly innocent manoeuver allows us to write $\Psi_{\{\lambda=m\}}$ as

$$\Psi_{\{\lambda=m\}}(x_1, x_2, \ldots, x_N) = \text{const.} e^{-i(N-1)\pi/L \sum x_i} \prod_{i<j} (e^{2\pi i x_i/L} - e^{2\pi i x_j/L})^m. \quad (2.9)$$

Once we drop the trivial $e^{-i(N-1)\pi/L \sum x_i}$ factor this is clearly of Laughlin form. We will use the notation

$$\Psi_m = \prod_{i<j} (e^{2\pi i x_i/L} - e^{2\pi i x_j/L})^m \quad (2.10)$$

for this state.

The dynamics of the Luttinger model is quite different from the FQHE edge. The edge has only a single branch of excitations, but the Luttinger fermion operator couples to both left and right goers, and so we must be prepared for both $x + v_f t$ and $x - v_f t$ dependence in the Green functions. Because of this we will focus initially on equal-time correlators since these depend only on ground state properties, and not on the spectrum.

Compare the integral

$$G_2(z, z') = Z_2^{-1} \int d^2 z_1 \ldots d^2 z_N \prod_i (\bar{z} - \bar{z}_i)^m \prod_i (z' - z_i)^m \prod_{i<j} |z_i - z_j|^{2m} e^{-\frac{1}{2} \sum_i |z|^2}, \quad (2.11)$$

which gives the FQHE one-particle density matrix, with that giving the same quantity in the Luttinger model

$$G_1(x, x') = Z_1^{-1} \int dx_1 \ldots dx_N \Psi_m^*(x, x_1, \ldots, x_N) \Psi_m(x', x_1, \ldots, x_N). \quad (2.12)$$

Motivated by 2.6 we might conjecture that they coincide on the boundary of the droplet

$$G_2(Re^{ix/R}, Re^{ix'/R}) \overset{?}{=} G_1(x, x'). \quad (2.13)$$

What evidence is there for this? Not much! The Fermion equal-time correlator for the Luttinger wavefunction (2.8) is [14]

$$\langle \Psi_{\{\lambda\}} | \psi^\dagger(x) \psi(x') | \Psi_{\{\lambda\}} \rangle = \frac{1}{(x - x')^{\frac{1}{2}(\lambda + \frac{1}{\lambda})}} \quad (2.14)$$

which does not reduce to (1.1) once $\lambda = m > 1$. The conjecture as stated cannot be true.

The heart of the problem lies in the seemingly innocent rewriting of (2.8) as (2.9). The two wavefunctions are algebraically identical, but in some sense the first changes by a factor of $e^{i\pi}$ under interchange of x_i with x_j, while the second changes by a factor of $e^{im\pi}$ (m, as always, is odd). Of course if we were *strictly* in one dimension this difference must be invisible. The FQHE particles however move in the plane and do not pass directly through one another, so they *can* perceive the distinction.

How can we build this phase into the evaluation of the Luttinger correlator? In the next section we will show that it may taken into account by altering the statistics of the operator we use to describe the fermion.

3. Haldane Fermions

The Luttinger model on an interval of period 2π is defined by the hamiltonian

$$\mathcal{H} = \int dx \left\{ \frac{1}{2} J_R^2 + \frac{1}{2} J_L^2 + \frac{g}{\pi} J_R J_L \right\}. \tag{3.1}$$

Here J_R, J_L are the currents for the left and right going fermions. They obey

$$[J_R(x), J_R(x')] = -[J_L(x), J_L(x')] = -\frac{i}{2\pi} \partial_x \delta(x - x'). \tag{3.2}$$

The interaction may be decoupled by introducing a new set of currents $\tilde{J}_L, \tilde{J}_R$ defined by

$$J_R = \cosh \alpha \tilde{J}_R + \sinh \alpha \tilde{J}_L$$
$$J_L = \sinh \alpha \tilde{J}_R + \cosh \alpha \tilde{J}_L. \tag{3.3}$$

If we set $-\tanh \alpha = g/\pi$ and express $\mathcal{H}$ in terms of $J_{L,R}$ the cross term disappears and

$$\mathcal{H} = \operatorname{sech} 2\alpha \int dx \left\{ \frac{1}{2} \tilde{J}_R^2 + \frac{1}{2} \tilde{J}_L^2 \right\}. \tag{3.4}$$

Both sets of currents obey the same commutation relations and are formal conjugates of each other — *i.e* we can write down a formal expression for a unitary operator U such that $J_{R,L} = U^\dagger \tilde{J}_{R,L} U$. As usual, U is a proper unitary transformation only in a theory with a cutoff.

We can write the currents as derivatives of two independent chiral boson fields

$$J_R = \frac{1}{2\pi} \partial_x \varphi_R \qquad J_L = \frac{1}{2\pi} \partial_x \varphi_L, \tag{3.5}$$

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with

$$[\varphi_R(x), \varphi_R(x')] = -[\varphi_L(x), \varphi_L(x')] = i\pi\mathrm{sgn}\,(x - x'), \tag{3.6}$$

and then bosonized expressions for fundamental fermions in the system are

$$\psi_R =: e^{i\varphi_R} : \qquad \psi_L =: e^{-i\varphi_L} : . \tag{3.7}$$

We calculate their correlators by introducing new $\tilde{\varphi}_{L,R}$ in the same manner as $\tilde{J}_{R,L}$

$$\begin{aligned} \varphi_R &= \cosh\alpha\tilde{\varphi}_R + \sinh\alpha\tilde{\varphi}_L \\ \varphi_L &= \sinh\alpha\tilde{\varphi}_R + \cosh\alpha\tilde{\varphi}_L. \end{aligned} \tag{3.8}$$

The $\tilde{\varphi}_{R,L}$ are independent *free* fields so substituting (3.8) in (3.7) allows us to compute correlators. For example,

$$\langle\psi_R^\dagger(x)\psi_R(x')\rangle = \frac{1}{(x - x')^{\cosh 2\alpha}}. \tag{3.9}$$

This coincides with (2.12) after one identifies λ with $e^{-2\alpha}$. This identification is confirmed by the computation of other Luttinger-Thirring correlators.

To define the higher statistics hyper-fermion operators we follow Haldane [16] and define two new fields

$$\begin{aligned} \theta(x) &= \frac{1}{2}(\varphi_R(x) + \varphi_L(x)) \\ \varphi(x) &= \frac{1}{2}(\varphi_R(x) - \varphi_L(x)), \end{aligned} \tag{3.10}$$

which obey

$$\begin{aligned} [\varphi(x), \varphi(x')] &= [\theta(x), \theta(x')] = 0 \\ [\varphi(x), \theta(x')] &= \frac{i}{2}\pi\mathrm{sgn}\,(x - x'). \end{aligned} \tag{3.11}$$

We then define

$$\Phi_m(x) =: e^{i\varphi(x)+im\theta(x)} : \tag{3.12}$$

For arbitrary m the operators in (3.12) change the total charge $J_R + J_L$ by one unit.

Clearly $\Phi_1(x) = \psi_R(x)$ and $\Phi_{-1}(x) = \psi_L(x)$. From (3.11) we see that

$$\Phi_m(x)\Phi_m(x') = e^{im\pi\mathrm{sgn}\,(x-x')}\Phi_m(x')\Phi_m(x), \tag{3.13}$$

so the Φ_m have Fermi statistics if m is an odd integer and Bose statistics if m is even.

The Φ_m are what we need to make the connection between the Luttinger model and the FQHE. It is they, not the fundamental fermions, whose equal time correlator corresponds to (2.11). We easily compute

$$\langle \Phi_m^\dagger(x)\Phi_m(x')\rangle = \frac{1}{(x-x')^{\frac{1}{2}(\lambda+\frac{m^2}{\lambda})}}. \tag{3.14}$$

Now when we set $\lambda = m$ we find that

$$\langle \Phi_m^\dagger(x)\Phi_m(x')\rangle_{\lambda=m} = \frac{1}{(x-x')^m}. \tag{3.15}$$

This coincides with (1.1) and supports our identification of the FQHE edge electron with the Luttinger hyper-fermion operator.

For $\lambda = e^{-2\alpha} = m$ we also find that

$$\Phi_m(x) =: e^{i\sqrt{m}\tilde\varphi_R(x)} : \tag{3.16}$$

showing that the hyper-fermion operator is the exponential of a field that, like the FQHE edge electron, does not suffer left/right mixing. This has the important *dynamical* consequence that Φ_m depends only on $x - v_f t$, and so the FQHE edge-electron *propagator* and not just the equal-time function coincides with the Luttinger correlator. We can use the other operator $\Phi_{-m} =: \exp -i\sqrt{m}\tilde\varphi_L :$ to represent the electrons on the other side of a Hall bar, or inner edge of an annulus.

Let us compare with Wen's construction of the FQHE edge-electron field [1]. He uses the known velocity of the excitations to argue that the commutator of the operators J_{edge} measuring the edge-electron number must obey (see also [18])

$$[J_{edge}(x), J_{edge}(x')] = \frac{-i}{2\pi m}\partial_x\delta(x-x'). \tag{3.17}$$

He then defines an edge-boson field φ_{edge} via

$$\frac{1}{2\pi m}\partial_x\varphi_{edge}(x) = J_{edge}(x), \tag{3.18}$$

and identifies the edge electron with

$$\psi_{edge}(x) =: e^{i\varphi_{edge}(x)} :, \tag{3.19}$$

since it changes the edge charge by unity, and has (hyper)-fermi statistics. Clearly then we should identify φ_{edge} with $\sqrt{m}\tilde{\varphi}_R(x)$, and the edge charge, J_{edge}, with $\tilde{J}_R(x)/\sqrt{m}$. In terms of the original Luttinger-Thirring fields

$$J_{edge}(x) = \frac{1}{2}(J_R + J_L) + \frac{1}{2m}(J_R - J_L). \tag{3.20}$$

The edge-electron operator Φ_m changes $\rho = (J_R + J_L)$ by unity, and $j = (J_R - J_L)$ by m, the combination resulting in a change of unity in J_{edge}.

Let us consider the significance of these operator correspondences for using the FQHE system as a paradigm for Luttinger liquid phenomenology. If we consider a real one-dimensional system, electrons in a mesoscopic wire for example, it may, under the right circumstances be modeled as a Luttinger liquid [3]. Consider then a point scattering impurity with a potential $V(x) = V_0 \delta(x)$. This contributes a term

$$\mathcal{H}_{imp} = V_0 \left(\psi_L^\dagger(0)\psi_R(0) + \psi_R^\dagger(0)\psi_L(0) \right) \tag{3.21}$$

to the effective Luttinger model. Here $\mathcal{H}_{imp}$ scatters the fundamental Luttinger fermions from one Fermi point to the other. This scattering is in addition to left-right mixing produced by the $J_L J_R$ term in (3.1). If we want to deconvolve these two processes we should express everything in terms of the $\tilde{\varphi}_{L,R}$. The bosonized version of (3.21) is

$$\mathcal{H}_{imp} = V_0 : \cos 2\theta(0) : . \tag{3.22}$$

This is most naturally expressed in terms of some new operators $\tilde{\psi}_{L,R}$ defined by

$$\begin{aligned}
\tilde{\psi}_R = (\Phi_m)^{\frac{1}{m}} &\overset{\text{def}}{=} \; : e^{i\varphi/m + i\theta} : \; = \; : e^{i\tilde{\varphi}_R/\sqrt{m}} : \\
\tilde{\psi}_L = (\Phi_{-m})^{\frac{1}{m}} &\overset{\text{def}}{=} \; : e^{i\varphi/m - i\theta} : \; = \; : e^{-i\tilde{\varphi}_L/\sqrt{m}} :,
\end{aligned} \tag{3.23}$$

as

$$\mathcal{H}_{imp} = V_0 \left(\tilde{\psi}_L^\dagger(0)\tilde{\psi}_R(0) + \text{h.c} \right). \tag{3.24}$$

In the absence of the impurity these two operators do not mix left and right moving excitations.

The operators $\tilde{\psi}_{L,R}^\dagger$ create an excitation with charge e/m and statistics π/m. When m is an odd integer these can be identifies with the Laughlin quasi-particle on the two edges of the FQHE fluid (in a Luttinger liquid m need *not* be an integer.). The operator

$\mathcal{H}_{imp}$ then hops a Laughlin quasi-particle from one edge of the FQHE to the other. Even in the ordinary Luttinger liquid a perturbation expansion in V_0 describes backscattering of quasi-particles of charge e/m at each order in V_0.

4. A one-dimensional one-component Plasma

In this section we return to the problem of finding the edge correlator directly from the Laughlin wavefunction. We know that naive dimensional reduction of the bulk wavefunction does not give the correct exponent. In the last section we saw that a solution was to keep the one-dimensional approximation, but to modify the Luttinger operators. Now we must answer the question of how to obtain the correlators of these modified operators from the one-dimensional Luttinger wavefunction. We should also ask whether the procedure makes physical sense. We find that it does when we use a prescription based on a method due to Hellberg and Mele [17].

The Luttinger model one-particle density matrix is given by the integral

$$G_1(x, x') = \int dx_1 \ldots dx_N \prod_i (\bar{z} - \bar{z}_i)^m (z' - z_i)^m \prod_{i<j} |z_i - z_j|^{2m}. \tag{4.1}$$

where $z = e^{2\pi i x/L}$, and similarly z'. We begin by considering a simpler problem

$$e^{-F(x,x')} = |z - z'|^{\frac{m}{2}} \int dx_1 \ldots dx_N \prod_i |z - z_i|^m |z' - z_i|^m \prod_{i<j} |z_i - z_j|^{2m}. \tag{4.2}$$

Finding F is equivalent to determining the force between a pair of logarithmically interacting charges of magnitude $1/\sqrt{2}$ inserted into a gas of similar charges of magnitude $\sqrt{2}$ which are confined to a circular loop of length L. The inverse temperature of the gas is m. The factor outside the integral is the mutual potential of the two charges.

If we assume that the one-component plasma completely screens the test charges at large distance, then F will become independent of the distance and

$$\int dx_1 \ldots dx_N \prod_i |z - z_i|^m |z' - z_i|^m \prod_{i<j} |z_i - z_j|^{2m} \propto \frac{1}{|z - z'|^{m/2}}. \tag{4.3}$$

This is the correct exponent for correlator of Luttinger bosons (see ref. [14] eq. 18).

The integral (4.1) giving the density matrix differs from (4.2) in containing extra phase factors. For the FQHE system we want to pick up a factor of $e^{\pm im\pi}$ for each particle lying between x and x'. In a genuinely one-dimensional problem the choice of sign would pose a problem. In two dimensions the ambiguity is resolved by the geometry. When we require z, z' to lie on the outer edge of a Hall droplet the electrons at z_i will always be passed on their right as z' circles the droplet counterclockwise, so the phase should increase by $m\pi$ each time z' passes by a z_i. If we were at the inner edge of an annulus then we would select the opposite sign.

Using this insight, the FQHE density matrix can be written

$$G_1^{\{m\}}(x, x') = e^{im\pi n_0(x'-x)} \frac{1}{|x-x'|^{m/2}} \left\langle e^{im\pi \int_x^{x'} \delta n(\xi) d\xi} \right\rangle, \qquad (4.4)$$

where the angular brackets denote an expectation value for the same Coulomb gas and $\delta n(\xi)$ is the excess number-density over its mean value n_0. The superscript m indicates that we are using the $+m\pi$ phase recipe. By introducing a new field χ with $\delta n(x) = \partial_x \chi$ we can write this as

$$\left\langle e^{im\pi \int_x^{x'} \delta n(\xi) d\xi} \right\rangle = \int d[\chi(x)] e^{m \int \partial_x \chi(x) \ln|x-x'| \partial_{x'} \chi(x') dx dx' + im\pi(\chi(x') - \chi(x))} \qquad (4.5)$$

The gaussian functional integral is easily performed by going to fourier space and using

$$\int_{-\infty}^{\infty} \frac{dk}{|k|} e^{ik(x-x')} = -2 \ln|x-x'| + \text{Constant}. \qquad (4.6)$$

We find

$$G_1^{\{m\}}(x, x') \propto e^{im\pi n_0(x'-x)} \frac{1}{|x-x'|^{m/2}} \frac{1}{|x-x'|^{m^2/2m}} = e^{im\pi n_0(x'-x)} \frac{1}{|x-x'|^m}. \qquad (4.7)$$

We have at last reproduced the desired exponent for the edge electron.

If we had taken the phase factor for the passage of z' past one of the other particles as being $e^{i\pi}$ only, we would have found instead

$$G_1^{\{1\}} \propto e^{i\pi n_0(x'-x)} \frac{1}{|x-x'|^{(m+\frac{1}{m})/2}}, \qquad (4.8)$$

which is the contribution to the Luttinger fermion one-particle density matrix from one of the two fermi points.

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Given these two results, it seems clear that our method of treating the integrals over the electron coordinates must be equivalent to the boson field theory manipulations of section 3. The connection is made by reviewing the derivation of the Luttinger model wavefunction presented in the appendix. To obtain the wavefunction we integrate over all the boson modes except for those on a single time slice. The functional integration over χ in (4.5) is the boson field integration over this final time slice.

5. Discussion

The principal result of this paper is a prescription for extracting the Luttinger liquid picture of the FQHE edge states directly from the Laughlin wavefunction: We first dimensionally reduce the wavefunction by constraining all its arguments to lie on the boundary of the two-dimensional electron gas. Then, motivated by the fact that the electrons are really some distance within the system, we smooth out the charges in the resulting one-dimensional Coulomb plasma so that the operators can distinguish between the passage of a particle that gives an $e^{i\pi}$ phase from the passage of one that gives an $e^{3i\pi}$ phase. This crucial step is the only relic of the direction perpendicular to the edge. Once we have performed these operations, we can use the Luttinger liquid *quantum placet.*

A comment about the resulting statistical transmutation is in order. Part of the lore of the FQHE is that electrons in Laughlin states bind to an even number of vortices which serve to transform them into hyper-fermions [20]. This is a dynamical effect. The operators ψ, $\psi^\dagger$ we use to create and annihilate the electrons are simple unadorned Fermi fields with conventional commutation relations. Since the electrons abandon their vortices as soon as they leave the FQHE system, it is the density matrix and other Green functions defined in terms of these conventional statistics operators that govern the interaction of the system with the outside world, and it is these we compute. When we examine the density matrix in the bulk we see no sign (beyond a reduced density) of the statistical dressing. Only as we approach the boundary of the electron gas does it begin to display the effects of the attached vortices. At the boundary, the density matrix, and indeed all time-dependent, conventional-statistics Green functions coincide with one-dimensional Luttinger liquid correlators of enhanced statistics operators.

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From the Coulomb gas analysis of the last section we begin to understand why we need the boundary to see the effects of the dynamical statistics change. Only near a boundary do we tend to have more electrons on one side than the other, and thus an opportunity to perceive the accumulating phase change.

The insight that comes from knowing how to implement the connection between the bulk wavefunction and the Luttinger liquid behaviour in the simple Laughlin states should be useful for understanding the connection between the edge behaviour of more complicated fractional phases and their candidate wavefunctions.

Acknowledgements

Appendix

In this appendix we will derive the many-body Luttinger-Thirring ground-state wavefunction $\Psi(\theta_1, \theta_2, \ldots, \theta_N)$ for N fermions living on a circle of circumference 2π. We will use a slightly modified form of the methods in ref. [12]. It is most convenient to use the Lorentz invariant Thirring form for this. We begin by finding the ground-state wavefunction Φ for the bosonized version. Because the bosonization of the Thirring model is most familar from the work of Coleman [19] we will use his conventions in this appendix. The boson fields here therefore differ in normalization by a factor of $2\sqrt{\pi}$ from those in the main text.

The bosonized action is

$$S = \frac{4\pi}{\beta^2} \int \frac{1}{2} (\partial\varphi)^2 d\tau d\theta \qquad (A.1)$$

Here β^2 is Coleman's parameterization of the interaction. For the free theory $\beta^2 = 4\pi$. To relate it to other paramterizations of the interaction it is best to compare correlators

calculated in the different schemes. We find $4\pi/\beta^2 = \lambda = e^{-2\alpha}$.

In a Schrödinger representation the wavefunction $\Phi(\varphi_c)$ is a functional of the boson field configuration. To compute it we take a path integral over φ's defined on the half-cylinder $\Omega = [-\infty, 0] \times S^1$, where the argument of the wavefunction, φ_c, appears as the boundary condition $\varphi_c(\theta) = \varphi(0, \theta)$ imposed on the circle at $\tau = 0$. The long euclidean time interval between $-\infty$ and 0 projects out the ground state, and the unnormalized wavefunction is given by

$$\Phi(\varphi_c) = \langle vac|\varphi_c\rangle = \int_0^{\varphi_c} d[\varphi] e^{-\frac{4\pi}{\beta^2}\int_\Omega \frac{1}{2}(\partial\varphi)^2 d\tau d\theta}. \qquad (A.2)$$

Being quadratic, the path integral may be performed by replacing the φ in the integrand with the solution to Laplace's equation for the given φ_c boundary values. Using the standard formula for the solution to the Dirichlet problem,

$$\varphi(\tau', \theta') = \oint d\theta \varphi_c(\theta) \partial_\tau G_\Omega(\tau, \theta; \tau'\theta')|_{\tau=0}, \qquad (A.3)$$

and integrating by parts, we find that the exponent

$$E = \int_\Omega d\tau d\theta \frac{1}{2}(\partial\varphi)^2 \qquad (A.4)$$

can be written in terms of the boundary data,

$$E = \frac{1}{2}\oint d\theta \varphi_c(\theta) \partial_\tau \varphi(0, \theta) = -\frac{1}{2}\int_{S^1}\int_{S^1} d\theta d\theta' \varphi_c(\theta) \varphi_c(\theta') \partial_\tau \partial_{\tau'} G_\Omega(\tau, \theta, \tau', \theta'). \qquad (A.5)$$

In these formulae G_Ω is the Dirichlet Green-function on the half cylinder, *i.e.*

$$\nabla_\mathbf{r}^2 G_\Omega(\mathbf{r}, \mathbf{r}') = \delta^2(\mathbf{r} - \mathbf{r}') \qquad (A.6)$$

and $G_\Omega(\mathbf{r}, \mathbf{r}') = 0$ if $\mathbf{r}$ is on the boundary circle.

The Green function G for the infinite cylinder is obtained as

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi}\mathrm{Re}\,\ln(e^{iz} - e^{iz'}) = -\frac{1}{2\pi}\ln|\sin(z - z')/2|, \qquad (A.7)$$

with $z = \tau + i\theta$, by a conformal transformation of the $\mathbf{R}^2$ Green function, $G_0(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi}\ln|\mathbf{r} - \mathbf{r}'|$. The half cylinder Green function, G_Ω, is then found from G by the method of images:

$$G_\Omega(\tau, \theta, \tau', \theta') = G(\tau - \tau', \theta - \theta') - G(\tau + \tau', \theta - \theta'). \qquad (A.8)$$

We now use Laplace's equation and the form of the arguments of G to trade the partial derivatives with respect to τ for partials with respect to θ

$$\partial_\tau \partial_{\tau'} G_\Omega(\tau, \theta, \tau'\theta') = -\partial_\theta \partial_{\theta'} G(\tau - \tau', \theta - \theta') - \partial_\theta \partial_{\theta'} G(\tau + \tau', \theta - \theta'). \qquad (A.9)$$

We need this expression only on $\tau = \tau' = 0$ where it is equal to $-2\partial_\theta \partial_{\theta'} G(0, \theta - \theta')$.

After a final integration by parts, the fruit of our labours is

$$E = \frac{1}{2} \int \int d\theta d\theta' \partial_\theta \varphi_c(\theta) \partial_{\theta'} \varphi_c(\theta') \frac{1}{\pi} \ln|\sin(\theta - \theta')/2|, \qquad (A.10)$$

and the exponential of this $\Phi = \exp(-4\pi E/\beta^2)$ is the harmonic oscillator like ground state of the Bose field.

We convert Φ to a many-body Fermi wavefunction Ψ expressed in terms of the particle locations by using the bosonization rule $\rho = \frac{1}{\sqrt{\pi}} \partial_\theta \varphi$, and replacing the density with its first quantized form $\rho(\theta) = \sum_{i=1}^{N} \delta(\theta - \theta_i)$. We find that

$$\Psi(\theta_1, \theta_2, \ldots, \theta_N) = |e^{-i(N-1)\sum \theta_i} \prod_{i<j} (e^{i\theta_i} - e^{i\theta_j})|^{4\pi/\beta^2} \qquad (A.11)$$

Since we only know that there is some particle at the location θ_i, and not which particular particle it is, this expression is to be used for the standard ordering $\theta_1 < \theta_2 \ldots < \theta_N$ only. The values of Ψ for other orderings of the arguments are found by imposing the antisymmetry.

References

[1] X. G. Wen, Phys. Rev. Lett. **64** (1990) 2206; Phys. Rev. **B41** (1990) 12838; **B43** (1991) 11025; **B44** (1991) 5708.

[2] K. Moon, H. Yi, C. L. Kane, S. M. Girvin, M. P. A. Fisher, Phys. Rev. Lett. **71** (1993) 4381.

[3] C. L. Kane, M. P. A. Fisher, Phys. Rev. Lett. **68** (1992) 1220; Phys. Rev. **B46** (1992) 7268, 15233.

[4] F. Milliken, C. P. Umbach, R. A. Webb, IBM preprint.

[5] F. D. M. Haldane, Phys. Rev. Lett. **45** (1980) 1358; J. Phys. **C14** (1981) 2585.

[6] S. Mitra, A. H. Macdonald, Phys. Rev. **B48** (1993) 2005.

[7] X. G. Wen, Int. J. Mod. Phys. **B6** (1992) 1711.

[8] R. B. Laughlin, Phys. Rev. Lett **50** (1983) 1395.

[9] S-C. Zhang, Int. J. Mod. Phys **6** (1992) 25.

[10] A. Karlhede, S. A. Kivelson, S. L. Sondhi, in *Correlated Electron Sytems* V. J. Emery ed. (World Scientific 1993).

[11] A. Lopez, E. Fradkin, Phys. Rev. Lett. **69** (1992) 2126.

[12] E. Fradkin, E. Moreno, F. A. Schaposnik, Nucl. Phys. **B392** (1993) 667.

[13] B. Sutherland, J. Math. Phys. **12** (1971) 246; Phys. Rev **A 4** (1971) 2019; **A5** (1972) 1372.

[14] N. Kawakami, S-K. Yang, Phys. Rev. Lett. **67** (1991) 2493.

[15] M. Stone, Ann. Phys. **207** (1991) 38; Phys. Rev. **B42** (1990) 8399; Int. J. Mod. Phys. **5** (1991) 509.

[16] F. D. M. Haldane, Phys. Rev. Lett. **47** (1981) 1840.

[17] C. S. Hellberg, E. J. Mele, Phys. Rev. Lett. **68** (1992) 3111.

[18] J. Martinez, M. Stone, Int. J. Mod. Phys. **B7** (1993) 4389.

[19] S. Coleman, Phys. Rev. **D11** (1975) 2088.

[20] B. I. Halperin, Helvetica Phys. Acta, **56** (1983) 75.