Contacts and edge-state equilibration in the fractional quantum Hall effect

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We develop a simple kinetic equation description of edge-state dynamics in the fractional quantum Hall effect (FQHE), which allows us to examine in detail equilibration processes between multiple edge modes. As in the integer quantum Hall effect, intermode equilibration is a prerequisite for quantization of the Hall conductance. Two sources for such equilibration are considered: edge-impurity scattering and equilibration by the electrical contacts. Several specific models for electrical contacts are introduced and analyzed. For FQHE states in which edge channels move in both directions, such as $\nu = 2/3$, these models for the electrical contacts *do not* equilibrate the edge modes, resulting in a nonquantized Hall conductance, even in a four-terminal measurement. Inclusion of edge-impurity scattering, which *directly* transfers charge between channels, is shown to restore the four-terminal quantized conductance. For specific filling factors, notably $\nu = 4/5$ and $\nu = 4/3$, the equilibration length due to impurity scattering diverges in the zero-temperature limit, which should lead to a breakdown of quantization for small samples at low temperatures. Experimental implications are discussed.

I. INTRODUCTION

An important lesson learned from studies of mesoscopic structures is that the transport properties of a system can be strongly influenced by the electrical contacts used to make the measurements.[1] The nature of the contacts can be particularly important in the quantum Hall regime, where transport takes place via edge states. Current carrying contacts feeding the edge states can alter their population, leaving the edge out of equilibrium. High impedance voltage contacts selectively measure the electrochemical potential of the edge modes to which they are most strongly coupled.

In the integer quantum Hall effect (IQHE), Büttiker[2] has generalized Landauer quantum transport theory[3] to incorporate the effects of multiple electrical contacts. A contact, or lead, is modeled as a large reservoir in equilibrium at an electrochemical potential μ. The edge-states in the sample are populated by electrons incident from the leads. Büttiker defines an "ideal" contact as one in which no scattering occurs at the contact. The edge-states emanating from such ideal contacts are thus populated in equilibrium at the chemical potential μ. Büttiker also discusses the more generic case of a disordered or "nonideal" contact, which is characterized by transmission and reflection matrices between the edge channels in the "sample" and in the "leads."

At integer filling factors $\nu > 1$, there are multiple edge channels. For generic "nonideal" contacts, the coupling to the different edge modes will be different. Nonideal current contacts will thus tend to populate the edge modes differently, putting them out of equilibrium with one another. In this case, the edge is *not* characterized by a unique chemical potential, and the Hall conductance measured using similar nonideal voltage contacts will *not* be quantized. For this reason, Büttiker emphasizes the important role played by interchannel electron scattering, which can reequilibrate the different modes. Provided the separation between current and voltage leads is greater than the equilibration length, the voltage lead will measure an equilibrated edge, giving a quantized Hall conductance.

By fabricating nonideal contacts that are close together, it is possible to study directly the equilibration between edge-states. Indeed, in beautiful experiments utilizing a quantum point contact, which selectively populates channels,[4–6] equilibration lengths of order 40 μm have been measured in the integer quantum Hall regime.

While the importance of contacts and edge-state equilibration is well appreciated for the integer quantum Hall effect, a suitable generalization to the fractional quantum Hall regime has been lacking. In the fractional quantum Hall effect (FQHE), the edge modes cannot be described in terms of a free-electron model, so that a Landauer-Büttiker description of edge transport is not possible. Recently, a powerful framework for describing edge states in the FQHE has been developed, based on the chiral Luttinger liquid model.[7] This description enables one to compute transport properties of edge-states in the FQHE. Recently, the effects of impurity scattering on edge-state equilibration has been discussed in the FQHE,[8,9] but the role of electrical contacts has not been adequately addressed.[10]

In this paper, we develop a simple theory for edge-state transport in the FQHE, which allows for the incorporation of electrical contacts and intermode equilibra-

tion. The approach is based on a simple kinetic equation for the edge-state dynamics, which closely resembles a linearized Boltzmann equation. Coupling to electrical contacts is incorporated by adding source terms to the kinetic equation, loosely analogous to scattering terms in the Boltzmann equation. Impurity scattering between multiple edge modes can also be simply incorporated into the kinetic equation. Several specific models for the contacts are considered, which are analogous to Büttiker's "ideal" and "nonideal" contacts in the IQHE.

For simplicity, we focus on quantum Hall states at filling $\nu = (p_1 - 1/p_2)^{-1}$ (p_1 odd, p_2 even), which correspond to the second level of the Haldane-Halperin hierarchy[11] and have two edge channels. As in the integer quantum Hall effect, we find that equilibration between the different edge channels is a prerequisite for the quantization of the Hall conductance. There are two sources for this equilibration, which we address separately: Impurity scattering along the edge and equilibration by the contacts themselves. In the former case, we find important differences with the IQHE. Specifically, for filling fractions with $|p_2| = 4, 6, ...$, such as $\nu = 4/3, 4/5, ...$, the intermode equilibration length, due to impurity scattering, is temperature dependent and *diverges* at low temperatures. In this case, for finite sized sample, quantization of the Hall conductance should break down at very low temperatures.

There are also important differences between the integer and fractional Hall effects with regards the equilibration taking place *at* the contacts. The differences are most pronounced when the two fractional edge modes are moving in opposite directions, for $p_2 < 0$ at filling $\nu = 2/3, 4/5, ...$. In this case, even "ideal contacts" are insufficient to equilibrate the two edge modes. The reason is that the two modes emanate from different reservoirs, at different chemical potentials. Then in the absence of any direct intermode impurity tunneling, the two channels on the same edge will be at different chemical potentials. Impurity scattering away from the contacts can still cause equilibration, but with an equilibration length diverging at low temperatures for $|p_2| \geq 4$. In contrast, when both edge modes propagate in the same direction ($p_2 > 0$), we show that "ideal contacts" can be defined, which completely equilibrate the two channels, just as in the IQHE.

In this paper, we focus almost exclusively on the linear response conductances, for a sample with *finite* width, L. Within linear response, both the Hall voltage drop V_H, and the Hall electric field $E = V_H/L$, are taken to zero, with *fixed* width L. In this limit, the edge currents give an order one contribution to the Hall conductance, and with long-ranged forces screened by a ground plane, dominate completely over bulk contributions. As we shall see, the conductance in this case is a "mesoscopic" quantity, which can depend on the nature of the electrical contacts and of the edge states that transport a current between them. In contrast, it is possible to define a "macroscopic" Hall conductance, which is a bulk property, independent of the edge dynamics. In the limit that $L \to \infty$, with fixed finite electric field, E, the edge contribution to the bulk Hall conductance vanishes as $1/L$.

Our paper is organized as follows. In Sec. II, we introduce the simple kinetic equation description of FQHE edge-state transport, for the case in which only a single edge mode is expected, ν^{-1} an odd integer. Various models for electrical contacts are discussed within this framework. In Sec. III, the description is generalized to describe hierarchical Hall states with two edge modes. Equilibration between the two edge modes both at the contacts, and due to edge-impurity scattering, is discussed. In Sec. IV, we describe several specific experimental consequences, and conclude in Sec. V.

II. SINGLE EDGE MODE

In this section, we introduce a simple kinetic equation description for the edge of a Laughlin state at filling $\nu = 1/m$. In this case, there is only a single edge mode, which satisfies a simple continuity equation. We then show how electrical contacts can be incorporated into this approach, and consider specific models for the contacts. Finally, we show how bulk electric fields, and bulk currents, can also be incorporated, without changing the conclusions. In Sec. III, we will turn our attention to hierarchical Hall states, which have multiple edge modes.

A. Kinetic equation

For filling $\nu = 1/m$ with odd integer m, a single chiral edge mode is expected.[7] The one-dimensional electron density, $n(x)$, satisfies a simple equation of motion,

$$\partial_t n + v\partial_x n = 0, \tag{2.1}$$

which describes waves moving in one direction at velocity v: $n(x,t) = f(x - vt)$, for arbitrary f. This can be written as a continuity equation,

$$\partial_t n + \partial_x J = 0, \tag{2.2}$$

with an edge current defined as

$$J = vn. \tag{2.3}$$

Equation (2.2) is a conservation law for electric charge at the edge. Together, (2.2) and (2.3) are a simple kinetic equation for edge charge transport.

Since the bulk is incompressible, a charge cannot pass from the edge into the bulk, at least in the linear response. In the presence of a large nonlinear driving field, though, bulk currents can flow, and it is necessary to augment Eq. (2.2). Moreover, charge can be added or removed from the edge mode at contacts. In these cases, source terms must be added to the right side of (2.2):

$$\partial_t n + \partial_x J = I_{\text{bulk}}(x) + I_c(x), \tag{2.4}$$

which describe charge being added or removed from the edge, via either bulk currents, or from contacts.

The kinetic equation (2.4) is analogous to the transport equation for quasiparticles in a Fermi liquid.[12] In the present case, the Fermi surface is replaced by a sin-

gle point. The left hand side describes collisionless transport, and follows directly from microscopic equations of motion, as we show below. The term on the right hand side coming from the contacts is analogous to a "collision term" in the Boltzmann equation. An explicit form for this term can be obtained by using Fermi's golden rule, as we show below, which is the rough equivalent of the relaxation time approximation. This treatment requires that the time between successive tunneling events from the contacts exceeds the dephasing time τ_ϕ. We shall describe the role of the bulk currents in Sec. II C.

We now show briefly that the collisionless terms (2.2) and (2.3) follow directly from a chiral Luttinger liquid[7] description of the edge mode. In terms of a boson field, ϕ, related to the electron density via $n(x) = (1/2\pi)\partial_x\phi$, the chiral Luttinger Hamiltonian is simply

$$H = \frac{v}{4\pi\nu}\int dx(\partial_x\phi)^2, \tag{2.5}$$

where the phase field satisfies a Kac-Moody commutation relation:

$$[\phi(x),\phi(x')] = i\pi\nu\mathrm{sgn}(x-x'). \tag{2.6}$$

From the Heisenberg equations of motion for the operator ϕ, it is straightforward to show that the density operator satisfies the kinetic equation (2.1).

The continuity equation (2.2) can be rewritten in the suggestive form: $\partial_x[J + (\partial_t\phi/2\pi)] = 0$, allowing us to identify the current operator as $J = -\partial_t\phi/2\pi$. It is useful to assume normal ordering, so that in equilibrium both the density and currents vanish. In the presence of a nonzero chemical potential, μ, however, currents will flow. The current response can be deduced by adding to the Hamiltonian a term of the form

$$\delta H = -\mu\int dx n, \tag{2.7}$$

and then evaluating the current, $J = -e\partial_t\phi/2\pi$, using the commutation relations (2.6). One deduces a nonvanishing transport current of the form

$$J = \nu\frac{e}{h}\mu. \tag{2.8}$$

The conductance is seen to be appropriately quantized, $G = eJ/\mu = \nu e^2/h$.

B. Contacts

We consider now incorporating electrical contacts into the above description of edge transport. We begin by discussing Büttiker's "ideal" contact[2] model generalized to the fractional quantum Hall regime. While this model is useful for performing simple calculations, it is rather unrealistic — particularly in the FQHE — since it assumes that the edge modes retain their integrity deep within the reservoirs. As an alternative, we consider a contact modeled as a tunnel junction to a metallic electrode. In Büttiker's terminology, this is an example of a "nonideal" contact. The point contact tunnel junction can be suitably generalized, however, into a tunnel junction "line contact." The "line contact" is shown to be "ideal," with vanishing contact resistance.

1. The ideal contact

In the Landauer-Büttiker approach to quantum transport, electrical contacts are modeled as reservoirs at

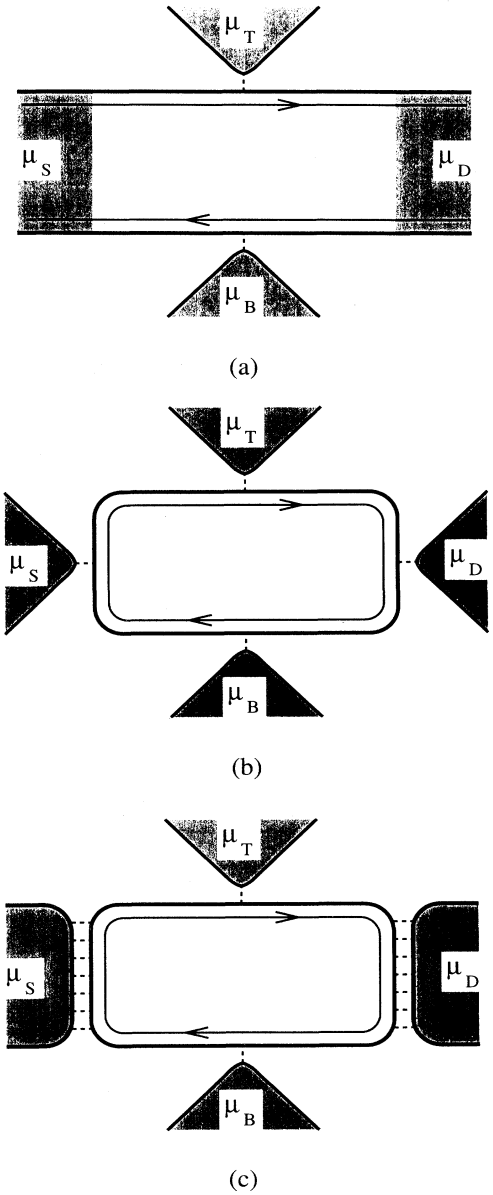

FIG. 1. Three different models for the source and drain contacts to a quantum Hall fluid. (a) "Ideal" contacts: the FQHE edge channels extend to infinity in both the source and drain electrodes; (b) tunnel junction point contacts; (c) tunnel junction line contacts. In (a), (b), and (c), the Hall voltage is measured via weakly coupled point contact tunnel junctions.

chemical potential μ. Transport is viewed as a scattering process. Electrons incident from the reservoirs enter the sample, scatter about, and then leave the sample back into the reservoirs. In this scheme, an "ideal contact" is one in which there is no electron backscattering during the process of entering or leaving the sample. The population of an each edge mode is then determined by the chemical potential of the reservoir from which it emanates.

For an interacting system, such as a FQHE edge mode, the transport cannot be described in terms of the population of free-electron states. Nonetheless, a Landauer-type formula for transport can be derived within a linear response Kubo formulation.[13,14] The "reservoir" is modeled as a semi-infinite strip of FQHE fluid connected to the "sample." The edge channels extend to infinity in the "reservoirs," as shown in Fig. 1(a). Following Fisher and Lee, the conductance may then be computed within linear response theory by applying a time dependent potential, $V(x)e^{i\omega t}$, where $V(x)$ is equal to μ_i in the ith reservoir, and then taking the $\omega \to 0$ limit. For noninteracting electrons this procedure is equivalent to the Landauer approach, but can be suitably generalized to the FQHE. It results in a nonequilibrium current flowing in the fractional edge channels, determined by the chemical potential of the reservoir from which they emanate, $J = \nu(e/h)\mu_i$. If the Hall voltage is measured on the top and bottom edges by similar "ideal contacts," the result is an appropriately quantized Hall conductance. Moreover, the two terminal conductance, defined as the ratio of the current to the source-drain voltage, is also quantized.

The assumption that the edge modes maintain their integrity deep within the reservoirs is highly unphysical. This feature is particularly worrisome in the FQHE, where the edge modes are gases of fractionally charged quasiparticles. One would expect an appreciable contact resistance as the electrons from the metallic electrodes splinter upon entering the sample, in contrast to the resistanceless "ideal contacts" considered above. We now consider more realistic models for electrical contacts, consisting of a metallic electrode coupled to the edge modes via tunnel junctions.

2. *The tunnel junction point contact*

Consider a metallic electrode, described by a Fermi liquid at chemical potential μ. The electrode is connected to the edge mode via a tunnel junction, at position $x = 0$. The tunneling process transfers an electron from metallic electrode to the edge with Hamiltonian:

$$H_{\rm tun} = -t_0\psi(x=0)\int dx\delta(x)e^{i\phi(x)/\nu} + \text{H.c.} \qquad (2.9)$$

Here, ψ is an electron destruction operator in the metallic electrode, and $e^{i\nu\phi}$ is the edge electron creation operator. This tunneling Hamiltonian leads to an additional term on the right side of the continuity equation (2.2),

$$\hat{I}_c = \delta(x)it_0\psi(x)e^{i\phi(x)/\nu} - \text{H.c.}, \qquad (2.10)$$

which describes tunneling of charge from electrode to edge. Since this operator is nonlinear, it is desirable to replace it by it's expectation value, so that (2.2) can still be used as a classical kinetic equation. This simplification requires that successive tunneling events are incoherent. In the limit $t_0 \to 0$, this should be the case, since the time between the tunneling events then exceeds the electron dephasing times both in the electrode and on the edge. In this limit, the average tunneling current is given by a contact conductance, G_c, times the chemical potential drop between the electrode and the edge mode. The chemical potential of the edge mode can be obtained from the upstream current entering the contact region: $\mu_{\rm edge} = J(x = 0^-)h/(\nu e)$. Thus, the kinetic equation can be written as a closed expression in terms of the edge density and current

$$\partial_t n + \partial_x J = I_c(x), \qquad (2.11)$$

with a contact current

$$I_c(x) = \delta(x)G_c\left(\mu - \frac{h}{\nu e}J(x)\right). \qquad (2.12)$$

When many contacts are present, at positions x_i, and chemical potentials μ_i, this generalizes to

$$I_c(x) = \sum_i \delta(x_i)G_c\left(\mu_i - \frac{h}{\nu e}J(x_i)\right). \qquad (2.13)$$

The tunneling conductance will, in general, be temperature dependent. Using Fermi's Golden rule perturbative in the tunneling matrix element t_0, gives $G_c \sim t_0^2 T^{(1/\nu-1)}$, which vanishes at low temperature in the FQHE. We will assume, in any event, that $G_c \ll e/h$.

The large resistance associated with the contacts will dominate the resistance in a two-terminal measurement. To see this explicitly from the kinetic equation, consider a two-terminal geometry [Fig. 1(b)], with the source electrode at μ_S and the drain at μ_D. In steady state, the time derivative can be dropped, and the kinetic equation can be readily solved. Denoting the current flowing along the top and bottom edges by $J_{T/B}$, one obtains from the kinetic equation,

$$J_T - J_B = G_c\left(\mu_S - \frac{h}{\nu e}J_B\right), \qquad (2.14)$$

$$J_B - J_T = G_c\left(\mu_D - \frac{h}{\nu e}J_T\right), \qquad (2.15)$$

where, for simplicity, we have assumed equal contact resistances, G_c^{-1}, for the source and drain electrodes. The transport current, $I = J_T - J_B$ is then found to be

$$I \simeq \frac{1}{2}G_c(\mu_S - \mu_D), \qquad (2.16)$$

under the assumption, $G_c \ll h/e$. As expected, the two-terminal resistance is simply a sum of the two contact

resistances, and is not quantized.

In a four-terminal Hall measurement there are two additional voltage contacts, on the top and bottom edges, as shown in Fig. 1(b). The chemical potentials in these contacts, denoted $\mu_{T/B}$, are set by the requirement that no net current flows from these electrodes into the sample. From (2.12), this implies $\mu_T = J_T h/(\nu e)$ and $\mu_B = J_B h/(\nu e)$. The transport current, $I = J_T - J_B$, is then

$$I = \frac{\nu e}{h}(\mu_T - \mu_B), \tag{2.17}$$

giving an appropriately quantized four-terminal Hall conductance.

As expected, we find a quantized four-terminal Hall conductance, independent of the contact resistance. As we show in Sec. III, this result breaks down when multiple edge modes are present — the four-terminal conductance is *not* quantized when measured with such nonideal contacts.

3. The tunnel junction line contact

The above tunnel junction point contact model provides an explicit realization of a "nonideal" contact, with large contact resistance. We now generalize this model to describe an "ideal contact," with vanishing contact resistance. Consider a metallic electrode, which is coupled to the edge mode via tunneling, along a segment of length L. We refer to this as a "line junction" contact. The validity of a kinetic equation description again requires that successive tunneling events from electrode to edge are incoherent. This will be satisfied provided the tunneling conductance per unit length is sufficiently small. Since the contact length, L, can be made large, however, the total conductance between electrode and edge need not be small.

The kinetic equation is again given by (2.11). However, the tunneling current from the electrode is now extended over a length L,

$$I_c(x) = \sigma_c \left(\mu - \frac{h}{\nu e} J(x)\right) \quad \text{for} \quad 0 < x < L, \tag{2.18}$$

where σ_c is the tunneling conductance per unit length. As before, σ_c may be temperature dependent.

In a steady state, the kinetic equation simplifies to $\partial_x J(x) = I_c(x)$, and can be readily solved. In the region $0 < x < L$, the solution for $J(x)$ is

$$J(x) = \frac{\nu e}{h}\mu + \left(J(x=0) - \frac{\nu e}{h}\mu\right) e^{-x/\ell_c}, \tag{2.19}$$

with an equilibration length defined by $\ell_c = \nu\sigma_c^{-1}$. Provided $\ell_c \ll L$, the edge mode equilibrates fully with the metallic electrode and $J(x > L) = \nu\mu$. The two-terminal conductance measured with "line contacts" is determined by considering currents on the top and bottom edges. Referring to Fig. 1(c), we have $J_T = (\nu e/h)\mu_S$ and $J_B = (\nu e/h)\mu_D$, which gives for the transport current $I = J_T - J_B$,

$$I = \frac{\nu e}{h}(\mu_S - \mu_D). \tag{2.20}$$

The two-terminal conductance is quantized, indicating that the "contact resistance" vanishes. The "line contact" thus provides an explicit realization of an "ideal contact" for FQHE states, with a single edge mode. Before discussing multiple edge modes, we briefly consider the role of bulk currents for ν^{-1} an odd integer.

C. Bulk currents

A confusing aspect of transport in the quantum Hall effect is the relative importance of edge versus bulk currents. If the long-ranged Coulomb interactions are screened by a ground plane, one expects the linear response transport current to be confined to the edges. However, when Coulomb forces are unscreened, or one is well outside the linear response regime, additional bulk currents are expected, along with bulk electric fields. The total transport current will be a sum of the edge and bulk contributions. Likewise, the measured Hall voltage will be a sum of the edge chemical potentials and the bulk electric potential drop.

The distinction between edge and bulk currents becomes clear in the IQHE for noninteracting electrons. Consider the schematic plot of energy levels[15] in the lowest two Landau levels, as one transverses the sample, see Fig. 2. In 2(a) there is no bulk electric field—the Lan-

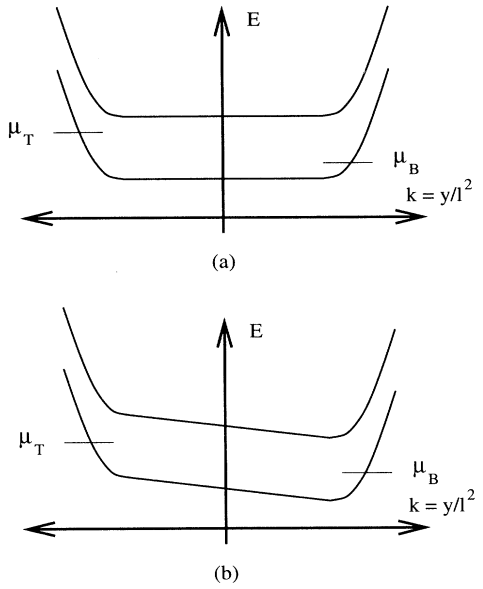

FIG. 2. Dispersion of energy levels for noninteracting electrons in a quantum Hall bar, as a function of the one-dimensional momentum k, which is related to the transverse position y. In (a) there is no electric field, so the bulk states in each Landau level are degenerate. In (b), there is a finite electric field in the bulk, which gives rise to a finite velocity $v = \partial E/\partial k$ for states in the bulk. In both cases the net current is determined by the electrochemical potentials at the two edges, μ_T and μ_B.

dau levels are flat and no bulk currents flow. In 2(b), the presence of a bulk electric field gives rise to a finite velocity $v = dE/dk$ of the bulk states, and hence a bulk current. In both cases, however, the total current, summing over both bulk and edge-states, is given by $I = (e/h)(\mu_T - \mu_B)$.

This picture can readily be generalized to the FQHE. The total transport current in the x direction is expressed as a sum of bulk and edge contributions

$$I = J_T - J_B + \int_{y_B}^{y_T} dy \;\; J^x_{\text{bulk}}, \tag{2.21}$$

where the bulk current density, defined between $y = y_B$ and $y = y_T$ near the top and bottom edges, is

$$\vec{J}_{\text{bulk}} = \sigma_{xy}\hat{z} \times \vec{E}. \tag{2.22}$$

The edge currents on the top and bottom are denoted J_T and J_B. The dividing lines, at y_T and y_B, between the edge and bulk are arbitrary, so that $J_{T/B}$ is only defined up to an additive constant. This constant may be chosen so that the edge currents vanish in equilibrium. This is equivalent to normal ordering the edge current operator, with respect to the equilibrium ground state.

In the presence of bulk electric fields, $\vec{E}$, the edge current is no longer conserved, since currents can flow from the edge into the bulk. The kinetic equation obeyed by the edge currents must, therefore, be modified,

$$\partial_t n_{T/B} + \partial_x J_{T/B} = \sigma_{xy} E_x(x, y_{T/B}), \tag{2.23}$$

for top and bottom, respectively.

However, it is now a simple matter to eliminate the dependence on electric fields in the above two equations. In terms of an electric potential, $\vec{E} = \vec{\nabla} V$, we define different currents:

$$\tilde{J}_{T/B} = J_{T/B} - \sigma_{xy} V(x, y_{T/B}). \tag{2.24}$$

These currents are conserved even with bulk electric fields present, since (2.23) can be rewritten: $\partial_t n_{T/B} + \partial_x \tilde{J}_{T/B} = 0$. Moreover, the total transport current becomes simply

$$I = \tilde{J}_T - \tilde{J}_B. \tag{2.25}$$

Notice that the currents, $\tilde{J}_{T/B}$, satisfy the same steady-state kinetic equations as the edge currents do in the absence of bulk electric fields. Thus, the results of this and the next sections are not modified by the presence of bulk electric fields.

III. MULTIPLE EDGE MODES

A. Kinetic equation

For hierarchical Hall states,[11] there are multiple modes on a given edge. The structure of the edge modes is set by the topological order in the bulk, which is characterized by a square symmetric matrix K,[16,17] with integer matrix elements. At the nth level of the hierarchy, the matrix K is an n by n matrix. In addition, there is a vector of integer "charges," t_i. The filling factor is given by

$$\nu = \sum_{ij} t_i K^{-1}_{ij} t_j. \tag{3.1}$$

The explicit form of the K matrix for a given quantum Hall state depends on the choice of basis, as do the integers t_i.[16,17] For convenience, we adopt throughout the "symmetric basis" in which $t_i = 1$ for all i.

The form of the K matrix determines the structure of edge excitations. In terms of n bosonic fields, ϕ_i, which satisfy commutation relations,

$$[\phi_i(x), \phi_j(x')] = i\pi K^{-1}_{ij} \text{sgn}(x - x'), \tag{3.2}$$

the appropriate edge Hamiltonian is

$$H = \int dx \frac{1}{4\pi} \sum_{ij} V_{ij} \partial_x \phi_i \partial_x \phi_j. \tag{3.3}$$

The matrix V_{ij} is a nonuniversal positive definite matrix, depending on the edge confining potential and edge electron interactions. The Hamiltonian describes n propagating chiral modes. The directions of propagation are determined by the signs of the eigenvalues of the K matrix.

The total electronic charge density at the edge is given by

$$n_\rho(x) = \sum_i n_i(x), \tag{3.4}$$

where the density in the ith mode is, $n_i = \partial_x \phi_i / 2\pi$.

A kinetic equation description of edge transport follows again from the Heisenberg equations of motion, which can be cast in the form,

$$\partial_t n_i + \partial_x J_i = 0, \tag{3.5}$$

with currents,

$$J_i = K^{-1}_{il} V_{lj} n_j. \tag{3.6}$$

The expression (3.6) relating currents and densities is analogous to the relationship between the current and density of quasiparticles in a Fermi liquid. The interaction terms V_{ij} play a role very similar to that of the Fermi liquid parameters.

Once again, it is useful to assume normal ordering for the densities, n_i, so that in equilibrium, all densities and currents vanish. With nonvanishing chemical potentials, μ_i, however, currents will flow. As before, the currents can be computed by adding to the Hamiltonian,

$$\delta H = -\sum_i \int dx \mu_i n_i, \tag{3.7}$$

and then evaluating the currents, $J_i = -e\partial_t \phi_i / 2\pi$, using the commutation relations (3.2). This gives a nonvanishing transport current,

$$J_i = (e/h)K_{ij}^{-1}\mu_j. \tag{3.8}$$

When the edge modes are in equilibrium at a common chemical potential, $\mu_i = \mu$, the total edge current is appropriately quantized, as follows from (3.1):

$$J_\rho = \sum_i J_i = \frac{\nu e}{h}\mu. \tag{3.9}$$

However, if the edge modes are fed by nonideal contacts, they will generally not all be at the same chemical potential. In this case, perfect Hall quantization can break down, as we detail below.

For simplicity we will focus hereafter on the special case of two edge modes, where K is a 2 by 2 matrix. This includes bulk states at filling $\nu = 1/(p_1 - 1/p_2)$, with p_1 and p_2 odd and even integers, respectively. The explicit form for K in the "symmetric" basis is

$$K = \begin{pmatrix} p_1 & p_1 - 1 \\ p_1 - 1 & p_1 + p_2 - 2 \end{pmatrix}. \tag{3.10}$$

When $p_1 = 1$ the K matrix is diagonal, with eigenvalues 1 and $p_2 - 1$. When p_2 is positive, as for $\nu = 4/3$ and $\nu = 2$, both modes propagate in the same direction. For negative p_2, however, the two modes are predicted to propagate in opposite directions. This includes fillings $\nu = 2/3$ and $\nu = 4/5$.

As we shall see below, quantization of the Hall conductance when multiple modes are present, generally requires that the different modes on a given edge are equilibrated at a common chemical potential. For this reason it will be convenient to transform to a set of fields that reveal more readily when an edge is equilibrated. To this end we define charge and neutral fields[8] via

$$\phi_\rho = \phi_1 + \phi_2, \tag{3.11}$$

$$\phi_\sigma = \phi_1 + (1 - p_2)\phi_2. \tag{3.12}$$

The charge and neutral fields commute with one another, and satisfy

$$[\phi_\rho(x), \phi_\rho(x')] = i\pi\nu\,\mathrm{sgn}(x - x'), \tag{3.13}$$

$$[\phi_\sigma(x), \phi_\sigma(x')] = i\pi p_2\,\mathrm{sgn}(x - x'). \tag{3.14}$$

Note that p_2 can be negative. The Hamiltonian becomes, $H = H_\rho + H_\sigma + H_{\mathrm{int}}$, with charge and neutral pieces,

$$H_\rho = \frac{v_\rho}{4\pi\nu}\int dx(\partial_x\phi_\rho)^2, \tag{3.15}$$

$$H_\sigma = \frac{v_\sigma}{4\pi|p_2|}\int dx(\partial_x\phi_\sigma)^2, \tag{3.16}$$

coupled together via

$$H_{\mathrm{int}} = \frac{2v_{\mathrm{int}}}{4\pi}\int dx\partial_x\phi_\rho\partial_x\phi_\sigma. \tag{3.17}$$

The velocities v_ρ, v_σ, and v_{int} depend on the original velocities, V_{ij} in (3.3).

The equations of motion for the charge and neutral fields can be obtained and expressed in terms of charge and neutral densities:

$$n_\alpha = \frac{1}{2\pi}\partial_x\phi_\alpha, \tag{3.18}$$

with $\alpha = \rho, \sigma$. Again, they can be written as continuity equations,

$$\partial_t n_\alpha + \partial_x J_\alpha = 0, \tag{3.19}$$

with currents defined as

$$J_\rho = v_\rho n_\rho + \nu v_{\mathrm{int}} n_\sigma, \tag{3.20}$$

and

$$J_\sigma = \mathrm{sgn}(p_2)v_\sigma n_\sigma + p_2 v_{\mathrm{int}} n_\rho. \tag{3.21}$$

Notice that for $p_2 < 0$, the neutral mode propagates in the direction opposite to the charge mode.

The charge and neutral currents can be given a simple physical interpretation. Using (3.11), the charge current can be expressed in terms of the original currents, J_i, as

$$J_\rho = J_1 + J_2. \tag{3.22}$$

Thus, J_ρ is simply the total electrical charge current along the edge. Likewise, the neutral current takes the form

$$J_\sigma = J_1 + (1 - p_2)J_2. \tag{3.23}$$

In the presence of nonvanishing chemical potentials, μ_i, this can be reexpressed using (3.8) as

$$J_\sigma = \frac{e}{h}(\mu_1 - \mu_2). \tag{3.24}$$

When the edge is equilibrated, $\mu_1 = \mu_2$, the neutral current vanishes, whereas a nonzero J_σ indicates an unequilibrated edge. Thus, J_σ can be interpreted as a current of (neutral) vortices, moving along the edge. The flux of vortices leads to a chemical potential gradient between the two edge modes.

B. Contacts

In this subsection, we consider models for contacts appropriate to quantum Hall fluids with multiple edge modes.

1. The tunnel junction point contact

We first consider leads which are connected to the sample via tunnel junction point contacts. When multiple edge channels are present, the electrons from the leads can tunnel onto the edge in different ways. For instance, for $\nu = 2$ an electron can tunnel into either of the two edge modes. Generally, the tunneling rates will be differ-

ent, depending on the details of the tunnel junction. As a result, the two edge modes will be populated differently, at different chemical potentials. As we shall show, this leads to a breakdown in the quantized Hall conductance (provided other processes do not equilibrate the modes—see below). This can be seen easily in the extreme limit that the tunneling is only into the outer edge channel, in which case the Hall conductance would be $1e^2/h$ rather than $2e^2/h$.

Consider then a metallic electrode at chemical potential μ connected to the two-channel edge via a tunnel junction point contact. In addition to the transfer of an electron to one of the two edge modes, the tunneling process may involve the simultaneous transfer of other electrons between the two modes. The most general charge $Q = 1$ process adds an electron to one channel, say channel one, and transfers m electrons from channel two to one. Here, m is an arbitrary integer. This process is equivalent to adding a unit charge to the charge mode, ϕ_ρ, while creating an instanton of amplitude $(2\pi)(1+mp_2)$ in the neutral field, ϕ_σ. This instanton can be interpreted as the addition of $(1 + mp_2)$ vortices. Upon using the commutation relations (3.13) and (3.14), one can show that this combined process may be accomplished via the operator $\exp i\phi_m(x_i)$, with

$$\phi_m = \frac{1}{\nu}\phi_\rho + \left(\frac{1}{p_2} + m\right)\phi_\sigma. \tag{3.25}$$

Electron charge transfer between the lead and the quantum Hall edge may then be introduced via a tunneling term in the Hamiltonian,

$$H_{\text{tun.}} = -\sum_m t_m \psi(x=0)\int dx\delta(x)e^{i\phi_m(x)} + \text{c.c.} \tag{3.26}$$

As in the single-channel case, the tunneling current from lead to edge may be expressed in terms of the chemical potential drop. However, in this case, the different channels may be at different chemical potentials. Consider the set of chemical potentials, μ_m, defined as the change in energy when a particle is added using the mth tunneling operator. These can be related to the chemical potentials of the original two channels as $\mu_m = \mu_1 + m(\mu_1 - \mu_2)$. Finally, upon using (3.8) and (3.22)–(3.23), these can be reexpressed in terms of the currents as

$$\mu_m = \frac{h}{e}\left[\frac{1}{\nu}J_\rho + \left(\frac{1}{p_2} + m\right)J_\sigma\right]. \tag{3.27}$$

The current tunneling from the lead to the edge in the mth tunneling channel is then

$$I_m(x) = \delta(x)\tilde{G}_m(\mu - \mu_m), \tag{3.28}$$

with μ the chemical potential of the metallic electrode. Again, the tunneling conductances G_m will, in general, be temperature dependent. At low temperatures, they are expected to vanish as a power of temperature, $\tilde{G}_m \approx t_m^2 T^{\Delta_m}$. For very low temperatures, it is possible that a single channel (with the smallest Δ_m) will dominate the tunneling.

We now modify the kinetic equations (3.19) to include tunneling of charge from the leads to the edge, by writing

$$\partial_t n_\rho + \partial_x J_\rho = I_\rho, \tag{3.29}$$

$$\partial_t n_\sigma + \partial_x J_\sigma = I_\sigma. \tag{3.30}$$

Here, I_ρ and I_σ denote the total tunneling rates for charge and vorticity, expressed as a sum over contributions from each of the tunneling channels,

$$I_\rho = \sum_m I_m, \tag{3.31}$$

$$I_\sigma = \sum_m (1 + mp_2)I_m. \tag{3.32}$$

The tunneling currents from lead to edge may now be reexpressed in terms of the edge currents themselves, J_ρ and J_σ, upon using (3.27) and (3.28). This gives

$$I_\rho = G_1\left(\frac{h}{\nu e}J_\rho(x) - \mu\right) + G_2\frac{h}{p_2 e}J_\sigma(x), \tag{3.33}$$

and

$$I_\sigma = G_2\left(\frac{h}{\nu e}J_\rho(x) - \mu\right) + G_3\frac{h}{p_2 e}J_\sigma(x), \tag{3.34}$$

where we have defined three conductances,

$$G_a = \sum_m \tilde{G}_m(1 + mp_2)^{a-1}, \tag{3.35}$$

with $a = 1, 2, 3$. The conductances G_a give a complete characterization of the tunnel junction between the lead and the quantum Hall edge. In general, G_1, G_2, and G_3, will be of comparable magnitudes. As in Sec. II B we will assume that $G_a \ll e^2/h$. Since $\tilde{G}_m > 0$, G_1 and G_3 are necessarily nonzero and positive. In contrast, G_2 can be positive or negative. Although a generic contact will have nonzero G_2, it is possible to imagine fine tuning a contact to make G_2 vanish.

Given the conductances G_a characterizing each contact, Eqs. (3.29,3.30), and (3.33,3.34) can be used to determine transport properties for multiterminal measurements. Again referring to Fig. 1, let $J_{T,B\rho,\sigma}$ denote the charge and neutral currents on the top and bottom edges. Consider first the two-terminal measurement shown in Fig. 1(b), with identical tunnel junctions connecting the sample to the source and drain electrodes, $G_{aS} = G_{aD}$. Solving the steady-state kinetic equations relates the net transport current, $I = J_{T\rho} - J_{B\rho}$, to the chemical potentials of the source and drain electrodes,

$$I = \frac{G_{1S}}{2}(\mu_S - \mu_D). \tag{3.36}$$

As in the single-channel case, the two-terminal resistance is dominated by the contacts, equaling the sum of the two contact resistances, G_1^{-1}.

Under the above transport conditions, in addition to

the flow of electrical current throughout the sample, there is also a flow of vortices—that is a nonvanishing neutral current J_σ — given by

$$J_{T\sigma} - J_{B\sigma} = \frac{G_{2S}}{2}(\mu_S - \mu_D) = \frac{G_{2S}}{G_{1S}} I. \tag{3.37}$$

Note that this vortex current is proportional to G_2, and will generically be nonzero, unless G_2 is fine tuned to zero. Since the neutral current is proportional to the difference of the chemical potentials of the two edge modes, (3.24), a nonvanishing neutral current indicates an absence of edge equilibration.

Next consider a four-terminal measurement, in which tunnel junction contacts are also used as voltage probes on the top and bottom edges of the sample, see Fig. 1(b). For simplicity, we assume that $G_{aT} = G_{aB} \ll G_{aS} = G_{aD}$. The chemical potentials of the voltage probes μ_T, μ_B are adjusted so that no net current flows through the contacts, $I_{\rho T} = I_{\rho B} = 0$. Upon using the steady-state kinetic equations for this four-terminal geometry, one finds

$$\mu_T - \mu_B = \frac{h}{\nu e} I + \frac{h G_{2T}}{p_2 e G_{1T}} \left(J_{T\sigma} - J_{B\sigma}\right), \tag{3.38}$$

where I is the source-to-drain transport current. The four-terminal Hall resistance, $R_H = (\mu_T - \mu_B)/(eI)$, is given by

$$R_H = \frac{h}{e^2}\left(\frac{1}{\nu} + \frac{G_{2T} G_{2S}}{p_2 G_{1T} G_{1S}}\right). \tag{3.39}$$

Again, unless $G_2 = 0$, the Hall resistance R_H is not quantized. Since the two edge channels are out of equilibrium, the sample edge does not have a well defined chemical potential. The voltage probes measure a weighted average of the chemical potentials, with the relative weights (from G_a) depending on nonuniversal details of the contacts. Other more complicated multiterminal geometries can also be easily analyzed using the kinetic equations.

2. The ideal contact

As we have seen above, for an edge with two modes, neither the two- nor four-terminal conductances is quantized when measured with tunnel junction point contacts. In Sec. II, we considered an "ideal contact," which gave a quantized conductance for both two- and four-terminal measurements, in the case of a Hall fluid with a single edge mode. Do these conclusions remain valid for a Hall fluid with multiple edge branches? We now show that they do, *provided* all of the channels on a given edge propagate in the same direction. For $\nu = 1/(p_1 + 1/p_2)$, this is the case for $p_2 > 0$ (e.g., $\nu = 2, 2/5, 4/3, \ldots$). However, for $p_2 < 0$ (e.g. $\nu = 2/3, 4/5, \ldots$), when the two modes propagate in opposite directions, we will show that the two- and four-terminal conductances are *not* quantized, even when measured with such "ideal contacts."

By definition, all edge modes that emanate from an "ideal contact" are in equilibrium at the reservoir chemical potential. When both channels move in the same direction, they will then share a common chemical potential, having emanated from the same reservoir. [The neutral current, J_σ in (3.24), will everywhere vanish.] It then follows from (3.9) that the net transport current along the edge will be appropriately quantized, as will the two- and four-terminal conductances.

In contrast, when the two edge modes move in opposite directions, they will generally be at different chemical potentials in a transport measurement, having emanated from different "ideal contacts." The two edge modes will not be in common equilibrium, and there will be a flow of vorticity along the edge: $J_\sigma \neq 0$. In Ref. 9, we showed that the two-terminal conductance measured with such "ideal contacts" is given by

$$G = (g_+ + g_-)\frac{e^2}{h}. \tag{3.40}$$

Here, $g_\pm$ are right (left) conductances, defined as the change in current in response to a chemical potential, which couples only to the right (left) moving modes. Both conductances $g_\pm$ are positive and satisfy $g_+ - g_- = \nu$, but they are nonuniversal and depend on the interaction matrix V_{ij} in (3.3). It follows that $G > \nu e^2/h$ is nonuniversal. One can likewise show that a four-terminal conductance measured using four "ideal contacts" is also nonuniversal.

In Sec. II, we demonstrated that an "ideal contact" could be realized more microscopically as a tunnel junction "line contact." The "line contact" junction also eliminated the need for fractional edge modes to exist inside the reservoirs. Does the "line junction," when generalized to an edge with multiple modes, restore universality absent with the "ideal contacts?" We now show that this is *not* the case.

3. The line junction contact

Consider then a tunnel junction "line contact" coupling to an edge with two modes. The line contact may be characterized by three conductivities σ_a, $(a = 1, 2, 3)$, defined as the tunneling conductances, G_a in (3.35), per unit length. For a contact with length L the tunneling currents from lead to the edge can then be expressed as

$$I_\rho(x) = \sigma_1\left(\frac{h}{\nu e} J_\rho(x) - \mu\right) + \sigma_2 \frac{h}{p_2 e} J_\sigma(x), \tag{3.41}$$

$$I_\sigma(x) = \sigma_2\left(\frac{h}{\nu e} J_\rho(x) - \mu\right) + \sigma_3 \frac{h}{p_2 e} J_\sigma(x), \tag{3.42}$$

for $0 < x < L$. The "line contact" is modeled by adding these spatially dependent source terms, to the right hand side of the kinetic equations (3.29) and (3.30).

In the steady state, the kinetic equations can be readily solved by diagonalizing a 2 by 2 matrix for the currents J_ρ and J_σ. In the region of the line contact $0 < x < L$, the solution takes the form

$$\begin{pmatrix} J_\rho \\ J_\sigma \end{pmatrix} = \frac{\nu e}{h}\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_1 e^{x/\ell_1} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + c_2 e^{x/\ell_2} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \tag{3.43}$$

where $\ell_{1,2}^{-1}$ are eigenvalues of the matrix and $a_{1,2}, b_{1,2}$ are its eigenvectors. Here, μ is the chemical potential of the contact. The constants c_1 and c_2 are determined by the boundary conditions at $x = 0, L$.

Using (3.35) and the explicit formula for the eigenvalues, it can be shown that $\mathrm{sgn}\ell_1^{-1}\ell_2^{-1} = \mathrm{sgn}p_2$. Thus, when $p_2 > 0$ and both edge modes move in the same direction, both solutions decay exponentially. Then provided $L \gg \ell_1, \ell_2$, the edge modes emanating from the "line contact" will be fully equilibrated with the contact: $J_\rho = (\nu e/h)\mu$, $J_\sigma = 0$. It then follows that all measured conductances will be appropriately quantized.

However, when $p_2 < 0$, one solution in (3.43) is growing exponentially, while the other is decaying. Then generically the neutral current J_σ will be nonzero at the end points of the "line contact." Again, the presence of a nonvanishing neutral current indicates that the two edge modes are not in equilibrium with one another. This in turn implies a nonuniversal Hall conductance, for both two- and four-terminal measurements, just as for the "ideal contact" model. However, in contrast to the "ideal contact," the value of the nonquantized conductance is determined by the ratios of the tunneling conductances σ_a and is independent of the nonuniversal interaction matrix V_{ij}.

So far, all the models of contacts that we have considered lead to an absence of conductance quantization for an edge with two modes moving in opposite directions. The lack of quantization is due to an absence of equilibration between the oppositely moving modes. Real quantum Hall samples show precise quantization, presumably due to processes along the sample edges, which allow for equilibration. We now turn to a discussion of impurity scattering along the edge, and show how it equilibrates and restores quantization.

C. Edge equilibration: Random impurities

Consider impurity scattering at the edge, which allows for nonmomentum conserving charge transfer processes between nearby edge modes. It is useful to distinguish two length scales. The first, a tunneling mean free path ℓ, denotes the distance an electron propagates along the edge before it is scattered by an impurity into a different channel. For noninteracting electrons, this scattering is elastic, and ℓ is temperature independent. For fractional quantum Hall edge channels, though, this length can be temperature dependent, and even divergent at zero temperature (see below). A second length, denoted ℓ_ϕ, is the length over which electrons lose their phase coherence within a single edge channel. In general, the dephasing length diverges at low temperatures. It arises both due to thermal dephasing (which gives $\ell_\phi \approx v/k_B T$) and due to inelastic scattering off phonons or other electrons, for which ℓ_ϕ diverges as a different power of the temperature.

In order to measure a quantized Hall conductance, the separation L between current and voltage leads must exceed both ℓ_ϕ and ℓ. On scales beyond ℓ, one expects the multiple modes to have equilibrated. However, L must also be larger than ℓ_ϕ for robust quantization, since in the regime $\ell_\phi > L > \ell$, sample specific mesoscopic fluctuations in the measured conductance are expected. True equilibrium is thus reached at the larger of ℓ and ℓ_ϕ.

We now describe the equilibration due to random impurities, which gives a finite length ℓ for interchannel mixing. The operator that transfers a unit charge between the two edge modes is $\exp i\phi_\sigma$, as can be deduced from the definitions in III A. Edge impurity scattering can thus be incorporated into (3.15)–(3.17), by adding a term to the Hamiltonian of the form,[8,9]

$$H_{\mathrm{random}} = \int dx \left[\xi(x) e^{i\phi_\sigma} + \mathrm{c.c.}\right], \tag{3.44}$$

where $\xi(x)$ is a (complex) spatially random tunneling amplitude. Since the total charge is conserved in such a scattering event, the kinetic equation for the charge sector is unchanged:

$$\partial_t n_\rho + \partial_x J_\rho = 0. \tag{3.45}$$

However, when an electron tunnels between the two channels, p_2 vortices are destroyed. The neutral sector thus contains a "collision term," $I_\perp(x)$.

$$\partial_t n_\sigma + \partial_x J_\sigma = I_\perp(x). \tag{3.46}$$

We compute the tunneling current, $I_\perp(x)$, using Fermi's golden rule. Again, such a description is appropriate in the limit that successive intermode tunneling events are incoherent. Provided the chemical potential difference, $J_\sigma = (e/h)(\mu_1 - \mu_2)$, between the two edge modes is sufficiently small, the response will be Ohmic, and the tunneling current per unit length will be linear in J_σ,

$$I_\perp = -\mathrm{sgn}(p_2) J_\sigma/\ell. \tag{3.47}$$

Here, ℓ is the intermode scattering length.

For the simple case of disorder with a δ-function correlated Gaussian distribution, the length ℓ may explicitly be computed perturbatively in the variance $W = \langle|\xi(x)|^2\rangle$. We find

$$\ell^{-1} \sim \ell_0^{-1} \left(\frac{T}{T_0}\right)^\alpha, \tag{3.48}$$

where $\ell_0^{-1} \propto W$ and T_0 is the high energy cutoff. The exponent α is related to the scaling dimension Δ of the tunneling operator $\exp i\phi_\sigma$ via $\alpha = 2\Delta - 2$.[8,9] For positive p_2 one has $\Delta = p_2/2$, so that $\alpha = p_2 - 2$. For negative p_2, however, $\Delta \geq |p_2|/2$ is nonuniversal and depends on the coupling constants V_{ij} in the Hamiltonian (3.3). In this case, $\alpha \geq |p_2| - 2$.

In the steady state, the extra term on the right hand side of (3.46) causes an exponential decay of the neutral mode with length ℓ,

$$J_\sigma(x) = J_\sigma(0) e^{-\mathrm{sgn}(p_2)x/\ell}. \tag{3.49}$$

Beyond this scale, the neutral current vanishes. Since the neutral current is proportional to the difference between chemical potentials of the two modes, (3.24), this

indicates that the edge equilibrates on this length scale. When the spacing between current and voltage probes is larger than ℓ, the conductance will be appropriately quantized regardless of the nature of the contacts.

Recall that for $p_2 < 0$, the neutral mode propagates *upstream*, in the direction opposite to the charge mode, J_ρ. It is interesting to note that the sign of p_2 also determines the direction in which the neutral mode decays. When $p_2 < 0$, J_σ likewise decays in the upstream direction. It then follows that when the contacts are further than ℓ apart, the departure from equilibrium will only be present within a distance ℓ upstream from the contacts.

Since Fermi's Golden rule, which lead to the result (3.47), can only be trusted when successive tunneling events are uncorrelated, these equations must be used with caution at very low temperatures, when quantum coherence effects can become important. There are two possible low temperature regimes, depending on the value of the exponent Δ.

For $\Delta < 3/2$, weak disorder is relevant, and the renormalization group flows are to a disorder dominated fixed point with $\Delta = 1$ at the fixed point. In this case, the above kinetic equation, which predicts only a single propagating mode, is not correct. Although the decay of the neutral mode J_σ on scale ℓ is correct, there is a hidden (neutral) mode which propagates at zero temperature. This hidden mode is not a vortex current, and can be present even on a fully equilibrated edge ($J_\sigma = \mu_1 - \mu_2 = 0$). As shown in Ref. 9, this hidden mode itself decays away at finite temperatures. Provided that we focus on those properties that are insensitive to the presence of this hidden mode, the kinetic equation provides an adequate description even in this regime.

For $\Delta > 3/2$, the length ℓ diverges in the zero-temperature limit, and at $T = 0$, both modes propagate. In this case, the kinetic equation gives a correct description even as $T \to 0$. This is plausible on physical grounds, since in this case $\ell(T)$ diverges more rapidly than the phase breaking length, $\ell_\phi \sim v/k_B T$, so that successive tunneling events should be incoherent. On a more technical level, the validity of (3.48) follows from the perturbative irrelevance of W in the renormalization group calculation.

The exponential decay of the neutral mode described above is a result of the linear (or Ohmic) intermode tunneling current in response to a nonequilibrium chemical potential difference between the two modes. Even when this Ohmic response vanishes at $T = 0$ (for $\Delta \geq 3/2$), a finite *non-Ohmic* tunneling current is expected, which will vanish faster than linearly with the chemical potential difference. This nonlinear tunneling current can also lead to equilibration between the edge channels. But in this case, the decay of J_σ is not exponential as in (3.49), but rather algebraic. Specifically, the decay of J_σ at $T = 0$ can be studied by analyzing a *nonlinear* kinetic equation. Consider the steady-state version of (3.46) with a nonlinear tunneling current,

$$I_\perp \propto \ell_0^{-1} \left(\frac{J_\sigma}{J_0}\right)^\alpha J_\sigma, \tag{3.50}$$

where $J_0 = k_B T_0 e/h$. Then (3.46) may be readily integrated to give

$$J_\sigma(x) = \frac{J_\sigma(0)}{\left[1 + \frac{\alpha x}{\ell_0}(J_\sigma(0)/J_0)^\alpha\right]^{1/\alpha}}. \tag{3.51}$$

At large distances, $J_\sigma(x)$ decays algebraically, varying as $J_0(\alpha x/\ell_0)^{-1/\alpha}$, independent of $J_\sigma(0)$. In Sec. IV, we will explore the consequences of this slow decay for quantum Hall states at filling $\nu = 4/3$ and $4/5$.

D. The compound contact

As we have seen, the absence of a quantized conductance can be traced to an absence of equilibration between the multiple edge modes. However, when impurity scattering is included along the edges, the different modes will equilibrate on a scale set by ℓ. Then, provided that the spacing between the current and voltage contacts is larger than ℓ, an appropriately quantized conductance will be found regardless of the nature of the contacts.

For a perfectly clean system with vanishing impurity scattering at the edge, one might ask whether it is still possible to construct a contact that populates all the edge modes in equilibrium. Such a *fully equilibrated* contact may be constructed, as illustrated in Fig. 3. Consider a "compound" contact that consists of a tunnel junction point contact, say, with edge impurity scattering regions localized within a length scale L on either side.[18] Provided L is larger than ℓ, all of the currents away from the compound contact will then be fully equilibrated. Outside the contact region, there will be a unique edge chemical potential, or equivalently a vanishing neutral current density. The key point, which distinguishes this "compound contact" from the earlier models for contacts, is that it allows for direct transfer of charges between the two oppositely moving edge modes. In this way, the current in the right moving mode, can lead to an adjustment of the left moving current, causing equilibration. It should be emphasized, though, that in some cases (e.g., $\nu = 4/5$), the length scale ℓ diverges at zero temperature, so such a compound contact would eventually cease to equilibrate at low enough temperatures.

A four-terminal Hall conductance measured with such "compound" contacts will be appropriately quantized, when $L \gg \ell$. Within the scattering region of size L at the compound current contacts there may be a nonequi-

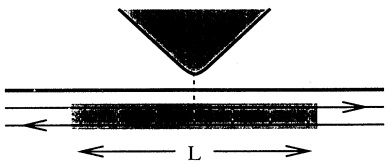

FIG. 3. A compound contact. Edge-impurity scattering allows for equilibration between the different edge channels within a distance L of the contact, as indicated by the shaded region.

librium current distribution, with $J_\sigma \neq 0$. However, they decay away in a distance ℓ, and at the voltage contacts $J_\sigma = 0$. The voltage probes will then measure $\mu = J_\rho/\nu$, giving a quantized Hall conductance.

One may also consider a compound contact made from a tunnel junction line contact. In this case, solving the kinetic equation shows that when $L \gg \ell$, the edge is fully equilibrated at the chemical potential of the "upstream" lead. (Here, "upstream" is defined in terms of the direction of propagation of the charge.) Thus, the compound line junction contact is an explicit realization of the Büttiker ideal contact even when $p_2 < 0$. The two-terminal conductance measured with such contacts should be appropriately quantized.

IV. EXPERIMENTAL IMPLICATIONS: LACK OF EQUILIBRATION FOR $\nu = 4/5, 4/3$

In this section we focus on specific predictions regarding edge-state equilibration in the FQHE, which would be interesting to test experimentally. In Sec. III C, we found for filling fractions with $|p_2| \geq 4$ (such as $\nu = 4/5$ or 4/3) that the edge-state equilibration length diverges at low temperatures. This should lead to a breakdown of the quantization of the Hall conductance, as the temperature is lowered.

In order to probe the equilibration between edge-states, consider the sample geometry shown in Fig. 4. A current is injected through a source contact, and the Hall voltage is measured using voltage probes on the top and bottom edges at distances L_T and L_B from the source. We assume that the drain contact is much further away, so that it has no effect on the equilibrium of the edge channels near the voltage probes. For simplicity, as in Sec. III B 1, we model the contacts as tunnel junction point contacts with conductances $G_{Ta}, G_{Ba} \ll G_{Sa}$. Further, suppose that the magnetic field points out of the page in Fig. 4, so that the direction of propagation of the charge J_ρ around the edge is counterclockwise.

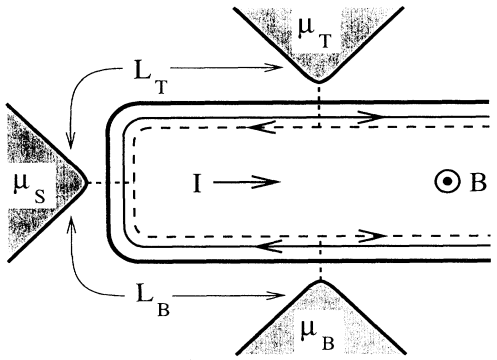

FIG. 4. Measurement geometry to probe-edge state equilibration. A current is injected through the source contact, and the Hall voltage is measured between the contacts T and B. As indicated by the solid line, charge propagates counterclockwise along the edge. For $p_2 < 0$, the neutral mode propagates (and decays) in the opposite direction, as indicated by the dashed line. For $p_2 > 0$, the neutral mode propagates clockwise, and the arrow on the dashed line should be reversed.

Since the source contact is nonideal, it populates the edge modes out of equilibrium. For $\nu = 4/3$ ($p_2 > 0$), all of the channels propagate counter clockwise, so that only the top edge is out of equilibrium. The lack of equilibrium on the top edge, characterized by $J_{T\sigma}$, is determined by I and decays away in an equilibration length ℓ. Following the analysis of Sec. III B 1, we then find that just above the source lead $J_{T\sigma}(0) = (G_{2S}/G_{1S})I$. It then follows that the measured Hall voltage is not quantized, and is given by

$$\mu_T - \mu_B = \frac{h}{e}\left(\frac{1}{\nu} + \frac{G_{2T}G_{2S}}{p_2 G_{1T}G_{1S}} e^{-L_T/\ell}\right) I, \qquad (4.1)$$

where ℓ is the temperature dependent equilibration length (3.48). Since G_2 in (4.1) can be either positive or negative, even the sign of the deviation from quantization is nonuniversal. Note that since the bottom edge is fully equilibrated, the nonuniversal linear Hall voltage is *independent* of the position L_B of the bottom contact.

When $\nu = 4/5$ ($p_2 < 0$), the neutral mode, J_σ propagates clockwise, in the opposite direction as J_ρ. It follows that in this case the bottom edge is out of equilibrium, whereas the top edge is fully equilibrated. The measured linear Hall voltage is given by (4.1) with T replaced by B. Surprisingly, even though the current is injected onto the top edge, the nonuniversal linear Hall voltage depends only on the position L_B of the bottom contact.

In order to estimate the temperature at which the lack of equilibration should be detectable, let us consider the temperature scale T^* at which the equilibration length is comparable to the distance L between contacts. From (3.48),

$$T^* = T_0 \left(\frac{\ell_0}{L}\right)^{1/\alpha}. \qquad (4.2)$$

As a rough estimate, we take the cutoff energy T_0 to be equal to the excitation gap for the bulk Hall fluid, $T_0 \approx 1$ K. For $\nu = 4/5$, the tunneling length in the absence of any effects due to quantum coherence, ℓ_0, depends on the impurity concentration near the edge and the physical separation between the two channels. With the rough estimate, $\ell_0 \approx 10$ nm, and with $L = 1$ μm and $\alpha = 2$, this gives a crossover temperature, $T^* \approx 100$ mK. For $\nu = 4/3$, however, the two channels reside in different Landau levels. Since the Zeeman energy is significantly less than the cyclotron energy, the two channels will have opposite spin. It follows that interchannel scattering can only occur via spin-orbit or spin flip scattering. The bare tunneling length ℓ_0 should, therefore, be substantially longer than for $\nu = 4/5$, and T^* substantially higher.

For temperatures $T > T^*$, the neutral current J_σ decays essentially to zero between the leads, and the edge channels effectively equilibrate, resulting in a quantized conductance. For $T < T^*$, however, full equilibration does *not* take place between the contacts and a nonuniversal Hall conductance given by (4.1) is expected.

While a nonuniversal linear conductance is expected at low temperatures, we now argue that increasing the

voltage can restore the quantization. In the following, we consider the case $\nu = 4/3$. The results for $\nu = 4/5$ are obtained, as above, by interchanging T and B. At zero temperature, $J_{T\sigma}$ decays according to (3.51) due to the non-Ohmic interchannel tunneling,

$$J_{T\sigma}(L_T) = \frac{J_{T\sigma}(0)}{[1 + (c_S I/I^*)^\alpha]^{1/\alpha}}, \tag{4.3}$$

where $c_S = \alpha^{1/\alpha} G_{2S}/G_{1S}$. Here, the characteristic current,

$$I^* = \frac{ek_B T_0}{h} \left(\frac{\ell_0}{L_T}\right)^{1/\alpha} = \frac{ek_B T^*}{h}, \tag{4.4}$$

sets the scale for the size of the linear response regime. Using the above estimates, we find $I^* \approx 0.5$ nA.

For $I < I^*$ there is no significant decay in the neutral current J_σ, by the time it reaches the top voltage contact, and the linear conductance is not quantized. However, at higher currents, the interchannel equilibration is enhanced by nonlinear tunneling. For $I \gg I^*$, the neutral current at the top voltage contact is $J_{T\sigma}(L) = \alpha^{-1/\alpha} I^*$. In this regime, the measured Hall voltage may be deduced from (3.38).

$$\mu_T - \mu_B = \frac{h}{\nu e}\left(I + d_T I^*\right), \tag{4.5}$$

where $d_T = \alpha^{-1/\alpha} \nu G_{2T}/p_2 G_{1T}$. This predicts a quantized Hall resistance when the source-drain current I is much larger than I^*. However, deviations from the quantized value, of order I^*/I, are present as a result of the incomplete equilibration between the two edge modes. These deviations lead to a slight offset of the linear I-V characteristic at high currents. The constant d_T, which can be of order 1, depends on the structure of the voltage contact, and can be either positive or negative.

V. CONCLUSION

As is well known in the integer quantum Hall effect, equilibration between multiple channels on the same edge is a prerequisite for quantization of the Hall conductance. There are two sources for this equilibration: Edge impurity scattering, and equilibration *at* the electrical contacts. In the IQHE, these can be analyzed using a free-electron model of the edge modes. In this paper, we have generalized to the FQHE, introducing a simple kinetic equation description of FQHE edge dynamics. This approach allows for a unified analysis of equilibration, due to both electrical contacts and edge-impurity scattering. More specifically, we have introduced and analyzed several concrete models for electrical contacts in the FQHE regime. This allows us to describe realistic transport geometries with multiple current and voltage contacts.

The important feature, which distinguishes FQHE edge dynamics from the IQHE, is the presence of edge modes which move in both directions along the edge, such as for filling $\nu = 2/3$. In this case, it is very difficult to equilibrate at the electrical contacts, as we have seen in detail by considering various specific models. Rather, equilibration requires direct interchannel charge transfer, from edge-impurity scattering. This is in contrast with the IQHE, for which multiple channels moving in the same direction can readily be brought into equilibrium by an electrical contact. Surprisingly, for certain quantum Hall states, notably $\nu = 4/5$ and $\nu = 4/3$, the length scale for equilibration between the edge channels due to impurity scattering diverges at low temperatures. This results in a breakdown of quantization for the Hall conductance at low temperatures in small samples. We hope that this work will help stimulate further experimental exploration of mesoscopic phenomena in the fractional quantum Hall regime.

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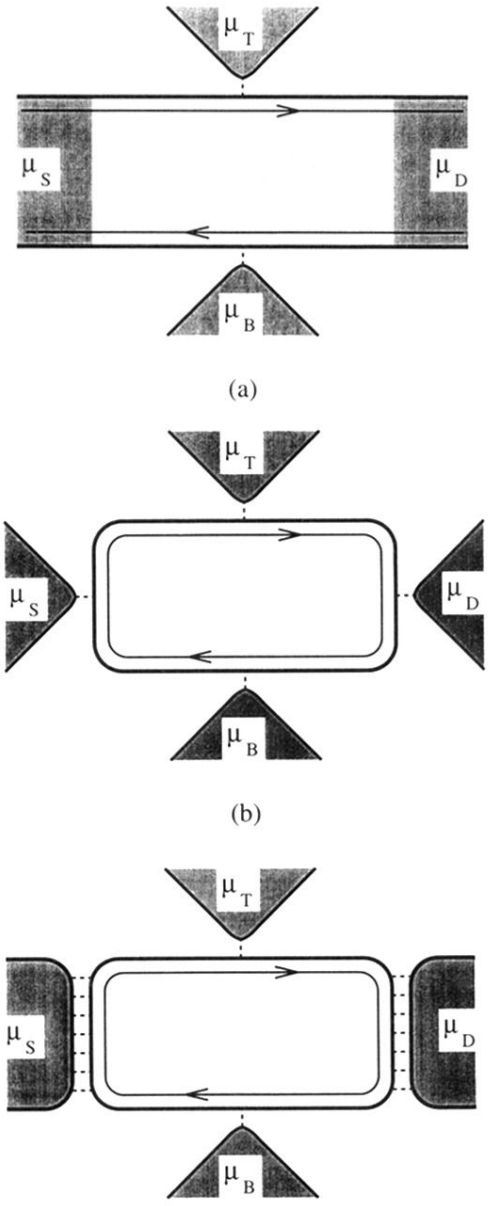

FIG. 1. Three different models for the source and drain contacts to a quantum Hall fluid. (a) "Ideal" contacts: the FQHE edge channels extend to infinity in both the source and drain electrodes; (b) tunnel junction point contacts; (c) tunnel junction line contacts. In (a), (b), and (c), the Hall voltage is measured via weakly coupled point contact tunnel junctions.

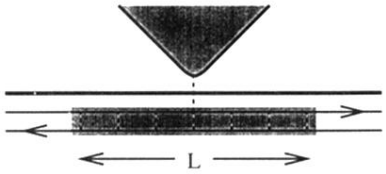

FIG. 3. A compound contact. Edge-impurity scattering allows for equilibration between the different edge channels within a distance L of the contact, as indicated by the shaded region.

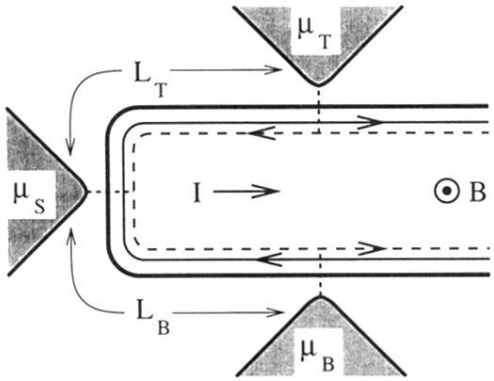

FIG. 4. Measurement geometry to probe-edge state equilibration. A current is injected through the source contact, and the Hall voltage is measured between the contacts T and B. As indicated by the solid line, charge propagates counterclockwise along the edge. For $p_2 < 0$, the neutral mode propagates (and decays) in the opposite direction, as indicated by the dashed line. For $p_2 > 0$, the neutral mode propagates clockwise, and the arrow on the dashed line should be reversed.