Truncation scheme for double-well systems with Ohmic dissipation

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In this paper we describe how a quantum system consisting of a single extended coordinate q in a double-well potential $V(q)$, which interacts with a dissipative environment, may be systematically reduced ("truncated") to an equivalent two-state system interacting with a modified environment. Although we concentrate on Ohmic dissipation, the method which we present is also applicable to other types of environmental spectral densities. The truncation scheme is applicable if there exists a wide separation of energy scales in the problem; we discuss the specific conditions which the system and environment must satisfy before such a scheme can be implemented. Our method proceeds by calculating the effects of the high-frequency environmental modes on the bare tunneling matrix element for transitions between the two wells. We are left with an equivalent two-state system with a renormalized tunneling matrix element, interacting with the remaining low-frequency environmental modes. The renormalized tunneling matrix element is calculated using path-integral techniques in a semiclassical approximation. Explicit results are given for a quartic double-well potential in the underdamped and overdamped regimes.

I. INTRODUCTION

The topic of quantum-mechanical tunneling in the presence of a dissipative environment is of current theoretical and experimental interest.[1] In particular, the case in which a quantum-mechanical particle (the "system") tunnels in a double-well potential, while interacting with a heat bath composed of harmonic oscillators (the "environment") is relevant to the discussion of macroscopic quantum coherence (MQC).[1,2] There has been little progress in treating the dynamics of such a system, even at zero temperature. However, we will show that at sufficiently low temperatures and for small well asymmetry it is possible to simplify the problem by restricting the Hilbert space of the system to the ground and first excited states in the double well. One may then work with an equivalent two-level system coupled to a bath of bosons (the spin-boson Hamiltonian). The dynamics of such a system has recently been studied by Chakravarty and Leggett[3] and others[4] (for the case of Ohmic dissipation).

It is the purpose of this paper to present a procedure for reducing the double-well problem to an equivalent two-state problem. Such a procedure has been studied by Sethna[5] in the context of tunneling centers in solids. In the tunneling-center problem the dissipation is furnished through a coupling of the defect to the long-wavelength phonons in the solid, and Sethna argues that the spectral density $J(\omega)$ of the phonons (to be defined below) at low frequencies behaves as $J(\omega)\propto\omega^3$. Such behavior makes the reduction procedure straightforward in principle, although in practice rigorous numerical results are difficult to obtain. The case of Ohmic dissipation, where $J(\omega)\propto\omega$ as $\omega\to 0$, is much more subtle from even a conceptual point of view, due to the infrared divergences which plague the theory. For these reasons we believe that a careful discussion of the reduction procedure for Ohmic dissipation is needed. The general "philosophy" of such a procedure is discussed further in Ref. 6.[7]

In Sec. II of this paper we will outline our procedure and examine when we expect such a truncation approximation to be valid. Section III will discuss a method for calculating the parameters which appear in the two-state Hamiltonian in terms of the physically accessible double-well parameters. Finally, in Sec. IV we use the results of Sec. III to obtain explicit results in the limits of large and small damping.

II. THE REDUCTION SCHEME

We consider a double-well potential for the system coordinate q, which is coupled linearly to a bath of harmonic oscillators with coordinates x_α. The Hamiltonian is given by

$$H=\frac{p^2}{2M}+V(q)+\sum_\alpha \tfrac{1}{2}(p_\alpha^2/m_\alpha+m_\alpha\omega_\alpha^2 x_\alpha^2) +q\sum_\alpha C_\alpha x_\alpha+q^2\sum_\alpha\frac{C_\alpha^2}{2m_\alpha\omega_\alpha^2} \tag{1}$$

with $V''(q=\pm q_0/2)=M\omega_0^2$ (see Fig. 1).[8] The spectral density is defined by

$$J_0(\omega)\equiv\frac{\pi}{2}\sum_\alpha\frac{C_\alpha^2}{m_\alpha\omega_\alpha}\delta(\omega-\omega_\alpha)\ , \tag{2}$$

where $J_0(\omega)=\eta\omega$ for Ohmic dissipation.[9]

In the absence of the dissipative coupling ($\eta=0$) the energy levels are shown schematically in Fig. 1. The bare tunneling splitting Δ_0 may be calculated in the limit $V_0\gg\hbar\omega_0$ using standard WKB techniques which give

$$\Delta_0\sim C\omega_0\exp(-AV_0/\hbar\omega_0)\ ,$$

where A and C are constants which depend upon the shape of the potential. We consider only the limit

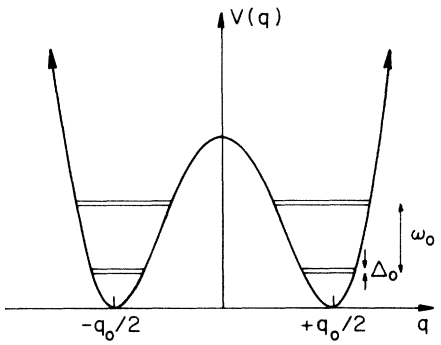

FIG. 1. Double-well potential $V(q)$ with a bare (undamped) tunnel splitting Δ_0 between the ground and first excited state.

$V_0 \gg \hbar\omega_0$ so that $\Delta_0 \ll \omega_0$.

First, consider coupling the system coordinate to only high-frequency oscillators with frequencies $\omega > \omega_c$, with ω_c chosen to satisfy $\omega_c \gg \Delta_0$. One expects that these oscillators will be able to follow the coordinate q during the tunneling process and will thus have the effect of adiabatically renormalizing the bare tunneling splitting Δ_0. A crude calculation (see Leggett[2] for details) gives

$$\Delta = \Delta_0 \exp\left[-\frac{q_0^2}{2\pi\hbar} \int_{\omega_c}^{\omega_b} \frac{J(\omega)}{\omega^2} d\omega \right] , \tag{3}$$

where the term in large parentheses is often referred to as the Franck-Condon factor and ω_b is the classical oscillation frequency in a single well, which for Ohmic dissipation is given by

$$\omega_b = \omega_0, \quad \eta \to 0 ,$$

$$\omega_b = M\omega_0^2/\eta, \quad \eta \to \infty .$$

For $J_0(\omega) \sim \omega^s$ $(\omega \to 0)$ with $s > 1$ the integral in (3) is well behaved; we may take $\omega_c \to 0$ and absorb all of the oscillators into an effective splitting. However, for $J_0(\omega) = \eta\omega$, we see that the integral is logarithmically divergent when $\omega_c \to 0$. Thus for Ohmic dissipation the oscillators with $\omega > \omega_c$ must be treated separately from the oscillators with $\omega < \omega_c$.

We will implement our reduction procedure in two stages. First, we couple the system to only high-frequency oscillators with $\omega > \omega_c$. The cutoff frequency ω_c is chosen such that the resulting energy spectrum of the system coupled to the high-frequency oscillators consists of two discrete low-lying states, with a renormalized level splitting $\Delta = \Delta(\eta, \omega_c)$, which are well separated in energy from the continuum of excited states. Since the continuum of excited states have energies $\gtrsim \hbar\omega_c$ we require that $\Delta(\omega_c) \ll \omega_c$. Now if we only consider temperatures T such that $K_B T \ll \hbar\omega_c$, the excited states will not be appreciably populated and can be ignored. In addition, if we were to consider a biased double well with bias energy ϵ between the well minima we must require that $\epsilon \ll \hbar\omega_c$. The second stage of the reduction scheme consists of coupling the remaining low-frequency oscillators with $\omega < \omega_c$ to the modified system. These oscillators will mediate transitions between the two low-lying states, and will also couple the low-lying states to the continuum of excited states. However, the latter coupling can be shown to be of order ω_c/ω_b.[6] Thus, if the parameters are such that ω_c can be chosen to simultaneously satisfy $\Delta(\omega_c) \ll \omega_c$ and $\epsilon, T \ll \omega_c \ll \omega_b$, we may ignore the continuum of states and replace the double-well Hamiltonian (1) by an effective two-state Hamiltonian with tunneling splitting $\Delta(\eta, \omega_c)$, coupled to the low-frequency oscillators. In the usual spin representation the Hamiltonian takes the form

$$\hat{H} = -\frac{\hbar\Delta}{2}\hat{\sigma}_x + \frac{\epsilon}{2}\hat{\sigma}_z + \sum_\alpha \tfrac{1}{2}(P_\alpha^2/m_\alpha + m_\alpha\omega_\alpha^2 x_\alpha^2) + \tfrac{1}{2}q_0\hat{\sigma}_z \sum_\alpha C_\alpha x_\alpha , \tag{4}$$

where the $\hat{\sigma}_i$'s are Pauli spin matrices and the sum over the oscillators now includes only those with frequencies less than ω_c.

If our procedure is correct then the unphysical cutoff frequency ω_c must cancel in our subsequent calculations of physical quantities using (4). We now demonstrate that that is indeed the case. First, we define a dimensionless damping parameter $\alpha = \eta q_0^2/2\pi\hbar$ and a dimensionless "barrier height" $v = M\omega_0^2 q_0^2/\hbar\omega_0$, where $v \gg 1$ in the WKB limit discussed in this work. We will show in Sec. III that to lowest order in ω_c/ω_b and $\Delta(\omega_c)/\omega_c$, Δ takes the form (for Ohmic dissipation)

$$\Delta(\omega_c/\omega_0, \alpha, v) = g(\alpha, v)(\omega_c/\omega_0)^\alpha \Delta_0(v) . \tag{5}$$

The precise functional form of $g(\alpha, v)$ depends on the shape of the potential barrier. To demonstrate explicitly the cancellation of ω_c, we consider the following quantities which were calculated using (4):[3,4]

$$\Gamma = \frac{\Delta^2}{2\omega_c}\left[\frac{2\pi T}{\hbar\omega_c}\right]^{2\alpha-1} \frac{\cosh(\beta\epsilon/2)}{\Gamma(2\alpha)} |\Gamma(\alpha + i\beta\epsilon/2\pi)|^2 , \tag{6}$$

$$\Delta_r = \Delta(\Delta/\omega_c)^{\alpha/(1-\alpha)} . \tag{7}$$

Here Γ is the relaxation rate from the upper well to the lower well (for $\alpha > 1$, or for $T, \epsilon \gg \Delta r$ with $\alpha < 1$) and Δ_r is a fully renormalized tunnel splitting (for $\alpha < 1$) which describes the typical frequency of coherent oscillations between the wells, which is expected to occur for $\alpha < \frac{1}{2}$. Since Δ and ω_c enter in (6) and (7) only in the combination $\Delta(\omega_c)/\omega_c^\alpha$, it is clear that upon insertion of (5) ω_c cancels out of the resulting expression. It is important to emphasize that this cancellation occurs only to leading order in ϵ/ω_c, T/ω_c, and ω_c/ω_b. This demonstrates explicitly that the reduction procedure is not valid beyond leading order in these small quantities, as anticipated.

III. CALCULATION OF Δ

In this section we will outline the calculation which leads to Eq. (5), one of our central results. For $\epsilon = 0$ and $\eta = 0$, Δ_0 may be calculated in a straightforward manner using instanton techniques.[10] The partition function for the system is written in terms of a path integral over the variable q. In the limit that the temperature $T \to 0$, this

partition function will yield the tunnel splitting. To evaluate the path integral, one considers instanton paths for which the system moves from $-q_0/2$ to $q_0/2$ in the inverted potential $-V(q)$. Summing contributions from paths with an arbitrary number of noninteracting instantons gives the result $\Delta_0 = A(v)\exp\{-S[q_{\rm cl}]/\hbar\}$, where $S[q_{\rm cl}]$ is the Euclidean action evaluated for a single instanton and $A(v)$ is a prefactor which is obtained by considering Gaussian fluctuations about the classical path and proper normalization of the "zero mode."

Our technique is a generalization of the above to the case where the system coordinate is coupled to the high-frequency oscillators. We closely follow the work of Caldeira and Leggett,[1] to which we refer the reader for details. Although this calculation is performed for $\epsilon=0$, it can be shown[6] that $\epsilon\neq 0$ only contributes corrections of relative order ϵ/ω_c to the $\epsilon=0$ result. Thus, our results are also applicable to the biased double-well potential as long as we only consider $\epsilon \ll \omega_c$. The oscillators are divided into high and low frequencies by writing $J_0(\omega)=J(\omega)+J'(\omega)$, where $J(\omega)$ [$J'(\omega)$] corresponds to the low- (high-) frequency oscillators. This is accomplished via the cutoff function $f(\omega;\omega_c)$, such that $f\to 0$ for $\omega \gg \omega_c$ and $f\to 1$ for $\omega\to 0$ [for instance, $f(\omega;\omega_c)=e^{-\omega/\omega_c}$]. Then

$$J(\omega)=\eta\omega f(\omega;\omega_c)\ , \tag{8a}$$

$$J'(\omega)=\eta\omega[1-f(\omega;\omega_c)]\ . \tag{8b}$$

The reduced equilibrium density matrix for the system (after the high-frequency oscillators have been integrated out) is given by

$$K(q_i,q_f;\beta)=K_0(\beta)\int_{q(0)=q_i}^{q(\beta)=q_f} Dq(\tau)\exp\{-S'_{\rm eff}[q(\tau)]/\hbar\}\ , \tag{9a}$$

where

$$S'_{\rm eff}[q]=\int_0^{\beta\hbar} d\tau\{\tfrac{1}{2}M\dot{q}^2+V(q) +\tfrac{1}{2}\int_{-\infty}^{\infty} d\tau'\alpha'(\tau-\tau')[q(\tau)-q(\tau')]^2\} \tag{9b}$$

with

$$\alpha'(\tau-\tau')=\frac{1}{2\pi}\int_0^{\infty} d\omega\, J'(\omega)e^{-\omega|\tau-\tau'|}\ . \tag{9c}$$

Within the "dilute-instanton" approximation, the level splitting is given by[6,10]

$$\Delta(\alpha,\omega_c/\omega_0,v)=A(\alpha,\omega_c/\omega_0,v)\exp\{-S'_{\rm eff}[q_{\rm cl}]/\hbar\} \tag{10a}$$

with

$$A=2(B/2\pi\hbar)^{1/2}\left[\frac{\det\hat{D}_0}{\det'\hat{D}_1}\right]^{1/2}\ , \tag{10b}$$

$$\hat{D}_0 q(\tau)=\left[-\frac{d^2}{d\tau^2}+\omega_0^2\right]q(\tau) +\frac{2}{M}\int_{-\infty}^{\infty}\alpha'(\tau-\tau')[q(\tau)-q(\tau')]d\tau'\ , \tag{10c}$$

$$\hat{D}_1 q(\tau)=\left[-\frac{d^2}{d\tau^2}+\frac{1}{M}V''(q_{\rm cl})\right]q(\tau) +\frac{2}{M}\int_{-\infty}^{\infty}\alpha'(\tau-\tau')[q(\tau)-q(\tau')]d\tau'\ , \tag{10d}$$

$$B=\int_{-\infty}^{\infty} d\tau\dot{q}_{\rm cl}^2\ . \tag{10e}$$

Here $q_{\rm cl}(\tau)$ is the solution of $\delta S'_{\rm eff}/\delta q=0$ (with $\beta=\infty$), satisfying the boundary condition $q_{\rm cl}(\pm\infty)=\pm q_0/2$, and the prime on the determinant in (10b) denotes the omission of the zero eigenvalue.

We emphasize that the dilute-instanton approximation is valid only when the typical instanton separation is much greater than the characteristic time of the instanton-instanton interaction. The interaction would be long ranged in the absence of the lower cutoff ω_c, but $f(\omega;\omega_c)$ cuts off the interaction at times of order ω_c^{-1}. The typical instanton separation is Δ^{-1}, so the dilute instanton approximation requires that $\Delta(\omega_c)\ll\omega_c$. For $\alpha>1$, we see from Eq. (5) that this condition will always be fulfilled provided that ω_c/ω_0 is taken sufficiently small. For $\alpha<1$, we must have $\Delta_0/\omega_c \ll (\omega_0/\omega_c)^{\alpha}$.

Let us now focus on the general structure of the action (9b). Putting $q(\tau)=q_0 z(\tau)/2$ into the second term of (9b) and changing variables, $x=\omega\tau$, gives

$$\Delta S'_{\rm eff}[q_{\rm cl}]/\hbar=\frac{q_0^2}{2\pi\hbar}\int_0^{\infty} d\omega\frac{J'(\omega)}{\omega^2}G(\omega)\ , \tag{11a}$$

where

$$G(\omega)\equiv\tfrac{1}{8}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx' e^{-|x-x'|} \times[z(x/\omega)-z(x'/\omega)]^2\ . \tag{11b}$$

Due to the boundary conditions, $z(\pm\infty)=\pm 1$, $G(\omega)\simeq 1$ for $\omega\ll\omega_b$. Since the integrals in (11b) will be cut off at high frequencies by the finite instanton width, $G(\omega)\to 0$ for $\omega\gg\omega_b$ [generally, we have $G(\omega)\sim(\omega_b/\omega)^2$ for $\omega\gg\omega_b$]. At low frequencies, $J'(\omega)$ cuts off the integral (11a) at ω_c.

We thus see that our result (11a) and (11b) reproduces our naive expectation (3), that [using (10) and (11)]

$$\Delta\sim\Delta_0\exp\left[-\frac{q_0^2}{2\pi\hbar}\int_{\omega_c}^{\omega_b}\frac{J_0(\omega)}{\omega^2}d\omega\right]\ .$$

For Ohmic dissipation, $J_0(\omega)=\eta\omega$, and $\Delta\sim\Delta_0(\omega_c/\omega_b)^{\alpha}$. We see that the upper cutoff on the integral emerges naturally as a consequence of the finite instanton width. The above result gives the leading behavior of $\Delta S'_{\rm eff}/\hbar$ for $\omega_c/\omega_b\to 0$. In addition, we will have some function $f'(\alpha,v)$ which depends upon the potential and the way in which the high- and low-frequency oscillators are separated. Since we have extracted the singular contribution in $\omega_c/\omega_b\to 0$, we expect, in general, that

$$\Delta S'_{\rm eff}/\hbar=-\alpha\ln(\omega_c/\omega_b)+f'(\alpha,v)+O(\omega_c/\omega_b)\ . \tag{12}$$

We will drop the terms of order ω_c/ω_b from now on, in accordance with the limit $\omega_c\ll\omega_b$. The frequency ω_b is equal to ω_0 times some function of α and v, which we absorb into $f'(\alpha,v)$, writing

$$\Delta S'_{\rm eff}/\hbar \sim -\alpha \ln(\omega_c/\omega_0)+f'(\alpha,v) .$$

There will also be a contribution to $S'_{\rm eff}[q_{\rm cl}]$ from the kinetic and potential energy terms, which is finite in the limit $\omega_c/\omega_b \to 0$. These results may be combined to give

$$\frac{S'_{\rm eff}[q_{\rm cl}]}{\hbar}=\frac{S_0}{\hbar}+\alpha \ln\left[\frac{\omega_0}{\omega_c}\right]+f(\alpha,v)+O(\omega_c/\omega_b) , \tag{13}$$

where S_0 is the action for a single instanton in the undamped ($\alpha=0$) case. It is straightforward to show that the prefactor in (10a) can be written

$$A(\alpha,v,\omega_c/\omega_b)=A(\alpha,v)+O(\omega_c/\omega_b) .$$

Using $\Delta_0=A(\alpha=0,v)e^{-S_0/\hbar}$ (the bare tunnel splitting) and collecting the results (10a) and (13) gives us the desired result (5) for the renormalized level splitting, with the function $g(\alpha,v)$ given by

$$g(\alpha,v)=\frac{A(\alpha,v)}{A(\alpha=0,v)}e^{-f(\alpha,v)} . \tag{14}$$

IV. CALCULATION OF $f(\alpha,v)$

In this section we will provide explicit results for the function $f(\alpha,v)$ for $\alpha \to 0$ and $\alpha \to \infty$. Let us first consider $\alpha \to 0$, specializing to the quartic double-well potential

$$V(q)=\frac{M\omega_0^2 q_0^2}{32}\left[\left[\frac{q}{q_0/2}\right]^2-1\right]^2 . \tag{15}$$

Define the following dimensionless variables:

$$\lambda=\omega_c/\omega_0, \quad z=q(\tau)/(q_0/2) ,$$

$$u=\omega_0\tau, \quad \sigma[z]=S_{\rm eff}[q]/(M\omega_0 q_0^2/8) .$$

The dimensionless action from (9b) is then

$$\sigma[z]=\int_{-\infty}^{\infty}\left[\left[\frac{dz}{du}\right]^2+\tfrac{1}{4}(z^2-1)^2\right]du + \frac{\alpha}{v}\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}du'\frac{1+2\lambda|u-u'|}{(u-u')^2(1+\lambda|u-u'|)^2} \times[z(u)-z(u')]^2 , \tag{16}$$

where we have taken $f(\omega:\omega_c)=e^{-\omega/\omega_c}$. The corresponding equation of motion, $\delta S/\delta q=0$, is

$$-2\frac{d^2z}{du^2}+(z^2-1)z + \frac{4\alpha}{v}\int_{-\infty}^{\infty}du'\frac{1+2\lambda|u-u'|}{(u-u')^2(1+\lambda|u-u'|)^2} \times[z(u)-z(u')]=0 , \tag{17}$$

which is subject to the boundary conditions $z(u=\pm\infty)=\pm 1$. If α is small, we may expand the instanton solution

$$z_{\rm cl}(u)=z_0(u)+\frac{\alpha}{v}z_1(u)+\cdots ,$$

with $z_0(u)=\tanh(u/2)$. Substituting this back into the action (16), we obtain, to lowest order in α,

$$\sigma[z_{\rm cl}]=\sigma_0+\frac{\alpha}{v}\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}du'\frac{1+2\lambda|u-u'|}{(u-u')^2(1+\lambda|u-u'|^2)} \times[z_0(u)-z_0(u')]^2 . \tag{18}$$

Note that z_1 does not contribute to $O(\alpha)$, since σ_0 is an extremum. The double integral in (18) may be evaluated asymptotically as $\lambda \to 0$. Putting the dimensions back in, we find

$$S'_{\rm eff}[q_{\rm cl}]/\hbar=S_0/\hbar-\alpha\ln(\omega_c/\omega_0)+f(\alpha) ,$$

with

$$f(\alpha)=[\tfrac{3}{2}+\gamma-\ln(2\pi)]\alpha+O(\alpha^2)=0.2392\alpha+O(\alpha^2) , \tag{19}$$

where $\gamma=0.5772\ldots$ is Euler's constant. We have not obtained any explicit results for the prefactor in (14), but for $\alpha \to 0$ we expect

$$A(\alpha,v)/A(\alpha=0,v)=1+O(\alpha/v) .$$

Since we are working in the WKB limit, $v=M\omega_0^2 q_0^2/\hbar\omega_0 \gg 1$, the contribution to Δ due to the prefactor may be ignored when compared to the exponential factor $\exp[-f(\alpha)]$. Combining (19) and (14) and (5), we obtain the exact result for the quartic double-well potential (in the WKB limit) to leading order in $\alpha \to 0$:

$$\Delta(\omega_c/\omega_0,\alpha,v)=(\omega_c/\omega_0)^{\alpha}e^{-0.24\alpha}\Delta_0(v) .$$

In the $\alpha \to \infty$ limit we may use a variational calculation to determine $f(\alpha,v)$. In the overdamped regime the relevant single-well frequency is $\omega_b=M\omega_0^2/\eta$, so that our small parameter is $\omega_c/\omega_b=\omega_c\eta/M\omega_0^2$; i.e., $\lambda\alpha/v \to 0$. For simplicity we chose as our variational function $z_v(u)=\tanh(u/2a)$ where a is a variational parameter which controls the width of the instanton. By substituting $z_v(u)$ into (16) and minimizing $\sigma[z_v]$ with respect to a, we find

$$S_{\rm eff}[q_{\rm cl}]/\hbar=-\alpha\ln(\omega_c/\omega_0)-\alpha\ln\alpha+\alpha\ln v -c\alpha+O(\ln(\alpha/v)) , \tag{20}$$

so that

$$f(\alpha,v)\sim -\alpha\ln\alpha+\alpha\ln v-c\alpha .$$

The constant c is given by $c=\ln(24\pi)-\frac{5}{2}-\gamma=1.24\ldots$. This result is an upper bound on the actual action. Note that the $\alpha\ln\alpha$ and $\alpha\ln v$ terms in $f(\alpha,v)$ are independent of the actual shape of the double-well potential and the form of the cutoff. These terms arise because the instanton width in the $\alpha \to \infty$ limit is proportional to $\omega_b^{-1}\sim(\alpha/v)\omega_0^{-1}$ rather than ω_0^{-1}, and it is this width which provides the high-frequency cutoff as described in Sec. II. In analogy with the cubic potential,[1] we expect

that in the large-α limit the prefactor in (14) will behave as

$$A(\alpha,v)/A(\alpha=0,v)\sim(\alpha/v)^{\nu},$$

where ν is a constant greater than zero. In the $\alpha\rightarrow\infty$ limit this is small compared to the $O(\alpha)$ term in the exponent.

In conclusion, we have presented a systematic framework for reducing a double-well problem to an equivalent two-level problem, for the important case of Ohmic dissipation. We have also pointed out the range of parameters for which such a truncation approximation is possible. Explicit results have been obtained for the quartic double well in the underdamped and overdamped regimes. While we have developed this technique with particular applications to superconducting quantum interference devices in mind, it may also prove useful in studies of tunneling phenomena in solids.

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