Chiral Metal as a Ferromagnetic Super Spin Chain

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Abstract

The electrons on the surface of a disordered multi-layer integer quantum Hall system constitute an unusual chiral metal with ballistic motion transverse to the field, and diffusive motion parallel to it. We present a non-perturbative analytic treatment of an appropriate model, consisting of disordered chiral Fermions in two dimensions. A supersymmetric generating functional is set up for the correlation functions of this system. The strong disorder limit is mapped into a supersymmetric spin chain, with ferromagnetic exchange coupling, reflecting the electron's chiral motion. The ferromagnetic ground state and the spin wave excitations, corresponding to the diffusion modes of the chiral metal, are found exactly. The parametric density of states correlator in the ergodic limit is computed from a Boltzmann-weighted sum over low energy spin states. The result is of a universal form and coincides with that for a Hermitian random matrix.

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1 Introduction

Disorder has a profound effect on low dimensional electron transport, generically leading to localization of all states in both one and two dimensions (2d)[1]. In the quantum Hall regime with a strong magnetic field, however, the behavior is richer[2]. Precisely at the transition between successive integer Hall plateaus, the 2d electron states are *not* localized, but quasi-extended with multi-fractal scaling characteristics[3]. Within a Hall plateau, the 1d edge states are extended, due to their purely chiral nature[4, 5]. Indeed, for a single edge mode, impurities simply lead to an unimportant forward scattering phase shift. However, when multiple edge modes are present, such as at the boundaries of hierarchical fractional quantum Hall effect (FQHE) states[6], interchannel impurity scattering can be important[7]. For example, backscattering between the two counter propagating modes of a $\nu = 2/3$ state is predicted to drive a transition into a phase with charge and neutral sectors decoupled[8]. Multiple edge modes are also present in multi-layer systems exhibiting a 3d bulk QHE in perpendicular applied field. Together, these edge modes comprise a conducting two-dimensional subsystem, which has been the focus of recent attention[9, 10]. Such systems can be realized by fabricating multi-layer GaAs heterostructures[11], but occur naturally in the Bechgaard salts, a class of quasi 1d compounds which exhibit a cascade of field-induced spin-density-wave transitions between bulk QHE phases[12].

The chiral 2d metal "living" on the surface of such bulk QHE systems can be probed via transport experiments along the field direction (z-axis). For the integer quantum Hall effect (IQHE), the edge state in each layer is a free (chiral) Fermion, so that uniform inter-layer tunnelling leads to coherent z-axis motion. The low energy surface excitations then comprise a 2d Fermi surface, which is equivalent to "half" of a conventional 2d open Fermi surface. A standard diagrammatic approach[10] shows that impurity scattering leads to diffusive motion

along the z-axis, with ballistic motion perpendicular to the field. Since backscattering is *not* possible in the ballistic direction, conventional localization effects are not expected. Indeed, a perturbative expansion about the diffusive metal shows an absence of any localization[10]. Moreover, random-walk arguments and numerics on an appropriate network model[9] suggest that this conclusion is valid generally.

In this paper we describe a non-perturbative analytic treatment of the IQHE surface sheath with impurity scattering. We show that an appropriate generating functional for the 2d electron correlations can be mapped onto a 1d quantum spin chain. The replica trick is avoided by employing Fermionic and Bosonic partners (Bosonic "ghosts") from the outset to perform the desired quenched average. The resulting spin chain is thus supersymmetric, involving both Bosonic and Fermionic spins. Since the edge modes all move in the same direction, the super spin chain has a ferromagnetic exchange interaction[13]. We show that the ground state of this super spin chain consists of all spins "aligned", analogous to the ground state of an ordinary ferromagnet. Moreover, there is a class of one-magnon excitations which are exact eigenstates of the super spin chain, leading to a sharp one-magnon pole in the spin-spin correlation function. This pole corresponds to the diffuson pole, describing the diffusive z-axis motion of electrons in the chiral 2d system. The exact pole confirms the complete absence of all localization corrections for this 2d chiral metallic system.

The use of supersymmetry has a long history in the theory of disordered electron transport. Efetov[14] reformulated earlier replica field theories[15] to obtain a supersymmetric non-linear sigma model (NLσM) description of disordered electron motion. This approach enabled the computation of several new properties, most notably electron localization in 1d and random matrix spectral correlations in 0d. To describe plateau transitions in the IQHE, a topological term is needed in the σ-model, as originally argued by Pruisken et. al.[16] and incorporated into the supersymmetric approach by Weidenmüller[17]. More recently, one of us[18] has shown that the 2d supersymmetric σ-model with topological term can be mapped into a 1d super spin chain Hamiltonian, with antiferromagnetic exchange interaction. This spin chain was, in turn, shown to be equivalent to the Chalker-Coddington network model[19, 20]. The antiferromagnetic exchange corresponds to counter propagating edge modes of the network model, in contrast to the ferromagnetic exchange for the chiral 2d metal studied here. Unfortunately, the AFM super spin chain model is *much* more complicated than the ferromagnetic model analysed below, and has so far eluded analysis.

In addition to obtaining the exact ground state and one-magnon excitations, we employ the ferromagnetic super spin chain to extract spectral correlations of the chiral metal. A finite length spin chain at non-zero temperatures corresponds to a 2d chiral metal with both dimensions finite. As with other zero-dimensional random electron systems, we expect interesting universal spectral correlations[21]. Working directly with the 1d spin Hamiltonian, we derive expressions for the two-point spectral correlation function and the parametric correlations. Not surprisingly, the results coincide with the universal random matrix theory results for the unitary ensemble, obtained previously from other methods[22, 23]. Our approach is notable for its (relative) simplicity, involving a simple sum over a large (super-)spin multiplet of quantum states.

This paper is organized as follows. After introducing the model in Section II, we construct the supersymmetric generating functional in Section III, and show that in the limit of strong disorder it can be mapped into a super spin chain, with ferromagnetic exchange. In Section

IV we consider the electron's motion in the thermodynamic limit. From the exact ground state and one-magnon spectrum of the spin chain, the diffuson (or density-density correlator) can be computed exactly, revealing diffusive motion parallel to the field, and ballistic motion perpendicular to it. In Section V we consider mesoscopic effects for a sample with finite dimensions. Due to the system's chirality and anisotropy there are three mesoscopic regimes. In the "zero-dimensional" ergodic regime, we compute the two-point spectral correlation function and the parametric correlations, by summing over a zero-energy multiplet of super spin states. Section VI is devoted to a summary and brief conclusions.

2 Model

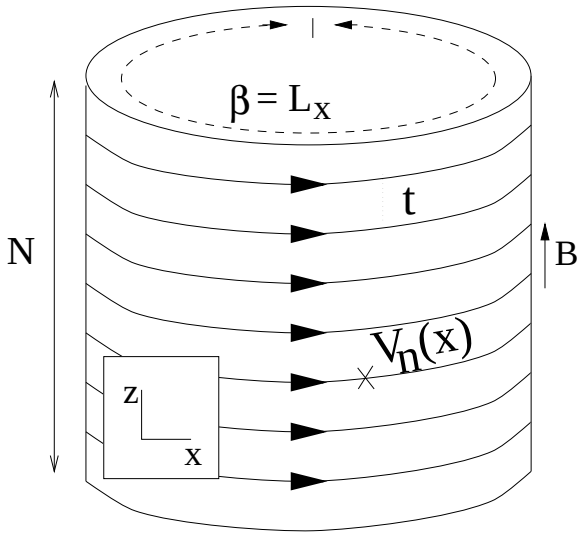

Figure 1: Geometry of a 3d quantum Hall sample. z–axis transport is included via the tunneling amplitude t, and impurity scattering by the random potential V. The circumference is denoted $\beta = L_x$, and the height in units of the layer spacing is N.

The system of interest (see Fig. 1) is a collection of N edge states, each described by a chiral Fermion, which are coupled together by an inter-edge hopping strength, t:

$$\mathcal{H}_0 = \sum_{n=1}^{N} \int_0^\beta dx [\psi_n^\dagger i\partial_x \psi_n - t(\psi_n^\dagger \psi_{n+1} + \text{h.c.})], \tag{1}$$

where $\beta = L_x$ is the circumference of the surface sheath (and the sample). Our choice of units is such that the edge velocity $v = 1$. The full Hamiltonian including random scattering is $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, where

$$\mathcal{H}_1 = \sum_n \int dx V_n(x) \psi_n^\dagger \psi_n. \tag{2}$$

For simplicity the random potential is assumed to have a Gaussian distribution with zero mean and

$$[V_n(x) V_{n'}(x')]_{\text{ens}} = 2u\delta_{nn'}\delta(x - x'), \tag{3}$$

where the square brackets denote an ensemble average over disorder realizations.

3 Generating functional

As usual, a generating functional can be introduced from which the electron Green's functions can be extracted[15]. Of interest are products of retarded and advanced Green's functions, denoted $G_\pm$, where

$$\begin{aligned} & G_\pm(n,n';x,x';\pm\omega/2) \\ = \; & \langle nx| \left[\mathcal{H} \pm i(\eta + i\omega/2)\right]^{-1} |n'x'\rangle. \end{aligned} \tag{4}$$

Here η is a positive infinitesimal, and $\omega/2$ is the electron energy. The appropriate generating functional can be written as an integral over a spinor of Grassmann fields, $\psi_{n\alpha}$, where $\alpha = \pm$ for retarded (advanced). To make the analogy with spin, we will sometimes use instead the notation $\alpha = \uparrow, \downarrow$. To avoid the use of the replica method, we adopt the supersymmetric approach, and consider an additional integration over a spinor of complex fields, $\phi_{n\alpha}$. The supersymmetric generating functional, $\mathcal{Z}$, introduced below, is then normalized, with $\mathcal{Z} = 1$ for every realization of the disorder potential. An average over an ensemble of disorder realizations can then be safely performed.

To insure convergence of the functional integrals over ϕ, the contribution to $\mathcal{Z}$ in the retarded sector is defined by exponentiating $i(\mathcal{H} - \omega/2 + i\eta)$, whereas in the advanced sector, $-i(\mathcal{H} + \omega/2 - i\eta)$. The appropriate supersymmetric generating functional then takes the form

$$\mathcal{Z} = \int D\psi D\overline{\psi} D\phi D\phi^* \exp(-S), \tag{5}$$

$$\text{with} \quad S = \int dx \sum_n \mathcal{L}_n. \tag{6}$$

The Lagrangian is expressed as a sum of Boson and Fermion contributions:

$$\mathcal{L}_n = L(\phi^*, \phi) + L(\overline{\psi}, \psi), \tag{7}$$

with

$$\begin{aligned} L(\phi^*,\phi) &= \phi_n^* \sigma_z (\partial_x - iV_n)\phi_n \\ &+ it(\phi_n^* \sigma_z \phi_{n+1} + \text{c.c.}) + \tilde{\eta}\phi_n^*\phi_n, \end{aligned} \tag{8}$$

where $\tilde{\eta} = \eta + i\omega/2$. In the future, we will drop the tilde, remembering to let $\eta \to \eta + i\omega/2$ when necessary in computing quantities at non-zero frequency. By construction, $\mathcal{Z} = 1$, independent of the random potential.

The single particle Green's functions can be expressed as averages of either the complex or Grassmann fields. Specifically,

$$G_\alpha(n,n';x,x') = -\alpha i < \phi_{n\alpha}(x)\phi^*_{n'\alpha}(x') > \tag{9}$$

$$= -\alpha i < \psi_{n\alpha}(x)\overline{\psi}_{n'\alpha}(x') >, \tag{10}$$

where $\alpha = \pm$ and the brackets denote an average taken with weight $\exp(-S)$. The density of states (DOS) follows from the imaginary part of the diagonal element,

$$\rho = \sum_{n=1}^{N} \frac{\beta}{2\pi i}(G_- - G_+)(n,n;x,x). \tag{11}$$

Note that an integral over x is unnecessary, since the DOS is independent of x. This follows from the chiral conservation law,

$$\partial_x |\Phi|^2 = \nabla_n \cdot J, \tag{12}$$

where the current is $J(x,n) = 2t\mathrm{Im}[\Phi^*(x,n)\Phi(x,n+1)]$. Eq. (12) is valid for every eigenfunction Φ of the Hamiltonian. To resolve the ambiguity which arises because the DOS involves G at equal points, $x = x'$, we take the symmetrized form,

$$4\pi\rho = \beta \sum_{n,\alpha} \sum_{\pm} < \phi_{n\alpha}(x \pm \epsilon)\phi^*_{n\alpha}(x) > \tag{13}$$

$$= \beta \sum_{n,\alpha} \sum_{\pm} < \psi_{n\alpha}(x \pm \epsilon)\overline{\psi}_{n\alpha}(x) >, \tag{14}$$

with ϵ a positive infinitesimal.

3.1 Coherent State

Following D.H. Lee[20], we reinterpret the spatial coordinate x as an imaginary time coordinate τ, with S the Euclidean action for a 1d quantum system. The Hamiltonian for this 1d quantum system, denoted H, acts as a transfer matrix in the x direction. In order to extract H, it is convenient to recast the action S into the canonical form for a coherent state path integral of a 1d system of Bosons and Fermions. Up to a sign in the advanced sector, the linear x-derivative term in (8) is already in the appropriate form. To "correct" this sign, we transform the advanced complex fields as

$$\phi_{n\downarrow} \leftrightarrow \phi^*_{n\downarrow}, \tag{15}$$

leaving the retarded fields, $\phi_{n\uparrow}$, unchanged. The first term in the Lagrangian then takes the canonical form: $\phi^*_n \partial_\tau \phi_n$. In the Fermion sector we similarly transform the Grassmann fields:

$$\psi_{n\downarrow} \rightarrow -\overline{\psi}_{n\downarrow}, \tag{16}$$

$$\overline{\psi}_{n\downarrow} \rightarrow \psi_{n\downarrow}, \tag{17}$$

again leaving the retarded Grassmann fields unchanged. The transformed Lagrangian can be written $\mathcal{L}_n = L_0 + L_t + L_V + L_\eta$, with

$$L_0 = \phi^*_n \partial_\tau \phi_n + \overline{\psi}_n \partial_\tau \psi_n \tag{18}$$

in canonical form. The term

$$L_\eta = \eta(\phi^*_n \phi_n + \overline{\psi}_n \psi_n), \tag{19}$$

and the random potential,

$$L_V = -iV_n(\tau)(\phi^*_n \sigma_z \phi_n + \overline{\psi}_n \sigma_z \psi_n), \tag{20}$$

are unchanged, whereas the tunnelling terms become

$$L_t = it(A_{n+1,n} + A_{n,n+1}) \tag{21}$$

with

$$A_{m,n} = \phi^*_{m\uparrow}\phi_{n\uparrow} - \phi^*_{n\downarrow}\phi_{m\downarrow} + (\phi \to \psi). \tag{22}$$

3.2 1d Hubbard model

We can now perform an average over the ensemble of random potentials. This gives

$$L_u = u\left(\phi^*_n\sigma_z\phi_n + \overline{\psi}_n\sigma_z\psi_n\right)^2. \tag{23}$$

The full action S is now equivalent to the Euclidian action for a 1d quantum system, and the 1d quantum Hamiltonian (transfer matrix) can be readily extracted. One simply replaces the Grassmann fields ψ_n by Fermion operators, f_n, and the complex fields, ϕ_n, by Bose operators, b_n. The resulting 1d Hamiltonian takes the form: $H = H_t + H_u + H_\eta$ with

$$H_\eta = \eta\sum_n (b^\dagger_n b_n + f^\dagger_n f_n), \tag{24}$$

$$H_u = u\sum_n \left(b^\dagger_n\sigma_z b_n + f^\dagger_n\sigma_z f_n\right)^2, \tag{25}$$

$$H_t = it\sum_n (A_{n+1,n} + A_{n,n+1}), \tag{26}$$

where

$$A_{n+1,n} = b^\dagger_{n+1\uparrow}b_{n\uparrow} - b^\dagger_{n\downarrow}b_{n+1\downarrow} + (b \to f), \tag{27}$$

and the Fermion and Boson operators satisfy canonical commutation relations

$$[b_{n\alpha}, b^\dagger_{n'\beta}] = \delta_{nn'}\delta_{\alpha\beta}, \tag{28}$$

$$[f_{n\alpha}, f^\dagger_{n'\beta}] = \delta_{nn'}\delta_{\alpha\beta}. \tag{29}$$

Here, and in the remainder of the paper, $[\mathcal{O}_1, \mathcal{O}_2]$ denotes the graded or super- commutator, defined as

$$[\mathcal{O}_1, \mathcal{O}_2] \equiv \mathcal{O}_1\mathcal{O}_2 - (-1)^{|\mathcal{O}_1||\mathcal{O}_2|}\mathcal{O}_2\mathcal{O}_1, \tag{30}$$

where $|\mathcal{O}| = 0$ if $\mathcal{O}$ is a Bosonic operator, while $|\mathcal{O}| = 1$ if it is a Fermionic one. This 1d Hamiltonian describes spinful Bosons and Fermions hopping on a 1d lattice, with hopping strength t, interacting via a quartic term with strength u. Large interaction u corresponds to strong disorder. For large disorder u, this term constrains the on-site Bose and Fermion densities to satisfy

$$b^\dagger_\uparrow b_\uparrow + f^\dagger_\uparrow f_\uparrow = b^\dagger_\downarrow b_\downarrow + f^\dagger_\downarrow f_\downarrow, \tag{31}$$

on each site n. There is a large energy cost (u) for configurations involving different numbers of particles in the retarded and advanced sectors, whereas retarded and advanced particles can move together. This reflects the phase cancellation between the retarded and advanced Green's functions. The positive infinitesimal restricts the total number of particles.

The similarity between the full Hamiltonian and more familiar interacting 1d models such as the Hubbard model[24] can be revealed by introducing a new set of operators. To this end, we define

$$B_\uparrow = b_\uparrow, \quad B_\downarrow = b_\downarrow^\dagger, \tag{32}$$

and

$$F_\uparrow = f_\uparrow, \quad F_\downarrow = f_\downarrow^\dagger. \tag{33}$$

The F operators are bona fide Fermion operators, satisfying

$$[F_{n\alpha}, F_{n'\beta}^\dagger] = \delta_{nn'}\delta_{\alpha\beta}, \tag{34}$$

but the operator $B_\downarrow$ does *not* satisfy the canonical Boson commutator, but rather, $[B_\downarrow, B_\downarrow^\dagger] = -1$. To restore the canonical form we define

$$\overline{B}_\downarrow = -B_\downarrow^\dagger, \tag{35}$$

and also $\overline{B}_\uparrow = B_\uparrow^\dagger$, which now satisfy

$$[B_{n\alpha}, \overline{B}_{n'\beta}] = \delta_{nn'}\delta_{\alpha\beta}. \tag{36}$$

As defined, B_n and $\overline{B}_n$ satisfy canonical Bose commutation relations, but it must be kept in mind that $\overline{B}_n$ is *not* the adjoint of B_n.

In terms of these new operators, the 1d Hamiltonian takes a form which closely resembles the Hubbard model, with spin-independent hopping and interaction:

$$\begin{aligned} H_t + H_u &= it\sum_n (A_{n+1,n} + A_{n,n+1}) \\ &\quad + u\sum_n \left(\overline{B}_n B_n + F_n^\dagger F_n\right)^2, \end{aligned} \tag{37}$$

where

$$A_{n+1,n} = \overline{B}_{n+1} B_n + F_{n+1}^\dagger F_n. \tag{38}$$

The positive infinitesimal term now acts like a small magnetic field, coupling to the z-component of the "spin":

$$H_\eta = \eta(\overline{B}_n \sigma_z B_n + F_n^\dagger \sigma_z F_n). \tag{39}$$

As with the Hubbard model, one anticipates that the interaction term will lead to a "charge gap", strongly suppressing fluctuations in $\overline{B}_n B_n + F_n^\dagger F_n$. Of interest are the remaining gapless "spin" excitations. For the conventional Fermionic Hubbard model, these spin excitations can be revealed by employing a transformation, to map the Hubbard model into a Heisenberg spin chain[24]. A similar transformation is desirable in this 1d supersymmetric model. To this end, we will return to the path integral representation, and perform a gauge transformation before averaging over disorder. This will allow us to obtain directly a (supersymmetric) spin chain model.

3.3 Gauge transformation

Consider the gauge transformation

$$\phi_n \to \exp(i\sigma_z \int_{-\infty}^{\tau} d\tau' V_n(\tau'))\phi_n, \tag{40}$$

and an identical transformation for ψ_n. Due to the linear derivatives in L_0, this eliminates the on-site random potential term in L_V, at the expense of introducing randomness into the hopping term. The full Lagrangian becomes $\mathcal{L}_n = L_0 + L_t + L_\eta$, with L_0 and L_η as given in Eq. (18) and Eq. (19), and

$$L_t = i(t_n A_{n+1,n} + t_n^* A_{n,n+1}). \tag{41}$$

The tunnelling amplitudes, $t_n(\tau)$, are both random and complex, given by

$$t_n(\tau) = t \exp\left(i \int_{-\infty}^{\tau} d\tau' [V_n(\tau') - V_{n+1}(\tau')]\right). \tag{42}$$

They satisfy

$$[t_n(\tau) t_{n'}^*(\tau')]_{\text{ens}} = \delta_{nn'} t^2 e^{-2u|\tau-\tau'|}. \tag{43}$$

In the following we will focus on the limit of strong disorder (large u), and approximate the above exponential with a delta function:

$$[t_n(\tau) t_{n'}^*(\tau')]_{\text{ens}} \to D\delta_{nn'}\delta(\tau-\tau'), \tag{44}$$

where $D = t^2/u$. The quantity D will play the role of the exchange interaction in the supersymmetric spin chain. The above large u limit is then analogous to passing from a Hubbard model to a spin chain.

To extract the 1d (spin chain) Hamiltonian, we again perform an ensemble average over disorder. Since the above delta function is really a short-range symmetric function of its argument, we will arrive at a symmetrized form. On passing again to Bose and Fermi operators, the full 1d Hamiltonian is given by $H = H_D + H_\eta$ where H_η was defined in Eq. (39), and

$$H_D = \frac{D}{2} \sum_n (A_{n+1,n} A_{n,n+1} + A_{n,n+1} A_{n+1,n}), \tag{45}$$

with $A_{n+1,n}$ as before, in Eq. (38).

Before recasting the Hamiltonian H_D into the form of a spin chain, it is instructive to re-express the DOS in terms of the Fermion and Boson operators. Recalling Eq. (13) we obtain for the density of states, $\rho = \sum_n \rho_n$ with

$$4\pi\rho_n/\beta = \langle b_{n\uparrow}^\dagger b_{n\uparrow} + b_{n\uparrow} b_{n\uparrow}^\dagger + b_{n\downarrow}^\dagger b_{n\downarrow} + b_{n\downarrow} b_{n\downarrow}^\dagger \rangle, \tag{46}$$

where $\langle ... \rangle$ is a "thermodynamic" expectation value with Hamiltonian H, see Sec. 3.5 for the details. A similar expression exists in terms of the Fermion operators $b \to f$. Equivalently we have

$$2\pi\rho_n/\beta = 1 + \langle b_n^\dagger b_n \rangle = 1 - \langle f_n^\dagger f_n \rangle, \tag{47}$$

which involves the on-site Boson and Fermion densities. These expressions can also be written as

$$2\pi\rho_n/\beta = \langle \overline{B}_n \sigma_z B_n \rangle = -\langle F_n^\dagger \sigma_z F_n \rangle. \tag{48}$$

As we shall now show, these correlators can be naturally interpreted as the z-component of an appropriately defined spin operator. In terms of these spin operators, the Hamiltonian H_D will take the form of a ferromagnetic (super-)spin chain.

3.4 (Super-)spin chain

To cast H_D in spin chain form, it is useful to introduce generalized spin currents. This can be done in a supersymmetric manner, by defining a four-component superfield,

$$\Psi_n = (F_n, B_n), \quad \overline{\Psi}_n = (F_n^\dagger, \overline{B}_n). \tag{49}$$

When necessary, we will use latin indices $(a, b, \cdots)$ to denote the Fermion/Boson label, i.e. $a = F, B \leftrightarrow 1, 2$. The Ψ obey the mixed statistics relation

$$\Psi_{ma\alpha}\overline{\Psi}_{nb\beta} = M_{ab}\overline{\Psi}_{nb\beta}\Psi_{ma\alpha} + \delta_{mn}\delta_{ab}\delta_{\alpha\beta}, \tag{50}$$

where $M_{ab} = \sigma^x_{ab} - \sigma^z_{ab}$. We may then define a four-vector super spin matrix via

$$\mathcal{J}^\mu_{ab} = \overline{\Psi}_a\gamma^\mu\Psi_b = \overline{\Psi}_{a\alpha}\gamma^\mu_{\alpha\beta}\Psi_{b\beta}, \tag{51}$$

where $\gamma = (\mathbf{1}, \boldsymbol{\sigma})/2$. In matrix form, this is

$$\mathcal{J}^\mu = \begin{pmatrix} F^\dagger\gamma^\mu F & F^\dagger\gamma^\mu B \\ \bar{B}\gamma^\mu F & \bar{B}\gamma^\mu B \end{pmatrix}. \tag{52}$$

The global current operators

$$\mathcal{J}^\mu_{\rm TOT} = \sum_n \mathcal{J}^\mu_n \tag{53}$$

generate symmetries of H_D, as can be easily seen by checking $[\mathcal{J}^\mu_{\rm TOT}, A_{n+1,n}] = 0$ using Eq. (50). Thus

$$[\mathcal{J}^\mu_{\rm TOT}, H_D] = 0. \tag{54}$$

It is now straightforward to obtain the spin chain representation. Noting that the hopping operator $A_{n+1,n} = \overline{\Psi}_{n+1}\Psi_n$, we see that substitution in Eq. (45) yields

$$H_D = D\sum_n \left(\overline{\Psi}_n\Psi_n - \overline{\Psi}_{na\alpha}\Psi_{nb\beta}\sigma^z_{bc}\overline{\Psi}_{n+1c\beta}\Psi_{n+1a\alpha}\right).$$

The identities $\overline{\Psi}_{na\alpha}\Psi_{nb\beta} = 2\mathcal{J}^\mu_{ab}\gamma^\mu_{\beta\alpha}$ and $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \delta^{\mu\nu}/2$ then give

$$H_D = 2D\sum_n \mathrm{Tr}\left[\mathcal{J}^0_n - \mathcal{J}^\mu_n\sigma_z\mathcal{J}^\mu_{n+1}\right]. \tag{55}$$

The minus sign on the spin-spin interaction indicates that this is a *ferromagnetic* (super-)spin chain.[1] The η term indeed acts as an infinitesimal field

$$H_\eta = 2\eta\sum_n \mathrm{Tr}\mathcal{J}^z_n. \tag{56}$$

The term $\sum_n \mathrm{Tr}\mathcal{J}^0_n = \mathcal{J}^0_{\rm TOT}$ vanishes on the Bose-Fermi vacuum, and commutes with H_D and H_η. It is therefore zero on all low energy states coupled by the Hamiltonian $H_D + H_\eta$, and will be dropped from all future expressions.

[1] Note that the presence of σ^z is equivalent in the second term to the conventional "supertrace" used in supersymmetric notation.

In order to perform calculations, it is useful to write out the interactions in a more explicit (but not manifestly supersymmetric) form. Dividing the terms into three sectors, $H_D = H_F + H_B + H_X$, one finds

$$H_F = -\frac{D}{2}\sum_n \left(F_n^\dagger F_n F_{n+1}^\dagger F_{n+1} + 4\boldsymbol{S}_n\cdot\boldsymbol{S}_{n+1}\right), \tag{57}$$

$$H_B = \frac{D}{2}\sum_n \left(\overline{B}_n B_n \overline{B}_{n+1} B_{n+1} + 4\boldsymbol{S}_n^B\cdot\boldsymbol{S}_{n+1}^B\right), \tag{58}$$

$$H_X = \frac{D}{2}\sum_n \Big(F_n^\dagger B_n \overline{B}_{n+1} F_{n+1} + 4\boldsymbol{S}_n^X\cdot\tilde{\boldsymbol{S}}_{n+1}^X - \overline{B}_n F_n F_{n+1}^\dagger B_{n+1} - 4\tilde{\boldsymbol{S}}_n^X\cdot\boldsymbol{S}_{n+1}^X\Big), \tag{59}$$

where

$$\boldsymbol{S}_n = F_n^\dagger\frac{\boldsymbol{\sigma}}{2}F_n, \qquad \boldsymbol{S}_n^B = \overline{B}_n\frac{\boldsymbol{\sigma}}{2}B_n, \tag{60}$$

$$\boldsymbol{S}_n^X = F_n^\dagger\frac{\boldsymbol{\sigma}}{2}B_n, \qquad \tilde{\boldsymbol{S}}_n^X = \overline{B}_n\frac{\boldsymbol{\sigma}}{2}F_n. \tag{61}$$

The Fermionic Hamiltonian, H_F, is precisely that of an ordinary spin-1/2 ferromagnetic spin chain[24]. In particular, the Fermionic currents obey the SU(2) algebra,

$$[S_m^\alpha, S_n^\beta] = i\epsilon^{\alpha\beta\gamma}S_n^\gamma\delta_{mn}. \tag{62}$$

Since H_F is invariant under global spin rotations, we expect that the total spin operator

$$\boldsymbol{S}_{\rm TOT} \equiv \sum_n \boldsymbol{S}_n \tag{63}$$

commutes with H_F. In fact, Eq. (54) implies

$$[\boldsymbol{S}_{\rm TOT}, H_F] = 0 = [\boldsymbol{S}_{\rm TOT}, H_D], \tag{64}$$

i.e. the full H_D is SU(2) invariant. H_η of course breaks the symmetry explicitly.

Although the Bosonic sector described by H_B appears also to be SU(2) symmetric, it is not so in the usual sense. Because of the unusual definition of conjugation ($\overline{B} \neq B^\dagger$), the currents $\boldsymbol{S}_n^B$ are not Hermitian. This non-hermiticity means that, even though there exists a set of currents forming an SU(2) Lie algebra and commuting with H_D, these cannot be used to generate *unitary* transformations $\exp i\boldsymbol{n}\cdot\boldsymbol{S}^B$. A Hermitian set can be defined by

$$J_n^x = iS_n^{By} = (b_{n\uparrow}^\dagger b_{n\downarrow}^\dagger + b_{n\uparrow}b_{n\downarrow})/2, \tag{65}$$

$$J_n^y = -iS_n^{Bx} = (b_{n\uparrow}^\dagger b_{n\downarrow}^\dagger - b_{n\uparrow}b_{n\downarrow})/2i. \tag{66}$$

$$J_n^z = S_n^{Bz} = (b_{n\uparrow}^\dagger b_{n\uparrow} + b_{n\downarrow}b_{n\downarrow}^\dagger)/2. \tag{67}$$

These obey instead the SU(1,1) algebra[25]

$$[J_n^x, J_n^y] = -iJ_n^z, \tag{68}$$

$$[J_n^y, J_n^z] = iJ_n^x, \tag{69}$$

$$[J_n^z, J_n^x] = iJ_n^y, \tag{70}$$

where the minus sign in the first relation distinguishes SU(1,1) from SU(2). In terms of these operators,

$$H_B = 2D\sum_n \Big(\tfrac{1}{4}(b_n^\dagger \sigma_z b_n - 1)(b_{n+1}^\dagger \sigma_z b_{n+1} - 1) + J_n^z J_{n+1}^z - J_n^x J_{n+1}^x - J_n^y J_{n+1}^y\Big). \quad (71)$$

The Lie algebra of SU(1,1) is a well-studied semisimple Lie algebra[25]. Quite generally, such Lie algebras have a so-called quadratic Casimir operator Λ, which commutes with all elements of the algebra, and is the analog of $\boldsymbol{S}\cdot\boldsymbol{S}$ for SU(2). For SU(1,1),

$$\Lambda = (J^z)^2 - (J^x)^2 - (J^y)^2. \quad (72)$$

It is straightforward to verify that $[\Lambda, \boldsymbol{J}] = 0$, and that therefore the global generator

$$\boldsymbol{J}_{\rm TOT} = \sum_n \boldsymbol{J}_n \quad (73)$$

commutes with H_B. As before, in fact,

$$[\boldsymbol{J}_{\rm TOT}, H_D] = 0, \quad (74)$$

as can be verified by explicit calculation.

3.5 Boundary Conditions

In the above development, we have implicitly assumed that the coordinate x, which runs along the chiral modes, is infinite. Since x has been replaced by imaginary time τ, this corresponds to zero "temperature", $\beta = \infty$, for the 1d (super-)spin chain. Consideration of mesoscopic effects requires a finite system, with both N and β finite. Since the chiral modes cannot end, when β is finite it is necessary to consider periodic boundary conditions in the x (or τ) direction, as depicted schematically in Fig. 1. The boundary conditions in the direction (n) transverse to the chiral currents, can be taken either open, corresponding to a cylinder, or periodic, corresponding to a torus.

When the system is periodic in the x-direction, care is needed in defining the boundary conditions on the complex and Grassmann fields. Specifically, the generating functional, $\mathcal{Z}$, will only equal one if the determinant from the Grassmann integration precisely cancels the inverse determinant from the integration over the complex fields. This requires that the boundary conditions on ψ and ϕ be identical: $\phi(0) = \phi(\beta)$ and $\psi(0) = \psi(\beta)$. However, in the standard imaginary time Grassmann path integral representation for Fermions, the Grassmann fields satisfy anti-periodic boundary conditions, $\psi(\beta) = -\psi(0)$. This discrepancy prevents us from making the transition from the path integral to the quantum Hamiltonian by simply replacing Grassmann fields by Fermionic operators. To remedy this, we perform a global gauge transformation,

$$\psi_n(x) \to e^{i\pi x/\beta}\psi_n(x), \quad (75)$$

in the Lagrangian Eq. (8), which transforms the boundary conditions for the Grassmann fields from periodic to anti-periodic, and puts the Grassmann path integral in the standard form we

desire. Under this transformation the Lagrangian picks up an additional term of the form

$$L_{bc} = \frac{i\pi}{\beta}\bar{\psi}_n \sigma_z \psi_n. \tag{76}$$

Since the transformation Eqs.(15-17) leaves this form unchanged, this corresponds to an additional term in the 1d Hamiltonian:

$$H_{bc} = \frac{i\pi}{\beta}\sum_n f_n^\dagger \sigma_z f_n. \tag{77}$$

Equivalently, what we need to do when computing thermodynamic averages with the Boltzmann weight $\exp(-\beta H)$, is to take instead of the usual trace the *supertrace*:

$$\langle O\rangle = \mathrm{STr} e^{-\beta H} O = \mathrm{Tr}(-1)^{N_F} O e^{-\beta H}, \tag{78}$$

counting the contributions from states containing an odd number of Fermions, N_F, with a minus sign. This latter procedure is the one adopted in the following. Thus we do *not* modify the Hamiltonian, but will remember to insert a minus sign for all states with odd Fermion number.

4 Diffuson

In this section, we calculate the chiral diffuson in the thermodynamic ($\beta, N \to \infty$) limit. It is defined by

$$\begin{aligned} K_n(x;\eta) &= \left[|G_+(n+n',n';x+x',x';0)|^2\right]_{\mathrm{ens}} \\ &= \left[G_+(n,0;x,0;0)G_-(0,n;0,x;0)\right]_{\mathrm{ens}}. \end{aligned} \tag{79}$$

It may be calculated in several ways, depending upon which fields (ϕ or ψ) are used to generate the Green's functions $G_\pm$. In particular,

$$\begin{aligned} K_n(\tau) &= \langle \mathcal{T}_\tau S_n^-(\tau) S_0^+(0)\rangle & (80)\\ &= \langle \mathcal{T}_\tau J_n^-(\tau) J_0^+(0)\rangle & (81)\\ &= -\langle \mathcal{T}_\tau A_n^-(\tau) \tilde{A}_0^+(0)\rangle & (82)\\ &= \langle \mathcal{T}_\tau \tilde{A}_n^-(\tau) A_0^+(0)\rangle, & (83) \end{aligned}$$

where

$$\begin{aligned} S_n^\pm &= S_n^x \pm i S_n^y, \qquad A_n^\pm = S_n^{Xx} \pm i S_n^{Xy}, & (84)\\ J_n^\pm &= J_n^x \pm i J_n^y, \qquad \tilde{A}_n^\pm = \tilde{S}_n^{Xx} \pm i \tilde{S}_n^{Xy}. & (85) \end{aligned}$$

These expressions have a simple interpretation when expressed in terms of the b and f operators,

$$\begin{aligned} S_n^+ &= f_{n\uparrow}^\dagger f_{n\downarrow}^\dagger, \qquad A_n^+ = f_{n\uparrow}^\dagger b_{n\downarrow}^\dagger, & (86)\\ J_n^+ &= b_{n\uparrow}^\dagger b_{n\downarrow}^\dagger, \qquad \tilde{A}_n^+ = b_{n\uparrow}^\dagger f_{n\downarrow}^\dagger. & (87) \end{aligned}$$

The operators create pairs (for the Fermions, these are singlets) of particles at a particular site. Eqs. (80-83) indicate that the diffuson is obtained by creating such a pair at the origin and propagating it to site n in time τ. The motion of a pair describes the long-range coherent propagation, in the chiral disordered metal, of one retarded and one advanced particle.

As discussed in Sec. 3.5, the angular brackets in Eqs. (80-83) indicate an average calculated by a trace with the weight $\exp(-\beta H)$. In the thermodynamic limit, $\beta \to \infty$, with η non-zero, only the zero energy (ground) states survive. Since the Hamiltonian is a sum of positive semi-definite operators, each of these operators must annihilate any prospective ground state. Satisfying this for the two terms in H_η in Eq. (24) therefore implies that the only state which contributes to the average is the $b-f$ vacuum $|0\rangle$, defined by

$$b_n|0\rangle = f_n|0\rangle = 0. \tag{88}$$

In the Fermionic sector, this is nothing other than the fully polarized spin-down ferromagnetic state, so that

$$S^z_{\rm TOT}|0\rangle = -\frac{N}{2}|0\rangle. \tag{89}$$

It follows that (as can be easily verified)

$$S^-_{\rm TOT}|0\rangle = 0, \tag{90}$$

i.e. $|0\rangle$ is the "lowest weight" state of this SU(2) (sub-) representation. Similarly, one has

$$J^-_{\rm TOT}|0\rangle = 0, \tag{91}$$

$$J^z_{\rm TOT}|0\rangle = \frac{N}{2}|0\rangle, \tag{92}$$

so this is also the lowest weight state of the SU(1,1) (sub-)representation, and it is the SU(1,1) analog of the fully polarized state. As in the case of the usual ferromagnet, this is the *exact* ground state.

Now consider the diffuson. From Eq. (80), we have

$$K_n(\tau) = \langle 0|S^-_n e^{-\tau H} S^+_0|0\rangle \tag{93}$$

$$= \int_0^{2\pi} \frac{dkdk'}{(2\pi)^2} \langle 0|S^-_k e^{-\tau H} S^+_{k'}|0\rangle e^{ikn}, \tag{94}$$

where the "single magnon" operator

$$S^\pm_k \equiv \sum_n S^\pm_n e^{\pm ikn} \tag{95}$$

simply creates a superposition of local spin-flips with wavevector k. To evaluate Eq. (94), we should study the state

$$|k\rangle = S^+_k|0\rangle. \tag{96}$$

Because no b Bosons have been added to the system, this state is still annihilated by H_B and H_X. The problem then effectively reduces to a purely Fermionic one, i.e. the usual spin-1/2

SU(2) spin chain. As in that case, *the state $|k\rangle$ is an exact eigenstate*! By direct computation, one finds

$$H|k\rangle = E_k|k\rangle, \tag{97}$$

where the dispersion relation is

$$E_k = 2D(1-\cos k) + 2\eta \approx 2\eta + Dk^2, \tag{98}$$

for $k \ll 1$. This is just the usual ferromagnetic spin-wave dispersion. It is thus straightforward to evaluate Eq. (94), to obtain

$$K_n(\tau \to x) = \theta(x) \int_0^{2\pi} \frac{dk}{2\pi} e^{-E_k x + ikn}. \tag{99}$$

Performing the Fourier transform on x gives the final (exact) result

$$\begin{aligned} K(k_x, k_z) &= \frac{1}{2\eta + ik_x + 2D(1-\cos k_z)} \\ &\approx \frac{1}{2\eta + ik_x + Dk_z^2}. \end{aligned} \tag{100}$$

We thus see that the ferromagnetic dispersion $E_k \approx Dk^2 + 2\eta$ corresponds to the expected form of the anisotropic diffuson for the chiral metal[10].

It is interesting to note that this result must also be obtained from the other formulations, Eqs. (81-83). In fact, one finds that there are three other sets of exact single super spin flip excitations, created by Fourier transforms of the operators J_n^+, A_n^X, and $\tilde{A}_n^+$ in Eqs. (86-87). The reader may amuse him or herself by carrying through this calculation explicitly.

5 Mesoscopic Regime

The above results describe the propagation of density fluctuations in the thermodynamic limit. We now consider the level statistics of a mesoscopic sample. A natural probe of these statistics is the density of states correlation function[23],

$$C(\omega; \beta, N) = [\rho(E+\omega/2)\rho(E-\omega/2)]_{\rm ens} - \overline{\rho}^2. \tag{101}$$

Of interest is the structure of this correlation function when the frequency ω is on the scale of the mean level spacing,

$$\Delta = \frac{2\pi}{N\beta}, \tag{102}$$

where βv is the system size in the x-direction, and v is the edge mode velocity. For isotropic metallic samples, $C(\omega)$ is a universal function of the ratio ω/Δ, described by random matrix theory[22, 23]. In the present case, the sample is both chiral and anisotropic, so care is needed to specify the random matrix theory regime. To this end it is useful to consider several important length scales.

5.1 Length scales

For finite β, the electron takes a finite time, $\tau_B \sim \beta$, to ballistically traverse the sample in the x-direction. Equating this with the time taken to diffuse across the system in the transverse direction, $\tau_D \sim N^2/D$, gives a (transverse) crossover length

$$L_0 = \sqrt{D\beta}, \tag{103}$$

measured in units of the z-axis lattice constant. This length scale has a simple interpretation in terms of the spin chain. For a chain with N sites, the energy of the lowest lying spin wave excitation, which has wavevector $k_{min} = 2\pi/N$, is $E_{min} \sim D(k_{min}^2) \sim \tau_D^{-1}$. This energy equals the spin chain "temperature", $E_{min} \sim T = 1/\beta$, for a spin chain of length L_0. The boundary $N = L_0$ thus demarkates the two following regimes: For a short spin chain with $N < L_0$, finite wavevector spin-wave excitations will not be appreciably excited, and the system will be effectively "zero-dimensional". The opposite limit, $N > L_0$, which we refer to as a "one-dimensional diffusive" regime, corresponds to the "classical limit" of the spin chain. The transverse electron motion is diffusive.

But there is another important length scale[9]. A 1d spin chain at finite temperature will have a finite spin-correlation length,

$$\xi \sim D\beta, \tag{104}$$

with D the spin-exchange constant. This length corresponds to the electron localization length along the z-axis. In the $\beta \to \infty$ limit of interest, one has $\xi \gg L_0$. The above "1d diffusive regime" requires $\xi > N > L_0$. When $N > \xi$ the system enters into a third regime - a 1d localized regime.

Of the three regimes, the "zero-dimensional" is the simplest and the one we focus on. This regime corresponds to random matrix theory, as verified below. Since we consider frequencies on the scale of Δ, the Heisenberg time $\omega^{-1} \sim \Delta^{-1} \sim N\beta$. In this regime, the ergodic condition that ω^{-1} exceeds both $\tau_B \sim \beta$ and τ_D requires simply $N \gg 1$. In the "1d diffusive regime", corrections to random matrix theory are anticipated, with wavefunctions possibly exhibiting multi-fractal scaling behavior, as in other low-dimensional diffusive systems[3, 26].

To evaluate the spectral correlator $C(\omega)$ at finite β requires performing a full thermodynamic trace over states of the supersymmetric spin chain – a complicated task. Fortunately, in the "zero-dimensional" regime, all finite wavevector spin-wave excitations are exponentially suppressed (by $e^{-\beta E_{min}} = e^{-(2\pi L_0/N)^2}$), and can be ignored. The trace can thus be restricted to a subset of states which are annihilated by the Hamiltonian H_D. This restricted trace is the operator analog of the zero-mode integration carried out on the conventional NLσM[14].

5.2 Zero energy multiplet

Consider then the *zero energy multiplet* of states that are annihilated by H_D, i.e. the set of ground states in the absence of the symmetry-breaking field. This set has a large degeneracy, because any particular vacuum (e.g. $|0\rangle$) may be "rotated" using the symmetry operators $\mathcal{J}^\mu_{\rm TOT}$ to obtain another zero energy state. These operators and the resulting multiplet form a representation of the U(2|1, 1) (super)symmetry group of H_D, which contains SU(2) and SU(1,1) sub-representations generated using the operators $\boldsymbol{S}_{\rm TOT}$ and $\boldsymbol{J}_{\rm TOT}$.

To describe the $E = 0$ multiplet, it is convenient to use the b Boson and f Fermion operators, for which $|0\rangle$ is the vacuum. Of the 16 elements of $\mathcal{J}$, 4 may be chosen diagonal. A natural choice is $\mathcal{J}^0_{11}$, $\mathcal{J}^3_{11}$, $\mathcal{J}^0_{22}$, $\mathcal{J}^3_{22}$, or, equivalently, the set of number operators,

$$n_{f\alpha} = \sum_n f^\dagger_{n\alpha} f_{n\alpha}, \qquad n_{b\alpha} = \sum_n b^\dagger_{n\alpha} b_{n\alpha}. \tag{105}$$

All elements of $\mathcal{J}$ commute with the total $b - f$ spin operator, which is

$$\mathrm{Tr}\mathcal{J}^0_{\mathrm{TOT}} = \tfrac{1}{2}\left(n_{f\uparrow} - n_{f\downarrow} + n_{b\uparrow} - n_{b\downarrow}\right) = 0, \tag{106}$$

where the final equality follows from evaluation in the $|0\rangle$ state. More fundamentally, this results from the Hubbard constraint Eq. (31).

The remaining 12 elements of $\mathcal{J}$ act as raising and lowering operators. From Eq. (54), we may act with any string of these operators upon $|0\rangle$ to produce another zero energy state. The 12 operators, however, are strongly redundant, and in fact only 4 suffice to generate the full $E = 0$ multiplet.

The first two of these are simply the raising operators in the SU(2) and SU(1,1) subalgebras, which create delocalized "spinless" pairs of Fermions and Bosons, respectively, i.e.

$$S^+_{\mathrm{TOT}} = \sum_n f^\dagger_{n\uparrow} f^\dagger_{n\downarrow}, \tag{107}$$

$$J^+_{\mathrm{TOT}} = \sum_n b^\dagger_{n\uparrow} b^\dagger_{n\downarrow}. \tag{108}$$

These are just the $k = 0$ magnon operators of the previous section. In addition, we must consider the Fermionic generators,

$$A^+_{\mathrm{TOT}} = \sum_n f^\dagger_{n\uparrow} b^\dagger_{n\downarrow}, \tag{109}$$

$$\tilde{A}^+_{\mathrm{TOT}} = \sum_n b^\dagger_{n\uparrow} f^\dagger_{n\downarrow}. \tag{110}$$

These create Fermionic $k = 0$ pairs. To simplify the notation we now drop the bulky subscript TOT.

To see that S^+, J^+, A^+, and $\tilde{A}^+$, are sufficient to generate the full $E = 0$ multiplet, note first that the remaining 8 off-diagonal generators all annihilate the vacuum (because they contain either b or f). Therefore, in any string of generators, they may be commuted or anticommuted to the right until they annihilate the vacuum, and it is straightforward to see that in the process all that may be left is a string of the four operators above. This proves the completeness of this set.

Furthermore, in any string of S^+, J^+, A^+, and $\tilde{A}^+$, the order is unimportant up to the overall sign. This is because the commutators

$$\begin{aligned} [S^+, J^+] &= [S^+, A^+] = [S^+, \tilde{A}^+] \\ = [J^+, A^+] &= [J^+, \tilde{A}^+] = [A^+, \tilde{A}^+] = 0 \end{aligned} \tag{111}$$

all vanish. A complete basis for the $E = 0$ multiplet is thus formed by the states

$$|n_F n_B n_A n_{\tilde{A}}\rangle = \left(S^+\right)^{n_F} \left(J^+\right)^{n_B} \left(A^+\right)^{n_A} \left(\tilde{A}^+\right)^{n_{\tilde{A}}} |0\rangle.$$

These states are non-vanishing only if n_A and $n_{\tilde{A}}$ are zero or one, as follows from the nilpotency of A^+ and $\tilde{A}^+$, e.g.

$$\begin{aligned}\left(A^+\right)^2 &= \sum_{m,n} f^\dagger_{m\uparrow} b^\dagger_{m\downarrow} f^\dagger_{n\uparrow} b^\dagger_{n\downarrow} \\ &= -\sum_{m,n} f^\dagger_{n\uparrow} b^\dagger_{n\downarrow} f^\dagger_{m\uparrow} b^\dagger_{m\downarrow} = -\left(A^+\right)^2 = 0. \end{aligned} \tag{112}$$

Likewise, $(\tilde{A}^+)^2 = 0$. It is thus natural to break up the states into four ladders,

$$\left(S^+\right)^{n_F} \left(J^+\right)^{n_B} |0\rangle, \tag{113}$$

$$\left(S^+\right)^{n_F} \left(J^+\right)^{n_B} A^+|0\rangle, \tag{114}$$

$$\left(S^+\right)^{n_F} \left(J^+\right)^{n_B} \tilde{A}^+|0\rangle, \tag{115}$$

$$\left(S^+\right)^{n_F-1} \left(J^+\right)^{n_B-1} A^+\tilde{A}^+|0\rangle. \tag{116}$$

The allowed values of n_F and n_B in each ladder are essentially determined by the exclusion principle. In the first sector, Eq. (113), one may add up to $2N$ Fermions to fill all N sites with both spin species, so that $n_F = 0, \ldots, N$. Since any number of Bosons may live at a site, the sum on $n_B = 0, \ldots, \infty$ is unbounded above. This result agrees with what we know from group theory: these states are simply direct product states of SU(2) and SU(1,1) representations. By computing

$$S^z|0\rangle = \frac{1}{2}\sum_n \left(f^\dagger_{n\uparrow} f_{n\uparrow} - f_{n\downarrow} f^\dagger_{n\downarrow}\right)|0\rangle = -\frac{N}{2}|0\rangle, \tag{117}$$

and recalling $S^-|0\rangle = 0$, we recognize that $|0\rangle$ is the lowest weight state of a spin $s = N/2$ SU(2) ladder, which has the well-known degeneracy $2s + 1 = N + 1$. This agrees with the assignment $S^z(n_F) = -N/2 + n_F$. Likewise, $J^-|0\rangle = 0$, $J^z|0\rangle = \frac{N}{2}|0\rangle$, and $J^z(n_B) = N/2 + n_B$. Thus $|0\rangle$ can also be regarded as the lowest weight state of a representation of SU(1,1), with spin $s = N/2$. This representation is irreducible, unitary and infinite-dimensional, where the third property is necessitated by the second one, since SU(1,1) is non-compact.

Application of the Pauli principle determines the allowed values of n_F in the other cases. In the second and third sectors, Eqs. (114-115), one Fermion is already present, so at most $2N - 1$ may be added. Since they come pairwise, we must restrict $n_F = 0, \ldots, N - 1$ in these cases. Finally, in the fourth sector, Eq. (116), $2N - 2$ Fermions may be added, suggesting that we need only $n_F \leq N$. However, if $n_F = N$, there are N Fermionic pairs present, and the Fermionic content of the state is uniquely determined. That is, the full state must be a direct product of a Bosonic part with the fully occupied Fermionic state. This implies that

$$\left(S^+\right)^N \left(J^+\right)^{n_B} |0\rangle \propto \left(S^+\right)^{N-1} \left(J^+\right)^{n_B-1} A^+\tilde{A}^+|0\rangle.$$

We therefore choose $n_F = 1, \ldots, N - 1$ in the last sector.

5.3 Supersymmetric quadruplets

To calculate traces over the $E = 0$ multiplet, it is beneficial to organize the ladders of states in a slightly different way. To that end, consider the Fermionic operators

$$Q_\alpha = \sum_n b^\dagger_{n\alpha} f_{n\alpha}, \tag{118}$$

$$\overline{Q}_\alpha = \sum_n f^\dagger_{n\alpha} b_{n\alpha}, \tag{119}$$

which are particular combinations of the global currents $\mathcal{J}^\mu_{\rm TOT}$, and belong to the set of twelve raising and lowering operators of which we have utilized only S^+, J^+, A^+, and $\tilde{A}^+$ so far. The set $\{Q_\alpha, \overline{Q}_\alpha, n_{f\alpha}, n_{b\alpha}\}$ where $\alpha = \uparrow$ or $\alpha = \downarrow$, closes under the graded commutator (i.e. the anticommutator for two Fermions, and the commutator in the other cases), and thus forms a subalgebra of the full algebra of global currents. This subalgebra is denoted $\mathcal{H} = \mathrm{u}(1|1) \oplus \mathrm{u}(1|1)$. (The direct sum is over spin.) The significance of $\mathcal{H}$ derives from the fact that its generators commute not only with H_D but also with H_η. Thus $H_D + H_\eta$ is a Casimir invariant for $\mathcal{H}$. Schur's lemma then states that $H_D + H_\eta$ is a multiple of unity on every irreducible multiplet of $\mathcal{H}$. Let us therefore construct these multiplets.

We interpret $Q_\uparrow$, $Q_\downarrow$ as lowering operators, and $\overline{Q}_\uparrow$, $\overline{Q}_\downarrow$ as raising operators for $\mathcal{H}$. A lowest weight state, $|\mathrm{LW}\rangle$, for $\mathcal{H}$ obeys

$$Q_\uparrow |\mathrm{LW}\rangle = Q_\downarrow |\mathrm{LW}\rangle = 0. \tag{120}$$

To solve these equations for $|\mathrm{LW}\rangle$, we take note of the commutation relations

$$[Q_\uparrow, S^+] = \tilde{A}^+, \qquad [\overline{Q}_\uparrow, \tilde{A}^+] = S^+, \tag{121}$$

$$[\overline{Q}_\downarrow, J^+] = \tilde{A}^+, \qquad [Q_\downarrow, \tilde{A}^+] = J^+, \tag{122}$$

$$[\overline{Q}_\uparrow, J^+] = A^+, \qquad [Q_\uparrow, A^+] = J^+, \tag{123}$$

$$[Q_\downarrow, S^+] = -A^+, \qquad [\overline{Q}_\downarrow, A^+] = -S^+. \tag{124}$$

All other graded commutators between the two sets $\{S^+, J^+, A^+, \tilde{A}^+\}$ and $\{Q_\uparrow, Q_\downarrow, \overline{Q}_\uparrow, \overline{Q}_\downarrow\}$ vanish. Using the above commutation relations and making the ansatz

$$\begin{aligned} |\mathrm{LW}\rangle &= \alpha \left(S^+\right)^{n_1} \left(J^+\right)^{n_2} |0\rangle \\ &+ \beta \left(S^+\right)^{n_1-1} \left(J^+\right)^{n_2-1} A^+ \tilde{A}^+ |0\rangle, \end{aligned} \tag{125}$$

one finds that the lowest weight condition can be satisfied with $\alpha = 1$ and $\beta = -n_1$. We denote the resulting lowest weight state by $|n_1 n_2, 00\rangle$.

Application of the raising operator $\overline{Q}_\uparrow$ to $|n_1 n_2, 00\rangle$ produces

$$\begin{aligned} |n_1 n_2, 10\rangle &= \left(S^+\right)^{n_1} \left(J^+\right)^{n_2-1} A^+ |0\rangle \\ &\propto \overline{Q}_\uparrow |n_1 n_2, 00\rangle, \end{aligned} \tag{126}$$

while application of $\overline{Q}_\downarrow$ gives

$$\begin{aligned} |n_1 n_2, 01\rangle &= \left(S^+\right)^{n_1} \left(J^+\right)^{n_2-1} \tilde{A}^+ |0\rangle \\ &\propto \overline{Q}_\downarrow |n_1 n_2, 00\rangle. \end{aligned} \tag{127}$$

When acting with the raising operator $\overline{Q}_\uparrow$ on $|n_1 n_2, 10\rangle$, or with $\overline{Q}_\downarrow$ on $|n_1 n_2, 01\rangle$, we get zero from $\overline{Q}_\uparrow^2 = 0 = \overline{Q}_\downarrow^2$ due to the Fermionic nature of these operators. A non-vanishing state results from applying $\overline{Q}_\downarrow$ to $|n_1 n_2, 10\rangle$:

$$\begin{aligned} |n_1 n_2, 11\rangle &= \left(S^+\right)^{n_1+1} \left(J^+\right)^{n_2-1} |0\rangle \\ &\quad +(n_2-1)\left(S^+\right)^{n_1} \left(J^+\right)^{n_2-2} A^+ \tilde{A}^+ |0\rangle \\ &\quad \propto \overline{Q}_\downarrow |n_1 n_2, 10\rangle. \end{aligned} \tag{128}$$

Exactly the same state is obtained by applying $\overline{Q}_\uparrow$ to $|n_1 n_2, 01\rangle$. The state Eq. (128) is annihilated by the action of both $\overline{Q}_\uparrow$ and $\overline{Q}_\downarrow$, which again follows from $\overline{Q}_\uparrow^2 = 0 = \overline{Q}_\downarrow^2$. Thus $|n_1 n_2, 11\rangle$ is a highest weight state for $\mathcal{H}$.

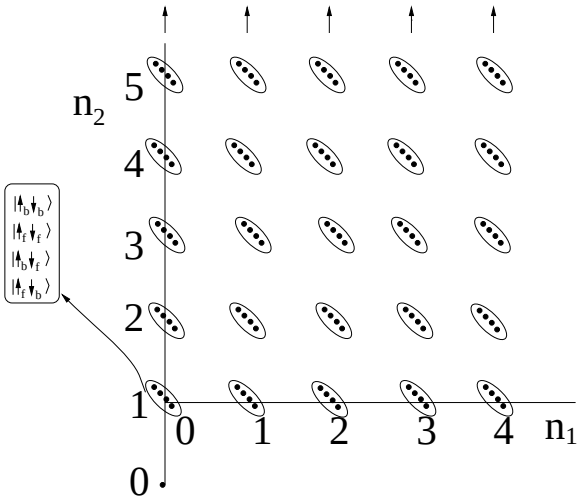

Figure 2: Illustration of the organization of states into $\mathcal{H}$-quadruplets, shown here for the case $N = 5$. Each quadruplet contains a state with n_1 Fermion pairs and n_2 Boson pairs, plus three partners with equal "spin" but differing Boson/Fermion content. The $n_1 = 0$, $n_2 = 1$ quadruplet is shown as an example. The vacuum state, with $n_1 = n_2 = 0$ is a unique $\mathcal{H}$-singlet.

The states $|n_1 n_2, \mu\nu\rangle$ $(\mu, \nu = 0, 1)$ so constructed are seen to be in one-to-one correspondence with the previous set Eqs. (113-116). What we have thus achieved is an organization of the zero energy sector into *irreducible $\mathcal{H}$-quadruplets* (see Fig. 2), with the exception of only the vacuum $|0\rangle$, which does not fit into the above scheme but figures as a separate $\mathcal{H}$-singlet. Each quadruplet consists of two Bosonic ($\mu\nu = 00, 11$) and two Fermionic ($\mu\nu = 01, 10$) states, reflecting the supersymmetry of the formalism. Recalling the earlier discussion based on the exclusion principle, we see that the allowed quantum numbers for the quadruplets are $n_1 = 0, 1, \ldots, N-1$ and $n_2 = 1, \ldots, \infty$. The pair of quantum numbers $n_1 = n_2 = 0$ corresponds to the $\mathcal{H}$-singlet vacuum state $|0\rangle$.

By construction, the quadruplet states $|n_1 n_2, \mu\nu\rangle$ taken together with the vacuum, constitute an orthogonal basis of the zero energy sector. As it stands, they are not normalized to unity. While it is not difficult to include the correct normalization factors, there is no need to do that

here. For our purposes, all we require is that we be able to calculate traces, and for that linear independence of the states is entirely sufficient.

5.4 DOS Correlation Function

We are finally in a position to calculate correlation functions. As a check, consider first the partition function within the zero energy multiplet. Since the full partition function equals unity for all values of the parameter d and approaches the zero mode contribution as $d \to \infty$, the zero mode contribution must itself be exactly normalized to unity. And, indeed,

$$\mathcal{Z}_0 = \mathrm{Tr}_0 (-1)^{N_F} e^{-\beta H} = 1, \tag{129}$$

where the subscript 0 signifies the restriction to the zero energy sector. The 1 on the right-hand side stems from the vacuum. All other contributions cancel as H is constant on each $\mathcal{H}$-quadruplet, and the two Fermionic states of a quadruplet come with a minus sign relative to the two Bosonic ones, cf. Sec. 3.5. Since H_η, by Eq. (24), just counts the total number of b Bosons and f Fermions, its action on the quadruplet states is

$$H_\eta |n_1 n_2, \mu\nu\rangle = 2\eta(n_1 + n_2)|n_1 n_2, \mu\nu\rangle. \tag{130}$$

Now consider the DOS correlator, Eq. (101). Defining

$$G_{\pm n}(\omega) = G_\pm(n, n; x, x; \omega) \tag{131}$$

and recalling Eq. (11), we get

$$\rho(\omega/2) = \frac{\beta}{2\pi i} \sum_n [G_{-n}(\omega/2) - G_{+n}(\omega/2)]. \tag{132}$$

The product therefore becomes

$$\begin{aligned} &4\pi^2 \rho(\omega/2)\rho(-\omega/2)/\beta^2 = \\ &\sum_{nn'} \Big[G_{+n}(\omega/2) G_{-n'}(-\omega/2) + G_{-n}(\omega/2) G_{+n'}(-\omega/2) \\ &- G_{+n}(\omega/2) G_{+n'}(-\omega/2) - G_{-n}(\omega/2) G_{-n'}(-\omega/2) \Big]. \end{aligned} \tag{133}$$

To proceed, we perform an ensemble average, which may be split into connected (cumulant) and disconnected pieces. To calculate the disconnected terms, we use the symmetric operator representation

$$G_{+n} = -\frac{i}{2}\left(f_{n\uparrow} f^\dagger_{n\uparrow} - f^\dagger_{n\uparrow} f_{n\uparrow} \right) = i f^\dagger_{n\uparrow} f_{n\uparrow} - i/2, \tag{134}$$

$$G_{-n} = -\frac{i}{2}\left(f^\dagger_{n\downarrow} f_{n\downarrow} - f_{n\downarrow} f^\dagger_{n\downarrow} \right) = -i f^\dagger_{n\downarrow} f_{n\downarrow} + i/2. \tag{135}$$

Therefore:

$$\begin{aligned} \overline{G}_{+n}(\omega/2)\overline{G}_{-n'}(-\omega/2) &+ \overline{G}_{-n}(\omega/2)\overline{G}_{+n'}(-\omega/2) \\ -\overline{G}_{+n}(\omega/2)\overline{G}_{+n'}(-\omega/2) &- \overline{G}_{-n}(\omega/2)\overline{G}_{-n'}(-\omega/2) = 1, \end{aligned} \tag{136}$$

where $\overline{G}_{\pm n} \equiv [G_{\pm n}]_{\rm ens} = \mp i/2$. These disconnected terms thus cancel the $-\overline{\rho}^2$ term in the definition of $C(\omega)$, Eq. (101). Then

$$\begin{aligned} C(\omega) &= \frac{\beta^2}{4\pi^2}\sum_{n,n'=1}^{N}\Big[G_{+n}(\omega/2)G_{-n'}(-\omega/2)+G_{-n}(\omega/2)G_{+n'}(-\omega/2) \\ &\quad -G_{+n}(\omega/2)G_{+n'}(-\omega/2)-G_{-n}(\omega/2)G_{-n'}(-\omega/2)\Big]_{{\rm ens},c}, \end{aligned} \tag{137}$$

where the subscript c indicates the connected average.

It is well known that the last two terms do not contribute to the correlator.[2] Using this simplification, we arrive at the form

$$C(\omega) = \frac{\beta^2}{2\pi^2}{\rm Re}\sum_{n,n'=1}^{N}\left[G_{+n}(\omega/2)G_{-n'}(-\omega/2)\right]_{{\rm ens},c}. \tag{138}$$

As with the diffuson, this may be calculated in several ways, equivalent by supersymmetry. We arbitrarily choose the Fermion representation

$$\begin{aligned} C(\omega) &= \frac{\beta^2}{2\pi^2}{\rm Re}\sum_{nn'}\langle f^\dagger_{n\uparrow}f_{n\uparrow}f^\dagger_{n'\downarrow}f_{n'\downarrow}\rangle \\ &= \frac{\beta^2}{2\pi^2}{\rm Re}\langle n_{f\uparrow}n_{f\downarrow}\rangle. \end{aligned} \tag{139}$$

The quadruplet states $|n_1 n_2, \mu\nu\rangle$ are eigenstates of $n_{f\uparrow}n_{f\downarrow}$ with eigenvalues n_1^2, $n_1(n_1+1)$, $(n_1+1)n_1$, and $(n_1+1)^2$ for $\mu\nu = 00$, 01, 10, and 11 respectively. Therefore the trace of $n_{f\uparrow}n_{f\downarrow}$ over the quadruplet with quantum numbers n_1, n_2 is

$$n_1^2 - 2n_1(n_1+1) + (n_1+1)^2 = 1.$$

Using this to evaluate the expectation value Eq. (139) as a trace against the Boltzmann weight, we obtain

$$C(\omega) = \frac{\beta^2}{2\pi^2}{\rm Re}\sum_{n_2=1}^{\infty}\sum_{n_1=0}^{N-1} e^{-\beta(2\eta+i\omega)(n_2+n_1)}. \tag{140}$$

While this sum may, of course, be evaluated exactly, it is simpler to take the continuum limit, valid for $\beta\omega \ll 1$. Measuring energy in units of the level spacing $\Delta = 2\pi/N\beta$, we define $\tilde{\omega} = N\beta\omega/2\pi$, and rescaled integration variables $x = n_2/N$, $y = n_1/N$. Equation Eq. (140) then becomes

$$C(\omega) = \frac{N^2\beta^2}{2\pi^2}{\rm Re}\int_0^\infty dx\int_0^1 dy\; e^{-2\pi i\tilde{\omega}(x+y)}. \tag{141}$$

After division by $\bar{\rho}^2 = \Delta^{-2}$, and omission of a $\delta(\tilde{\omega})$ term, this result takes the universal form

$$C(\omega)/\bar{\rho}^2 = -\frac{\sin^2(\pi\tilde{\omega})}{(\pi\tilde{\omega})^2}, \tag{142}$$

which is well-known from random matrix theory[22]. Similar manipulations lead to identical results when $C(\omega)$ is evaluated using the purely Bosonic or mixed expressions analogous to Eq. (139).

[2] If they are calculated directly using a modified supersymmetric functional, the resulting Hamiltonian has a unique ground state even for $\eta = 0$, indeed leading to this conclusion.

5.5 Extension to parametric correlations

The above calculation can be extended to the case where the random potential $V_n(x)$ depends on a parameter, λ:

$$\mathcal{H}_1(\lambda) = \sum_n \int dx\ V_n(x,\lambda)\psi_n^\dagger \psi_n, \tag{143}$$

$$V_n(x,\lambda) = V_n(x) - \lambda W_n(x). \tag{144}$$

The extra term $\lambda W_n(x)$ will be called the "background potential". For simplicity we take it to be independent of x: $W_n(x) = h_n$. (It can be shown that the final result (158) remains essentially unchanged when this assumption is lifted.) We assume that there are no correlations between the background potential λh_n and the random potential $V_n(x)$. For the moment the other statistical properties of the numbers h_n are left unspecified.

The object of interest is the parametric density of states correlation function

$$C(\omega,\lambda) = [\rho(\omega,\lambda)\rho(0,0)]_{\rm ens} - \bar{\rho}^2. \tag{145}$$

By the same steps as in the previous section, its calculation reduces to that of

$$\sum_{n,n'} [G_{+n}(\omega/2,\lambda/2)G_{-n'}(-\omega/2,-\lambda/2)]_{{\rm ens},c}\,.$$

The calculation bears much similarity to the one we did for $\lambda = 0$, and we will now focus on the new features arising for $\lambda \neq 0$.

The background potential gives rise to an additional piece in the Lagrangian,

$$\mathcal{L}_n \to \mathcal{L}_n + \frac{i}{2}\lambda h_n \left(\phi_n^* \sigma_z \phi_n + \bar{\psi}_n \sigma_z \psi_n\right), \tag{146}$$

and in the spin Hamiltonian,

$$H \to H + \lambda H_{\rm M}, \quad H_{\rm M} = i\sum_n h_n (S_n^z + J_n^z), \tag{147}$$

which acts like the coupling to an imaginary magnetic field. Note that adding $\lambda H_{\rm M}$ is the same as substituting $\omega \to \omega + \lambda h_n$. To prevent the background potential from causing an overall shift of the frequency, we shall require that $\sum_n h_n$ vanishes identically for every realization of the disorder.

What is the effect of $H_{\rm M}$ inside the zero energy multiplet? By the requirement $\sum_n h_n = 0$, the zero mode approximation $S_n^z + J_n^z \to N^{-1}\sum_n (S_n^z + J_n^z)$ for $H_{\rm M}$ gives a vanishing result to linear order in λ. To see the effect of the background potential, we must go to higher order and do a calculation similar in spirit to that performed by Simons and Altshuler[23] for the case of time-reversal invariant disordered metallic grains. What the perturbation $H_{\rm M}$ does is to couple the zero energy multiplet with the one-magnon sector. By second-order perturbation theory, this coupling results in an effective Hamiltonian, H_λ, acting on the states of the zero energy multiplet:

$$H_\lambda = H_{\rm M}\left(-\sum_{k\neq 0}\frac{\Pi_k}{Dk^2}\right)H_{\rm M}, \tag{148}$$

where Π_k is the projector onto the one-magnon sector with momentum k. In the appendix it is shown that for N large, H_λ reduces to

$$H_\lambda = \gamma \mathcal{C}^{+-}, \quad \gamma = \frac{\lambda^2}{N} \sum_{k \neq 0} \frac{|\tilde{h}_k|^2}{Dk^2}, \tag{149}$$

where

$$\mathcal{C}^{+-} = S^+ S^- + J^+ J^- + \tilde{A}^+ A^- - A^+ \tilde{A}^-, \tag{150}$$

and

$$\tilde{h}_k = \frac{1}{N} \sum_n e^{ikn} h_n \tag{151}$$

is the Fourier transform of the background potential. In order for second-order perturbation theory to apply, the change in energy due to the perturbation must be much smaller than the smallest energy denominator. Because the gap for single magnon excitations is of the order of D/N^2, and H_λ will be seen below to shift the energy by an amount of the order of γN^2, the condition

$$\gamma \ll D/N^4 \tag{152}$$

on γ is sufficient for the perturbative formula (148) to be valid. To give this result some added physical meaning, consider the simple case in which the numbers $h_n = \hat{h}_n - \bar{h}$, where the $\hat{h}_n$ are chosen to be independent, identically distributed random variables with zero mean and unit variance, and $\bar{h} = N^{-1} \sum_n \hat{h}_n$ is chosen to maintain zero spatial average. The ensemble average of the quantity $|\tilde{h}_k|^2$ is $1/N$ (for $k \neq 0$), and

$$[\gamma]_{\text{ens}} = \frac{\lambda^2}{N^2} \sum_{k \neq 0} \frac{1}{Dk^2} = \frac{\lambda^2}{12D} \tag{153}$$

in the large N limit. More generally, we are led to define a dimensionless, order one, γ_0 by

$$\gamma =: \frac{\lambda^2}{D} \gamma_0. \tag{154}$$

In the particular example here, comparison of Eqs. (153-154) gives $[\gamma_0]_{\text{ens.}} = 1/12$. The condition on the smallness of γ then becomes

$$\lambda \ll D/N^2, \tag{155}$$

i.e. the typical strength of fluctuation of the background potential must be much smaller than the one-magnon gap.

We are now ready to compute the parametric density of states correlator, $C(\omega, \lambda)$. From Eqs. (121-124) and similar commutation relations involving the lowering operators S^-, J^-, A^-, and $\tilde{A}^-$, one finds that $\mathcal{C}^{+-}$ commutes with the Fermionic charges Q_α and $\overline{Q}_\alpha$, which means that $\mathcal{C}^{+-}$ is a Casimir invariant for $\mathcal{H} = \mathrm{u}(1|1) \oplus \mathrm{u}(1|1)$. Therefore $\mathcal{C}^{+-}$ must be proportional to unity on the quadruplets:

$$\mathcal{C}^{+-} |n_1 n_2, \mu\nu\rangle = \Gamma(n_1, n_2) |n_1 n_2, \mu\nu\rangle. \tag{156}$$

To calculate the eigenvalue $\Gamma(n_1, n_2)$, we may work out the action of $\mathcal{C}^{+-}$ on any one of the four states $(\mu, \nu = 0, 1)$, say $|n_1 n_2, 01\rangle$. A straightforward calculation gives

$$\begin{aligned} \Gamma(n_1, n_2) &= n_2(N + n_2 - 1) \\ &+ n_1(N - n_1 - 1). \end{aligned} \tag{157}$$

By including the contribution from $H_\lambda = \gamma \mathcal{C}^{+-}$ into the Boltzmann weight, we obtain for the parametric density of states correlator,

$$\begin{aligned} C(\omega, \lambda) &= \frac{\beta^2}{2\pi^2} \mathrm{Re} \sum_{n_2=1}^{\infty} \sum_{n_1=0}^{N-1} e^{-i\beta\omega(n_2+n_1)} \\ &\times e^{-\beta\lambda^2 D^{-1}\gamma_0 \Gamma(n_1,n_2)}. \end{aligned} \tag{158}$$

For $\beta N \lambda^2 D^{-1} \gamma_0 \ll 1$ or, equivalently, $\lambda^2 D^{-1} \gamma_0 \ll \Delta$, we may again take the continuum limit and replace the sum by an integral. By performing a simple but revealing substitution of integration variables, we get

$$\frac{C(\omega, \lambda)}{\bar{\rho}^2} = \int_{|p|\geq\pi} \int_{|q|\leq\pi} e^{i\tilde{\omega}(p-q) - \tilde{\lambda}^2(p^2-q^2)} \frac{dpdq}{(2\pi)^2}, \tag{159}$$

with the rescaled parameter $\tilde{\lambda}$ given by

$$\tilde{\lambda}^2 = \beta \frac{N^2}{D} \lambda^2 \gamma_0 / (2\pi)^2. \tag{160}$$

Upon making the identification of $\tilde{\omega}$ with position and $\tilde{\lambda}^2$ with imaginary time, the expression for $C(\omega, \lambda)$ is seen to coincide with the dynamical density-density correlation function of a free Fermi gas. For the case of small metallic particles, this coincidence was recently pointed out by Simons et al.[21]

6 Conclusions

6.1 Summary of results

We have studied the zero-temperature properties of the surface states of a three-dimensional layered integer quantum Hall sample. Previous work[9, 10] showed that for weak disorder in the thermodynamic limit this system is a *dirty chiral metal*, with finite conductivity along the field axis at zero temperature, and ballistic transport transverse to it.

In this paper, several new results are obtained and placed in a more powerful conceptual framework. By constructing an appropriate generating functional for disorder averaged correlations, we showed that this system can in fact be viewed as a kind of supersymmetric (SUSY) ferromagnetic spin chain. Like for an ordinary ferromagnet, the ground state and single magnon excitations can be obtained exactly. This allows an exact calculation of the diffuson (or density-density correlator). The conventional $z = 2$ dynamical scaling for ferromagnetic spin waves corresponds to the combination of finite conduction (transverse diffusion) and ballistic

transport discussed above. In the thermodynamic limit, therefore, we have demonstrated the absence of localization in the 2d chiral metal, even in the limit of strong disorder.

In addition, the quantum formulation of the SUSY ferromagnet provides a novel method to study the mesoscopics of the chiral metal. As an example, we have determined the correlation function of the density of states at two energies and for two slightly different realizations of disorder, the so-called parametric correlations. As expected, in the ergodic regime this takes on the universal form appropriate for the unitary ensemble, i.e. a free Fermion density-density correlation function[23].

6.2 Connection to the non-linear σ model

It is useful to connect the approach taken here to the more conventional NLσM method[14]. This connection may be made quite explicit using the SUSY coherent states developed in the appendix. We will construct a coherent state path integral representation for the partition function $\mathcal{Z}$. We begin by considering the problem of decoupled edges, i.e. with $D = 0$. Then

$$\mathcal{Z} = \prod_n \mathcal{Z}_n, \tag{161}$$

where

$$\mathcal{Z}_n = \mathrm{STr} e^{-\beta H_n}. \tag{162}$$

The single super spin Hamiltonian for $\lambda = 0$ is determined by the condition $H_\eta = \sum_n H_n$, so by Eq. (56),

$$H_n = 2\eta \left(S_n^z + J_n^z\right). \tag{163}$$

The coherent state path integral can then be simply obtained using the resolution of unity, Eq. (174), and the formula for the supertrace, Eq. (175). One finds

$$\mathcal{Z}_n = \int D(Z_n, \tilde{Z}_n) e^{-S_n[Z_n]}, \tag{164}$$

where the single mode action is

$$S_n[Z_n] = \int_0^\beta d\tau \ \langle Z_n | \partial_\tau + H_n | Z_n \rangle \equiv \int_0^\beta d\tau \mathcal{L}_n.$$

The single spin Lagrangian is given by

$$\begin{aligned} \mathcal{L}_n &= \langle Z_n | \partial_\tau + H | Z_n \rangle \\ &= -\mathrm{STr}(1 + \tilde{Z}_n Z_n)^{-1} \tilde{Z}_n (\partial_\tau + 2\eta) Z_n. \end{aligned} \tag{165}$$

This expression can be put into a simpler form by defining the 4×4 supermatrices

$$g_n = \begin{pmatrix} 1 & Z_n \\ -\tilde{Z}_n & 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{166}$$

and

$$Q_n = g_n \Lambda g_n^{-1}. \tag{167}$$

In terms of these fields, the single spin Lagrangian becomes

$$\mathcal{L}_n = \frac{1}{2}\mathrm{STr}\left(\Lambda g_n^{-1}\partial_\tau g_n\right) + \frac{\eta}{2}\mathrm{STr}\left(\Lambda Q_n\right). \tag{168}$$

The first (dynamical phase) term is of the Wess-Zumino type, and *cannot* be written in a globally non-singular form in terms of the Q field. A Wess-Zumino term also occurs in the coherent state path integral for an ordinary SU(2) spin[24], and could have been expected here on general grounds. It is necessary to obtain the $z = 2$ dynamics appropriate for a ferromagnet. Inclusion of such a term allows us here to obtain a true NLσM formulation, in contrast to the earlier Q-matrix formulation in Ref.[10], in which only an expansion in the ordered (metallic) phase was determined.

Eq. (168) holds for a single super spin. It is now straightforward to include the exchange coupling H_D to obtain

$$\mathcal{Z} = \int D(Z, \tilde{Z}) e^{-S} \tag{169}$$

where

$$S = \int_0^\beta d\tau \sum_n \left(\mathcal{L}_n + \mathcal{L}_{\mathrm{int},n}\right), \tag{170}$$

and

$$\mathcal{L}_{\mathrm{int},n} = \frac{D}{4}\mathrm{STr}\left(Q_n Q_{n+1}\right). \tag{171}$$

Eq. (170) is the full NLσM action for the 2d system. The universal level correlations, usually discussed in this context, are properties of the zero-dimensional quantum limit, in which it is appropriate to make a "zero mode" approximation, neglecting spatial and temporal (i.e. $\tau \equiv x$) variations of Q. The zero mode theory then becomes a single super-integral over Q_0. This formulation can be connected back to the operator one of the text, by noting that the resulting integral is simply another representation of the restricted trace over the zero energy multiplet of completely polarized super spin states.

6.3 Questions and open problems

We conclude with a discussion of various questions which remain open to future investigation. One interesting issue is to understand in more detail the nature of the crossovers between the three mesoscopic regimes described in Section V. In the language of the ferromagnetic super spin chain, which has finite length N and is at finite temperatures ($\beta < \infty$), the three regimes correspond to: (i) a "zero-dimensional" regime in which all (finite wavevector) spin-waves are absent, being too costly in energy, (ii) a "1d diffusive" regime in which the spins behave classically but are still ferromagnetically ordered, and (iii) a "localized" regime in which the spin chain length exceeds the ferromagnetic spin correlation length. For ordinary (non-super) ferromagnetic spin chains, recent progress has been made in computing various scaling functions connecting these regimes[27]. The apparent generality of the technique suggests that progress might likewise be made for the super spin chain.[29]

Also of interest in terms of crossover behavior are the scaling properties of the *wavefunctions* in various limits. Although we do not expect interesting scaling properties in the thermodynamic limit (because the 2d system is truly a stable (chiral) metal), interesting possibilities

arise in the intermediate mesoscopic regime. Specifically, in the "1d diffusive" regime, with $L_0 = \sqrt{D\beta} \ll N \ll \xi = D\beta$, the electron motion is ergodic on frequency scales $\omega \sim \Delta$. Nevertheless, there is a breakdown of the zero-mode approximation, which requires $L_0 \gg N$. We then expect the wavefunctions to become very broadly distributed random variables[26]. It would be interesting to see if the moments exhibit multifractal scaling in this regime, as well as to look at scaling of multi-point correlators.

Thirdly, it would be interesting to calculate the conductance fluctuations[30], which would involve applying appropriate boundary conditions to the ends of the super spin chain and calculating an eight-Fermion (or eight-Boson, etc.) correlator (product of four Green's functions). Preliminary work using diagrammatic techniques by Mathur[31] suggests rather interesting crossover phenomena in the variance of the conductance.

Lastly, these methods may be useful to connect with earlier work on chiral classical wave propagation in a disordered medium. As pointed out in Ref. [32], the Schrödinger equation for electrons in the surface sheath may be viewed as a classical wave equation for "directed" propagation. It is then of interest to study the spreading and deflection of a point source at an initial $x = 0, n = 0$ to some larger x. The beam width is closely related to our diffuson, but (like the conductance fluctuations) the motion of the beam center involves the product of four Green's functions, but may also be tractable[31].

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Appendix

This appendix is in two parts. In the first part we are going to develop some of the mathematical structures underlying the coherent state path integral for the super spin Hamiltonian. In the second part we will derive the expression (149) for the effective zero mode Hamiltonian H_λ, which results from treating the background potential by second-order perturbation theory.

Our first item will be to discuss the group of canonical transformations of the Bose and Fermi operators. For that purpose we introduce a more economical notation, by setting

$$\begin{aligned} c_{B\uparrow} &= B_\uparrow, \quad c_{F\uparrow} = F_\uparrow, \quad c_{B\downarrow} = B_\downarrow, \quad c_{F\downarrow} = F_\downarrow, \\ \overline{c}_{B\uparrow} &= \overline{B}_\uparrow, \quad \overline{c}_{F\uparrow} = F_\uparrow^\dagger, \quad \overline{c}_{B\downarrow} = \overline{B}_\downarrow, \quad \overline{c}_{F\downarrow} = F_\downarrow^\dagger. \end{aligned}$$

Now let A, B, C and D be complex 2×2 supermatrices, i.e.

$$A = \begin{pmatrix} A_{FF} & A_{FB} \\ A_{BF} & A_{BB} \end{pmatrix} \quad \text{etc.},$$

where A_{FF} and A_{BB} are complex numbers, while A_{FB} and A_{BF} are Grassmann numbers. Consider then the transformation

$$\begin{aligned} \overline{c}_{a\uparrow}^n &\to \overline{c}_{b\uparrow}^n A_{ba} + \overline{c}_{b\downarrow}^n C_{ba}, \\ \overline{c}_{a\downarrow}^n &\to \overline{c}_{b\uparrow}^n B_{ba} + \overline{c}_{b\downarrow}^n D_{ba}, \end{aligned}$$

where the superscript $n = 1, ..., N$ indexes the chiral modes and, here and below, the summation convention is used. By setting $\{\overline{c}_\alpha\} = \{\overline{c}_{F\uparrow}, \overline{c}_{B\uparrow}, \overline{c}_{F\downarrow}, \overline{c}_{B\downarrow}\}$, we write this in the abbreviated form

$$\overline{c}^n_\alpha \to \overline{c}^n_\beta g_{\beta\alpha}, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

If g has an inverse, we can also transform the annihilators, by

$$c^n_\alpha \to (g^{-1})_{\alpha\beta} c^n_\beta.$$

When rearranging products, we follow the convention that Grassmann numbers not only anticommute among themselves, but also anticommute with the Fermionic operators. It is then easy to see that the transformation $c^n_\alpha \to (g^{-1})_{\alpha\beta} c_\beta$, $\overline{c}^n_\alpha \to \overline{c}^n_\beta g_{\beta\alpha}$, is canonical, i.e. preserves the graded commutator.

Canonical transformations have an inverse, and a succession of two canonical transformations is again canonical. Therefore such transformations[33] form a group, which in the present context is the Lie supergroup Gl(2|2), obtained by complexifying U(2|1, 1). The elements g of Gl(2|2) can be written in the form of a Gauss decomposition,

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_\uparrow & 0 \\ 0 & h_\downarrow \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{Z} & 1 \end{pmatrix},$$

where, from an easy calculation,

$$\begin{aligned} Z &= BD^{-1}, \quad h_\downarrow = D, \\ \tilde{Z} &= D^{-1}C, \quad h_\uparrow = A - BD^{-1}C. \end{aligned}$$

Every canonical transformation g can be realized by a Fock space operator T_g that is obtained by exponentiating some bilinear $\overline{c}c$, and acts by $c^n_\alpha \to T_g c^n_\alpha T_g^{-1}$, $\overline{c}^n_\alpha \to T_g \overline{c}^n_\alpha T_g^{-1}$. In detail the correspondences are as follows:

$$\begin{aligned} T_{\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}} &= \exp\left(\overline{c}^n_{a\uparrow} Z_{ab} c^n_{b\downarrow}\right), \\ T_{\begin{pmatrix} 1 & 0 \\ \tilde{Z} & 1 \end{pmatrix}} &= \exp\left(\overline{c}^n_{a\downarrow} \tilde{Z}_{ab} c^n_{b\uparrow}\right), \\ T_{\begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix}} &= \exp\left(\overline{c}^n_{a\uparrow} A_{ab} c^n_{b\uparrow}\right), \\ T_{\begin{pmatrix} 1 & 0 \\ 0 & e^D \end{pmatrix}} &= \exp\left(\overline{c}^n_{a\downarrow} D_{ab} c^n_{b\downarrow}\right). \end{aligned}$$

By construction, the correspondence $g \to T_g$ defines a representation of Gl(2|2) on Fock space. Therefore every relation that is valid for supermatrices g, and uses no more than the Lie supergroup structure of Gl(2|2), also holds for the Fock space operators T_g. In particular, by applying the Gauss decomposition to the product

$$\begin{pmatrix} 1 & 0 \\ \tilde{Z} & 1 \end{pmatrix} \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix},$$

we get

$$\begin{aligned}
& \exp\left(\overline{c}^n_{a\downarrow}\tilde{Z}_{ab}c^n_{b\uparrow}\right)\exp\left(\overline{c}^n_{a\uparrow}Z_{ab}c^n_{b\downarrow}\right) \\
= \; & \exp\left(\overline{c}^n_{a\uparrow}[Z(1+\tilde{Z}Z)^{-1}]_{ab}c^n_{b\downarrow}\right) \\
& \times\exp\left(\overline{c}^n_{a\downarrow}[\ln(1+\tilde{Z}Z)]_{ab}c^n_{b\downarrow}-\overline{c}^n_{a\uparrow}[\ln(1+Z\tilde{Z})]_{ab}c^n_{b\uparrow}\right) \\
& \times\exp\left(\overline{c}^n_{a\downarrow}[\tilde{Z}(1+Z\tilde{Z})^{-1}]_{ab}c^n_{b\uparrow}\right),
\end{aligned}$$

which is called a "disentangling" identity, since it moves the Gl(2|2) lowering operators all the way to the right, and the raising operators all the way to the left.

Consider now the generalized coherent states[34]

$$\begin{aligned}
|Z\rangle &= \exp\left(\overline{c}^n_{a\uparrow}Z_{ab}c^n_{b\downarrow}\right)|0\rangle\ \mathrm{SDet}(1+\tilde{Z}Z)^{N/2}, \\
\langle Z| &= \mathrm{SDet}(1+\tilde{Z}Z)^{N/2}\langle 0|\exp\left(\overline{c}^n_{b\downarrow}\tilde{Z}_{ba}c^n_{a\uparrow}\right).
\end{aligned}$$

The superdeterminant of a 2×2 supermatrix,

$$\begin{aligned}
& \mathrm{SDet}\begin{pmatrix} M_{FF} & M_{FB} \\ M_{BF} & M_{BB}\end{pmatrix} \\
= \; & \left(M_{BB}-M_{BF}{M_{FF}}^{-1}M_{FB}\right)/M_{FF} \\
= \; & M_{BB}/\left(M_{FF}-M_{FB}{M_{BB}}^{-1}M_{BF}\right),
\end{aligned}$$

satisfies $\ln\mathrm{SDet}M=\mathrm{STr}\ln M$, where $\mathrm{STr}M=-M_{FF}+M_{BB}=:\sum_a(-1)^{|a|}M_{aa}$, with $|a|=0$ if $a=B$ (Boson) and $|a|=1$ if $a=F$ (Fermion), is the supertrace. Using this, the disentangling identity, and the relations

$$\overline{c}^n_{a\uparrow}c^n_{b\uparrow}|0\rangle=0, \qquad \overline{c}^n_{a\downarrow}c^n_{b\downarrow}|0\rangle=\delta_{ab}(-1)^{|a|+1}N|0\rangle,$$

it is easy to check that the generalized coherent states are normalized,

$$\langle Z|Z\rangle=1.$$

Another useful way of writing the generalized coherent states is

$$|Z\rangle=T_g|0\rangle, \qquad \langle Z|=\langle 0|T_g^{-1},$$

where

$$\begin{aligned}
g &= \begin{pmatrix}1 & Z\\ 0 & 1\end{pmatrix}\begin{pmatrix}(1+Z\tilde{Z})^{+1/2} & 0\\ 0 & (1+\tilde{Z}Z)^{-1/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ -\tilde{Z} & 1\end{pmatrix} \\
&= \begin{pmatrix}(1+Z\tilde{Z})^{-1/2} & Z(1+\tilde{Z}Z)^{-1/2}\\ -\tilde{Z}(1+Z\tilde{Z})^{-1/2} & (1+\tilde{Z}Z)^{-1/2}\end{pmatrix},
\end{aligned} \tag{172}$$

and

$$g^{-1}=\begin{pmatrix}(1+Z\tilde{Z})^{-1/2} & -Z(1+\tilde{Z}Z)^{-1/2}\\ \tilde{Z}(1+Z\tilde{Z})^{-1/2} & (1+\tilde{Z}Z)^{-1/2}\end{pmatrix}. \tag{173}$$

The zero energy multiplet of Sec. 5.2 coincides with the space of states obtained by acting repeatedly with the Gl(2|2) raising operators $\bar{c}^n_{a\uparrow}c^n_{b\downarrow}$ on the vacuum $|0\rangle$. In[35] it was shown that, if $D(Z,\tilde{Z}) = Dg_H$ is the uniform superintegration measure on the supermanifold $G/H = \mathrm{Gl}(2|2)/\mathrm{Gl}(1|1)\times\mathrm{Gl}(1|1)$ parameterized by the coherent states $|Z\rangle = T_g|0\rangle$, the unit operator on the zero energy multiplet can be resolved by

$$1 = \int Dg_H T_g|0\rangle\langle 0|T_g^{-1} = \int D(Z,\tilde{Z})|Z\rangle\langle Z|. \tag{174}$$

(Note that the conventions of that reference differ from ours by $\tilde{Z}\to -\tilde{Z}$ and the exchange of the Fermionic and Bosonic sectors.) The integral is over

$$\tilde{Z}_{FF} = \bar{Z}_{FF}, \qquad \tilde{Z}_{BB} = -\bar{Z}_{BB},$$

with the integration domain being

$$0 \le |Z_{FF}|^2 < \infty, \qquad 0 \le |Z_{BB}|^2 < 1.$$

These relations mean[18] that the variable Z_{FF} is a complex stereographic coordinate for the two-sphere S^2 (as is well-known for SU(2) coherent states) while Z_{BB} parameterizes a two-hyperboloid H^2 (as is appropriate for the non-compact group SU(1,1)). Of course Z_{BF}, Z_{FB}, $\tilde{Z}_{BF}$, and $\tilde{Z}_{FB}$ are Grassmann variables.

Given the resolution of unity (174), we can convert supertraces over the zero energy multiplet into integrals over the generalized coherent states:

$$\langle O\rangle_0 = \mathrm{Tr}_0(-1)^{N_F}O = \int D(Z,\tilde{Z})\langle Z|O|Z\rangle. \tag{175}$$

The coherent state path integral for the partition function of a super spin system with Hamiltonian H, is now obtained in the usual manner by inserting resolutions of unity between infinitesimal imaginary time slices of the Boltzmann weight $\exp(-\beta H)$. The Lagrangian of the resulting path integral is $\mathcal{L} = \langle Z|\partial_\tau + H|Z\rangle$, whose explicit form is easily calculated by using the definition of the coherent states and the disentangling identity. For example,

$$\begin{aligned}
& \langle Z|\partial_\tau|Z\rangle - \partial_\tau \ln \mathrm{SDet}(1+\tilde{Z}Z)^{N/2} \\
= \;& \mathrm{SDet}(1+\tilde{Z}Z)^N\langle 0|\exp(\bar{c}^n_{b\downarrow}\tilde{Z}_{ba}c^n_{a\uparrow})\partial_\tau\exp(\bar{c}^n_{a\uparrow}Z_{ab}c^n_{b\downarrow})|0\rangle \\
= \;& \mathrm{SDet}(1+\tilde{Z}Z)^N\frac{\partial}{\partial s}\Big|_{s=0}\mathrm{SDet}(1+\tilde{Z}Z+s\tilde{Z}\partial_\tau Z)^{-N} \\
= \;& -N\mathrm{STr}\left[(1+\tilde{Z}Z)^{-1}\tilde{Z}\partial_\tau Z\right].
\end{aligned}$$

The Wess-Zumino (or linear in ∂_τ) term in (165) is obtained on setting the number of chiral modes N equal to one and omitting a total τ derivative.

Below we will need an explicit expression for the coherent state expectation value of the operator $\mathcal{C}^{+-}$ appearing in the formula (149) of H_λ. To calculate it we write

$$\begin{aligned}
\mathcal{C}^{+-} &= S^+S^- + J^+J^- + \tilde{A}^+A^- - A^+\tilde{A}^- \\
&= S^-S^+ + J^-J^+ + \tilde{A}^-A^+ - A^-\tilde{A}^+.
\end{aligned}$$

The second equality is invalid for both SU(2) and SU(1,1), since $S^+S^- \neq S^-S^+$ and $J^+J^- \neq J^-J^+$, but it does hold in the present case as a result of cancellations due to supersymmetry. Using the second form of $\mathcal{C}^{+-}$ we easily find

$$\begin{aligned} & \mathrm{SDet}(1+\tilde{Z}Z)^{-N}\langle Z|\mathcal{C}^{+-}|Z\rangle \\ = & \sum(-1)^{|d|+1}\frac{\partial^2}{\partial Z_{cd}\partial\tilde{Z}_{dc}}\ \langle 0|\exp(\bar{c}^n_{b\downarrow}\tilde{Z}_{ba}c^n_{a\uparrow})\exp(\bar{c}^n_{a\uparrow}Z_{ab}c^n_{b\downarrow})|0\rangle \\ = & \sum(-1)^{|b|+1}\frac{\partial^2}{\partial Z_{ab}\partial\tilde{Z}_{ba}}\ \mathrm{SDet}(1+\tilde{Z}Z)^{-N}, \end{aligned}$$

which yields

$$\langle Z|\mathcal{C}^{+-}|Z\rangle = -N^2\mathrm{STr}\left[\tilde{Z}Z(1+\tilde{Z}Z)^{-2}\right] + \mathcal{O}(N^1). \tag{176}$$

We now turn to the effective zero mode Hamiltonian

$$H_\lambda = H_{\mathrm{M}}\left(-\sum_{k\neq 0}\frac{\Pi_k}{Dk^2}\right)H_{\mathrm{M}},$$

and show that it reduces to (149) in the large-N limit. The first step is to construct the projector on the space of momentum-k single magnon excitations, Π_k. Magnons can be created on top of the vacuum or any other state of the zero energy multiplet. We define the single magnon creation and annihilation operators by

$$\begin{aligned} \mathcal{J}^+_{ab}(k) &= \sum_n e^{+ikn}\ \overline{c}^n_{a\uparrow}c^n_{b\downarrow}, \\ \mathcal{J}^-_{ab}(k) &= \sum_n e^{-ikn}\ \overline{c}^n_{a\downarrow}c^n_{b\uparrow}, \end{aligned}$$

and consider the states

$$|ab,k\rangle = \frac{1}{\sqrt{N}}\mathcal{J}^+_{ab}(k)|0\rangle.$$

The corresponding projection operator is

$$\pi_k = \sum_{ab}|ab,k\rangle\langle ab,k| = \frac{1}{N}\sum_{ab}(-1)^{|b|+1}\mathcal{J}^+_{ab}(k)|0\rangle\langle 0|\mathcal{J}^-_{ba}(k).$$

It is easy to verify that, with the normalization and sign factors chosen, π_k satisfies $\pi_k\pi_k = \pi_k$, and is a singlet with respect to the (complexified) algebra $\mathcal{H} = \mathrm{gl}(1|1)\oplus\mathrm{gl}(1|1)$ generated by the zero mode operators $Q_\uparrow$, $Q_\downarrow$, $\overline{Q}_\uparrow$, $\overline{Q}_\downarrow$, $n_{f\uparrow}$, $n_{f\downarrow}$, $n_{b\uparrow}$, and $n_{b\downarrow}$. The states $|ab,k\rangle$ are single magnon excitations built on the vacuum. Other one-magnon states (not based on the vacuum) are generated by applying a global rotation,

$$|ab,k\rangle \to T_g|ab,k\rangle.$$

The projector π_k transforms as

$$\pi_k \to T_g\pi_k T_g^{-1}.$$

The projector on the entire one-magnon sector is obtained by averaging over all global rotations,

$$\Pi_k = \int Dg_H\ T_g\pi_k T_g^{-1}. \tag{177}$$

The integrand is well-defined as a function on the coset space G/H since, by the $\mathcal{H}$-singlet property of π_k,

$$T_{gh}\pi_k T_{gh}^{-1} = T_g T_h \pi_k T_h^{-1} T_g^{-1} = T_g \pi_k T_g^{-1}$$

for $h \in H = \mathrm{Gl}(1|1) \times \mathrm{Gl}(1|1)$.

To go further, we need to take advantage of the large-N limit. Let us evaluate the expectation value of $H_{\mathrm{M}}\Pi_k H_{\mathrm{M}}$ in a coherent state $|Z\rangle \equiv T_g|0\rangle$,

$$E := \int Dg'_H \ \langle 0|T_g^{-1} H_{\mathrm{M}} T_{g'} \pi_k T_{g'}^{-1} H_{\mathrm{M}} T_g|0\rangle.$$

For large N we expect the dominant contributions to the g' integral to come from the immediate vicinity of $g = g(Z,\tilde{Z})$, so we substitute variables $g' = g\exp X$ (with X subject to the transversality condition $\Lambda X + X\Lambda = 0$) and approximate Dg'_H by its linearization DX at g. The integration over X is trivial since

$$\langle 0|T_g^{-1} T_{g\exp X}|0\rangle = \langle 0|T_{\exp X}|0\rangle \simeq e^{-N\mathrm{STr}X^2/4} \to \delta(X)$$

reduces to a δ-function in the large-N limit. Thus, doing the g' integral to leading order in the large parameter N, we get

$$E = \langle 0|T_g^{-1} H_{\mathrm{M}} T_g \pi_k T_g^{-1} H_{\mathrm{M}} T_g|0\rangle.$$

The large-N limit is semiclassical in nature, and it is not hard to see that all off-diagonal matrix elements of H_λ become negligible for $N \to \infty$. Therefore, it is in fact sufficient to consider the diagonal ones, which is what we are doing.

The next step is to compute the rotated perturbation $T_g^{-1} H_{\mathrm{M}} T_g$ from the expression $H_{\mathrm{M}} = i\sum_{n,a} h_n(\overline{c}^n_{a\uparrow} c^n_{a\uparrow} - \overline{c}^n_{a\downarrow} c^n_{a\downarrow})/2$. For this we set

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix},$$

and use the transformation laws (see the beginning of this appendix)

$$\begin{aligned}
T_g^{-1}\overline{c}^n_{a\uparrow} T_g &= \overline{c}^n_{a'\uparrow}\tilde{A}_{a'a} + \overline{c}^n_{a'\downarrow}\tilde{C}_{a'a}, \\
T_g^{-1}\overline{c}^n_{b\downarrow} T_g &= \overline{c}^n_{b'\uparrow}\tilde{B}_{b'b} + \overline{c}^n_{b'\downarrow}\tilde{D}_{b'b}, \\
T_g^{-1} c^n_{a\uparrow} T_g &= A_{aa'} c^n_{a'\uparrow} + B_{aa'} c^n_{a'\downarrow}, \\
T_g^{-1} c^n_{b\downarrow} T_g &= C_{bb'} c^n_{b'\uparrow} + D_{bb'} c^n_{b'\downarrow},
\end{aligned}$$

which yield

$$\begin{aligned}
T_g^{-1} H_{\mathrm{M}} T_g \ = \ & \frac{i}{2}\sum_n h_n \Big(\overline{c}^n_\uparrow (\tilde{A}B - \tilde{B}D) c^n_\downarrow \\
& + \overline{c}^n_\downarrow (\tilde{C}A - \tilde{D}C) c^n_\uparrow + \ldots \Big).
\end{aligned}$$

On taking the matrix element with the one-magnon creation operator, we obtain

$$\langle 0|T_g^{-1} H_{\mathrm{M}} T_g \mathcal{J}^+_{ab}(k)|0\rangle = \frac{iN}{2}\tilde{h}_k (-1)^{|a||b|+1} (\tilde{C}A - \tilde{D}C)_{ba}.$$

Similarly,

$$\langle 0|\mathcal{J}^-_{ba}(k)T_g^{-1}H_{\rm M}T_g|0\rangle = \frac{iN}{2}\tilde{h}_{-k}(-1)^{|a||b|+1}(\tilde{A}B - \tilde{B}D)_{ab}.$$

We now switch notation from g to $Z, \tilde{Z}$. From (172,173) we read off the relations

$$\begin{aligned}\tilde{C}A - \tilde{D}C &= 2\tilde{Z}(1+Z\tilde{Z})^{-1},\\ \tilde{A}B - \tilde{B}D &= (1+Z\tilde{Z})^{-1}2Z,\end{aligned}$$

whence the coherent state expectation value of H_λ takes the form

$$\langle Z|H_\lambda|Z\rangle = -N\sum_{k\neq 0}\frac{\tilde{h}_k\tilde{h}_{-k}}{Dk^2}\times {\rm STr}\left[Z\tilde{Z}(1+Z\tilde{Z})^{-2}\right].$$

Comparison with (176) gives $H_\lambda = \gamma\mathcal{C}^{+-}$, as claimed in (149).

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