Nodal Liquid Theory of the Pseudo-Gap Phase of High-T_c Superconductors

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We introduce and study the *nodal liquid*, a novel zero-temperature quantum phase obtained by quantum-disordering a d-wave superconductor. It has numerous remarkable properties which lead us to suggest it as an explanation of the pseudo-gap state in underdoped high-temperature superconductors. In the absence of impurities, these include power-law magnetic order, a T-linear spin susceptibility, non-trivial thermal conductivity, and two- and one-particle charge gaps, the latter evidenced, e.g. in transport and electron photoemission (which exhibits pronounced fourfold anisotropy inherited from the d-wave quasiparticles). We use a $2+1$-dimensional duality transformation to derive an effective field theory for this phase. The theory is comprised of gapless *neutral* Dirac particles living at the former d-wave nodes, *weakly coupled* to the fluctuating gauge field of a dual Ginzburg-Landau theory. The nodal liquid interpolates naturally between the d-wave superconductor and the insulating antiferromagnet, and our effective field theory is powerful enough to permit a detailed analysis of a panoply of interesting phenomena, including charge ordering, antiferromagnetism, and d-wave superconductivity. We also discuss the zero-temperature quantum phase transitions which separate the nodal liquid from various ordered phases.

I. INTRODUCTION

The discovery of the cuprate high-temperature superconductors in 1986[1] was a watershed in the recent history of condensed matter physics, an event which stimulated intense experimental and theoretical activity. As sample quality and experimental precision have advanced, these materials' rich phase diagram and phenomenology have come into focus.[2] However, many theoretical efforts have not reached fruition because of a serious obstacle, namely, that these materials are apparently in a strongly-coupled, non-perturbative regime. To put it more bluntly: there is no obvious small parameter which facilitates an expansion about a tractable model. In this paper, we promulgate the existence of a *weakly-coupled* 'dual' description of a particularly exotic region of the phase diagram. This description paves the way for controlled calculations of experimentally measurable quantities.

Before plunging into our exegesis of this dual description and its consequences, let us briefly review the phenomena which we wish to explain. The phase diagram as a function of temperature T and doping x (the nature of the doping varies from material to material, but is generally believed to be proportional to the effective hole concentration in each CuO_2 layer) is indicated schematically in Fig. 1. Best understood and in many cases very well characterized are the undoped ($x = 0$) materials, which are insulating antiferromagnets (AFs) below the Néel temperature T_N. At moderate dopings ($0.1 \lesssim x \lesssim 0.3$) and low temperatures ($T < T_c(x)$), superconductivity occurs. For many phenomenological purposes the superconducting phase is adequately described by the same Ginzburg-Landau and London theory standard for conventional superconductors (SCs).[3] One important distinction between the high-T_c and conventional SCs, suspected for many years and now generally accepted after a number of beautiful and compelling experiments,[4,5] is their $d_{x^2-y^2}$ (d-wave) pairing symmetry. This pairing symmetry is a crucial ingredient in a zero-temperature quantum description. In particular, d-wave symmetry leads to gapless quasiparticles residing at the four nodes of the pair wavefunction.
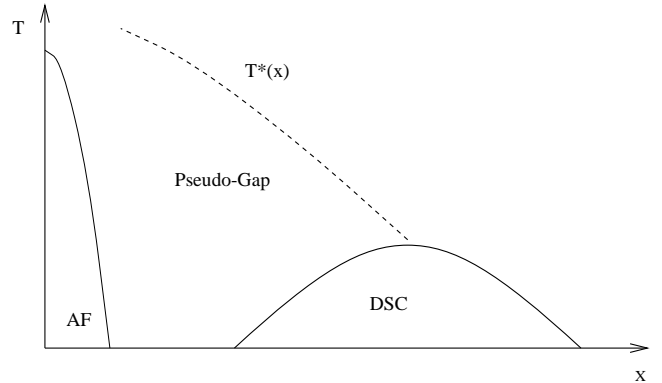

Fig. 1: Schematic phase diagram of a high-temperature superconductor as a function of doping x and temperature T.

Recent studies have revealed puzzling behavior in the *underdoped* region between the AF and SC. Below the high-temperature dotted line ($T^*(x)$) in Fig. 1, angle-resolved photoemission (ARPES),[6,7] transport,[8] NMR,[9,10] optical conductivity,[11,12] and other measurements[2] indicate a dramatic reduction of low-energy (single-)electronic and spin degrees of freedom.

Furthermore, the ability of ARPES measurements to resolve wavevector dependence exposes an angular variation similar to that of d-wave quasiparticles in the SCing phase[6,7]. This portion of the phase diagram is commonly called the *pseudo-gap* region. The ultimate nature of the corresponding underlying quantum ground state is an intriguing theoretical puzzle, and a principal subject of this paper.

To proceed, we look to the experiments for guidance. They indicate three types of excitations which are important below the dotted line in Fig. 1: the ordering fields related to antiferromagnetism and superconductivity, *and* d-wave quasiparticles near the four nodes. Conspicuously absent from this list are electrons and holes at an ordinary Fermi-surface. The physics of this omission is that pairing occurs (due to unspecified strong interaction physics) at the high energy $T^*(x)$. Given these ingredients, one natural theoretical strategy is to attempt to approach the pseudo-gap state by increasing x from the AF at half-filling. Many researchers have already attempted this approach, but it remains inconclusive.

We, instead, tackle the pseudo-gap state from the right, literally. To do this, we must contemplate quantum-disordering the d-wave superconductor. For simplicity, we will assume for the moment a purely two-dimensional model of a single CuO_2 plane. We imagine that pairing establishes a local superconducting d-wave order parameter $\Delta(\vec{x}, t) = |\Delta|e^{i\varphi}$, where $\vec{x}$ is the two-dimensional coordinate and t is time. The experimental properties of the pseudo-gap state can be interpreted as an indication that $|\Delta|$ is large in this region, so that quantum fluctuations of the phase of the order-parameter φ must be responsible for the lack of off-diagonal long-range-order ($\langle\Delta\rangle = 0$), even as $T \to 0$. The important long-distance dephasing is accomplished by *vortex* loops and lines, around which φ winds by $\pm 2\pi$. To destroy the long-range correlations in Δ, we must unbind vortex loops of arbitrarily large size, just as vortex-antivortex pairs unbind above the Kosterlitz-Thouless transition temperature in a two-dimensional superfluid. To implement this unbinding, it is extremely helpful to use the $2 + 1$-dimensional *duality*[13,14] relating an XY-model for Δ and a Ginzburg-Landau model with complex field Φ ("disorder parameter") interacting with a gauge field **a**. The duality interchanges Cooper pairs and vortices, so that the desired quantum disordered d-wave state is the ordered (condensed) phase, $\langle\Phi\rangle \neq 0$, of the Ginzburg-Landau theory. The imaginary-time effective action for this dual theory is nothing but the Ginzburg-Landau free energy functional for a three-dimensional superconductor at finite-temperature.

The fate of the d-wave quasiparticles in this construction requires particular care, since these are strongly coupled to the fluctuating pair field. In Sec. II A, we show how the quasiparticles can be treated by extracting the $U(1)$ phase from the bare electron operators. Once the phase is extracted, one is left with a set of gauge-invariant fermion operators which create electrically neutral (but

spin-ful) quanta we call *nodons*. In the dual variables, the fundamental Lagrangian of our model is

$$
\begin{aligned}
\mathcal{L} =\ & \psi_1^\dagger[i\partial_t - v_F\tau^z i\partial_x - v_\Delta\tau^x i\partial_y]\psi_1 + (1 \leftrightarrow 2, x \leftrightarrow y) \\
& + \frac{\kappa_\mu}{2}|(\partial_\mu - ia_\mu)\Phi|^2 - V_\Phi(|\Phi|) \\
& + a_\mu\epsilon_{\mu\nu\lambda}\partial_\nu(A_\lambda - \kappa_\lambda^{-1}J_\lambda) + \frac{1}{2\kappa_0}(e_j^2 - b^2) \\
& + 2\lambda J_0 + \mathcal{L}_N.
\end{aligned} \tag{1.1}
$$

In the sections which follow, we will elucidate the physics of this Lagrangian in some detail, so we restrict ourselves, in this introduction, to a whirlwind tour. The fields $\psi_j^\dagger, \psi_j$, $j = 1, 2$ are the nodon creation and annihilation operators at the two antipodal pairs of nodes. J_λ is a bilinear in the nodon operators which has an interpretation in the d-wave superconducting phase as the quasiparticle electrical 3-current. Φ is the complex scalar field representing the vortices, and a_μ is the gauge field which is dual to the phase of the superconducting order parameter. The term proportional to λ describes the effects of particle/hole asymmetries, and the $\mathcal{L}_N$ term describes the coupling of the nodons to antiferromagnetism, which we will return to presently.

A remarkable result of calculations with Eq. 1.1 is that gapless nodons survive the quantum disordering of the SC! The nodons are like the smile of the Cheshire cat: the $d_{x^2-y^2}$ order parameter is gone, but the nodes remain. The consequent *Nodal Liquid* (NL) described by Eq. 1.1 is a distinct and novel zero-temperature quantum phase with a number of fascinating properties. For simplicity, consider first a hypothetical NL phase at half-filling in the absence of impurities. The possibility of AF ordering will be included later via $\mathcal{L}_N$. We hypothesize that antiferromagnetism might be avoided and the NL achieved in a half-filled square lattice model by adjusting an attractive nearest-neighbor interaction and second-neighbor electron hopping amplitude. The NL is a nominally insulating state, with non-zero gaps Δ_1 and Δ_2 for adding both individual electrons/holes and Cooper pairs, respectively. Gapless nodons with anisotropic ballistic dispersion ($\omega \sim k$), however, persist, and can carry both spin and current. With particle/hole symmetry ($\lambda = 0$ in Eq. 1.1), we expect power law ($\sim 1/|x|^4$) spin correlations at $(0, 0)$, (π, π), $(\pi, 0)$, and symmetry-related points in the Brillouin zone. Scaling arguments lead us to expect a weak dissipative dynamic contribution to the conductivity which, in the presence of particle/hole symmetry, would vary as $\text{Re}\,\sigma(\omega, T = 0) \sim \omega^6$. Nodons also contribute a quadratic specific heat $C_{\text{nodon}} \sim aT^2$. Despite the similarity of the nodons to d-wave quasiparticles, the single-particle spectral function is predicted to show a gap at the Fermi energy in the NL. We expect, however, this gap to be strongly angle-dependent: of order the pairing scale T^* away from $(\pm\pi/2, \pm\pi/2)$ and reduced to $\Delta_1 \ll T^*$ at these special points. A particle/hole asymmetric NL should exhibit similar behavior, but with singularities shifted from $(\pm\pi/2, \pm\pi/2)$ in momentum space, and an even smaller contribution to the

low-frequency dissipative conductivity which, in the absence of umklapp scattering and impurities must come from interactions with phonons.

Consider next doping the NL. Naively, this can be modeled via an increase in the in-plane hole concentration, although the actual transfer of charge to the CuO_2 layers may be not be complete. In the grand canonical ensemble, charge is added by increasing $\mu = A_0/2$. In the dual theory (see Eq. 1.1), this chemical potential acts like an external magnetic field $h = 2\mu$ in the Ginzburg-Landau theory (but, unlike a magnetic field, it of course does not break time-reversal invariance). For small μ, the system remains in the Meissner phase, and no dual flux penetrates – i.e. no charge is added to the system within the charge gap. Following the analogy with Ginzburg-Landau theory,[3] we expect that the nature of doping depends upon the Abrikosov parameter $\kappa_v = \lambda_v/\xi_v$, where λ_v and ξ_v are effective dual penetration and coherence lengths, respectively. For $\kappa_v \gtrsim 1/\sqrt{2}$, type II doping occurs, and the "field" penetrates first for $\mu > \mu_{c1}$ in a dual flux lattice. Dual flux tubes are in fact Cooper pairs, so this is a paired Wigner crystal (PWC) state. We show in Sec. III that gapless nodons survive the doping and coexist with the PWC. We generally expect the displacements of the Wigner crystal to be pinned either by the periodic lattice potential or disorder (either is effective when arbitrarily weak), so that this phase remains insulating. As doping increases from zero, the characteristic nodon momenta shift further from $(\pm\pi/2, \pm\pi/2)$. Continued doping to $\mu > \mu_{c2}$ leads to another transition into the "normal" state of the dual theory, which is nothing but the d-wave SC. Neglecting disorder, and with weak lattice effects, quantum fluctuations are expected to drive this $2+1$-dimensional flux lattice melting transition weakly first order. For $\kappa \lesssim 1/\sqrt{2}$, one has instead type I doping at a single "critical field" μ_c. This is a first order transition, accompanied by a jump in the hole concentration from 0 to x_c at $\mu = \mu_c$. In the canonical ensemble with fixed $0 < x < x_c$, one expects two-phase coexistence, i.e a "mixed" phase in the dual Ginzburg-Landau theory. Taking into account long-range Coulomb interactions, one arrives at the frustrated phase separation physics discussed at length by Emery and Kivelson,[15] and all the consequent issues. Crude arguments (see Sec. III) suggest $\kappa \sim 1 - 3$, in the weakly type II limit, but close enough to the threshhold value to allow for different scenarios in different materials. Regardless, type II and some type I schemes imply NL *phases* at finite doping before the onset of superconductivity.

An intermediate NL phase is extremely appealing from the theoretical point of view, as it offers a compelling interpolation between the undoped AF and the d-wave superconductor. Consider the following three important energies : the single-particle gap, Δ_1, the minimum energy required to add a charge $\pm e$ and spin $s = 1/2$ to the system; the two-particle gap, Δ_2, the minimum energy to add charge $\pm 2e$ and spin $s = 0$; and the spin gap, Δ_s, the energy required to add spin $s = 1$ but no charge. In

the AF, both single- and two-particle gaps are non-zero, but the spin gap vanishes due to low frequency magnons. In the d-wave superconductor, the two-particle gap vanishes, since the pairs have condensed, but spin and single-particle gaps are "almost" non-zero (which we call "0^+"), since only the quasiparticles carry these quantum numbers, and their density of states vanishes with the energy. Passing from the SC to the NL, Δ_2 changes from zero to non-zero, and Δ_1 changes from 0^+ to a true gap. The transition to the AF occurs then simply by developing a non-zero staggered magnetization.

This transition and other magnetic physics is discussed in Sec. IV, using the Néel Lagrangian density

$$\mathcal{L}_N = \frac{K_\mu}{2}|\partial_\mu \mathbf{N}|^2 - V_N(|N|)$$
$$+ g[\mathbf{N} \cdot \psi^\dagger \tau^z \sigma^y \boldsymbol{\sigma} \psi^\dagger + \text{h.c.}], \qquad (1.2)$$

where g measures the strength of the coupling between the Néel order parameter and the nodons. Eq. 1.2 can be obtained by introducing a $2k_F$ density-density interactions between the nodons and decoupling the antipodal terms with the Néel vector $\mathbf{N}$. Let us once again consider first the case of half-filling with particle/hole symmetry. For sufficiently strong interaction g, or when the quadratic coefficient r_N in $V_N(N)$ is negative, one obtains an AF phase with $\langle \mathbf{N} \rangle \neq 0$. In this phase the nodons develop a gap and low-energy spin quanta are carried entirely by spin waves. Depending upon the "mass" r_Φ of V_Φ, this is either a simple AF or AF order coexisting with a d-wave SC. Decreasing g or increasing r_N destroys the long-range AF order and liberates the nodons. This interesting phase transition is discussed in Sec. IV. Increasing r_Φ results in a further transition to the d-wave superconductor, which we believe is in the three-dimensional inverted-XY universality class. Tuning $r_\Phi = r_N = 0$ describes a multicritical point connecting directly the AF and d-wave SC phases.

Without particle/hole symmetry, $\lambda \neq 0$, and another possible phase exists: the coexisting AF and Nodal Liquid (AF/NL), with long-range AF order at (π, π) and power-law spin-density-wave correlations from the nodons at incommensurate wavevectors. This phase may be difficult to distinguish experimentally from the pure AF, and it seems possible that some of the well-studied *undoped* cuprate materials might well be in the AF/NL phase rather than the pure AF. In any event, the model of Eqs. 1.1,1.2 provides a simple basis for understanding the suppression of Néel order upon doping. To see this, assume that at half-filling $r_N > 0$. Agreement with experiment then requires that g be sufficiently strong to induce AF order. As the chemical potential μ is increased above the charge gap to induce holes into the system, the hole density or dual "internal field" becomes non-zero. From Eq. 1.1, this creates an effectively larger particle/hole asymmetry $\lambda_{\text{eff}} - \lambda \sim x$. This presents a competition. By ordering the Néel vector the system can create a gap for the nodons and reduce their kinetic energy. However,

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at finite λ_{eff} the nodon Dirac point would prefer to move away from $(\pm\pi/2, \pm\pi/2)$, which reduces the gain in kinetic energy. As x increases, therefore, we may expect to drive transitions from the AF to AF/NL and pure NL phases. Of course, there are in fact many different scenarios for type I and type II doping, small or large intrinsic λ, etc. These are discussed in Sec. IV. Once magnetism has been discussed, we conclude with a summary of the main points of the paper, open issues, relations to other work, and a brief discussion of experimental implications. Finally, two appendices include technical details of microscopic and ϵ-expansion calculations.

II. D-WAVE AND DUALITY

A. Model and Symmetries

Consider a tight binding model of electrons hopping on a square lattice, with a local Hamiltonian satisfying certain general symmetries. We will assume the system is both $U(1)$ and $SU(2)$ invariant (i.e. we neglect spin-orbit coupling), and has time-reversal, reflection, and four-fold rotational symmetry. Sometimes it will also be convenient to specialize to models which possess an additional discrete particle/hole symmetry. We denote lattice electron creation and annihilation operators as $c_\alpha^\dagger(\vec{x})$ and $c_\alpha(\vec{x})$, where $\vec{x}$ is the two-dimensional coordinate in the frame with $x = x_1$ and $y = x_2$ parallel to the a and b crystalline axes (i.e. to the Cu–O bonds). Here α is a spin label. In momentum space the kinetic energy takes the usual form,

$$H_0 = \sum_{k\alpha} \epsilon_k c_{k\alpha}^\dagger c_{k\alpha}, \quad (2.1)$$

and at this stage we allow for general electron interactions:

$$H_{int} = \sum_{k,q,k'} V(k,q,k') c_{-k+q,\alpha}^\dagger c_{k\beta}^\dagger c_{-k'+q,\beta} c_{k',\alpha}. \quad (2.2)$$

A discrete particle/hole transformation is implemented by

$$c_\alpha(\vec{x}) \xrightarrow{p/h} e^{i\vec{\pi}\cdot\vec{x}} c_\alpha^\dagger(\vec{x}), \quad (2.3)$$

with $\vec{\pi} = (\pi, \pi)$. Many common models (e.g. the Hubbard and t-J) are invariant under a particle/hole transformation at half-filling. Invariance of the kinetic energy implies that $\epsilon_k = -\epsilon_{k+\vec{\pi}}$, a form valid with near neighbor hopping. However, a second neighbor hopping term violates particle/hole symmetry.

As discussed in the introduction, we wish to describe the physics below the relatively strong d-wave pairing scale T^*, in order to approach the pseudo-gap phase from the superconducting side. To do so, we imagine introducing a d-wave order parameter

$$\Delta_k(\vec{q}) = \sum_{k'} V(k,q,k')\langle c_{-k'+q\downarrow} c_{k'\uparrow} \rangle. \quad (2.4)$$

BCS theory[16] can be implemented in terms of the spatially varying pair field, obtained via Fourier transformation, $\Delta_k(\vec{x})$. The self-consistent gap equation is usually solved for a *spatially uniform* order parameter, with $\Delta_k \equiv \Delta_k(\vec{x})$. Singlet pairing implies $\Delta_k = \Delta_{-k}$, and in a d-wave superconductor Δ_k has four zero's or nodes as $\vec{k}$ varies around the Fermi surface. Our strategy will be to obtain an effective field theory which has a *local* d−wave gap function, determined by strong coupling physics below some length scale Λ^{-1} of say 5-10 lattice spacings, but which can fluctuate quantum mechanically on longer spatial scales. These longer length scale quantum fluctuations will be responsible for quantum disordering the d-wave superconductor, and will allow us to access a new phase – the nodal liquid. As we shall see, an important role is played by the d−wave quasiparticles, which survive the quantum disordering. To implement this approach, we first briefly recapitulate the properties of quasiparticles in the d−wave superconductor.

B. Quasiparticles

With spatially uniform d-wave order given by Δ_k, the effective Hamiltonian for the quasiparticles is $H = H_0 + H_1$, with H_0 the kinetic energy and

$$H_1 = \sum_k [\Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \Delta_k^* c_{-k\downarrow} c_{k\uparrow}]. \quad (2.5)$$

Since $\Delta_k = \Delta_{-k}$ for singlet pairing, it is natural to break sums into positive and negative k_y. To do so, consider a four component fermion field, $\Upsilon_{a\alpha}(\vec{k})$, at each wavevector with $k_y > 0$ positive:

$$\Upsilon_{a\alpha}(\vec{k}) = \begin{bmatrix} \Upsilon_{11} \\ \Upsilon_{21} \\ \Upsilon_{12} \\ \Upsilon_{22} \end{bmatrix} = \begin{bmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \\ c_{k\downarrow} \\ -c_{-k\uparrow}^\dagger \end{bmatrix}. \quad (2.6)$$

In the second column vector the minus sign has been introduced so that $\Upsilon_{a\alpha}$ transforms like a spinor under an $SU(2)$ rotation, i.e. as $\Upsilon_{a\alpha} \to U_{\alpha\beta}\Upsilon_{a\beta}$. Here $U = \exp(i\boldsymbol{\theta} \cdot \boldsymbol{\sigma})$ is a global spin rotation with Pauli matrices $\boldsymbol{\sigma}_{\alpha\beta}$.

In these variables, the quasiparticle Hamiltonian becomes

$$H_{qp} = \sum_k{}' \Upsilon^\dagger(\vec{k})[\tau^z \epsilon_k + \tau^+ \Delta_k + \tau^- \Delta_k^*]\Upsilon(\vec{k}), \quad (2.7)$$

where the prime on the summation denotes over k_y positive, only, and we have introduced a vector of Pauli matrices, $\vec{\tau}_{ab}$ acting in the particle/hole subspace. Also, we are employing the notation $\tau^\pm = (\tau^x \pm i\tau^y)/2$.
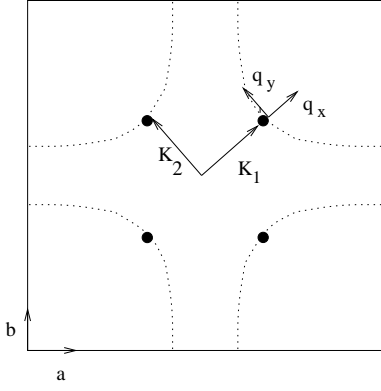
Fig. 2: The wavevectors $\vec{K}_i$, q_x, q_y in relation to the a,b axes. The dotted line represents the putative Fermi surface.

With approximate particle/hole symmetry, the d-wave nodes are located near the special wavevectors $\pm\vec{K}_j$, with $\vec{K}_1 = (\pi/2, \pi/2)$ and $\vec{K}_2 = (-\pi/2, \pi/2)$. Since our aim is to obtain an effective description at low energies and long lengthscales, it is sufficient to focus on the gapless modes near these points, integrating out the electrons far away in the Brillouin zone. It is then convenient to introduce two continuum fields Ψ_j, one for each pair of nodes, expanded around $\pm\vec{K}_1, \pm\vec{K}_2$:

$$\Psi_{ja\alpha}(\vec{q}) = \Upsilon_{a\alpha}(\vec{K}_j + \vec{q}). \qquad (2.8)$$

Here, the wavevectors $\vec{q}$ are assumed to be small, within a circle of radius Λ around the origin. With this definition, the particle/hole transformation is extremely simple,

$$\Psi \to \Psi^\dagger. \qquad (2.9)$$

For this reason it is convenient to *always* define the continuum fields Ψ around $\pm\vec{K}_j$, and account for deviations of the node momenta from these values by a particle/hole symmetry-breaking parameter λ.

Once we have restricted attention to the momenta near the nodes, it is legitimate to linearize in the quasiparticle Hamiltonian. The resulting continuum theory is more conveniently written in coordinates perpendicular and parallel to the Fermi surface, so we perform the rotation via $x \to (x - y)/\sqrt{2}$ and $y \to (x + y)/\sqrt{2}$, correspondingly transforming the momenta q_x and q_y (see Fig. 2). Linearizing near the nodes, we put $\epsilon_{K_1+q} = v_F q_x$ where v_F is the Fermi velocity and $\Delta_{K_1+q} = \tilde{\Delta} q_y$, where $\tilde{\Delta}$ has dimensions of a velocity. An identical linearization is possible around the second pair of nodes, except with $q_x \leftrightarrow q_y$. Upon Fourier transforming back into real space, $\Psi_j(\vec{q}) \to \Psi_j(\vec{x})$, we arrive at a compact form for the Hamiltonian density of the quasiparticle excitations in the d-wave superconductor: $\mathcal{H}_{qp} = \mathcal{H}_\Psi + \mathcal{H}_\lambda$ with

$$\mathcal{H}_\Psi = \Psi_1^\dagger[v_F \tau^z i\partial_x + (\tilde{\Delta}\tau^+ + \tilde{\Delta}^*\tau^-)i\partial_y]\Psi_1$$
$$+ (1 \leftrightarrow 2; x \leftrightarrow y), \qquad (2.10)$$

and the particle/hole symmetry breaking term,

$$\mathcal{H}_\lambda = \lambda\Psi_j^\dagger\tau^z\Psi_j. \qquad (2.11)$$

The quasiparticle Hamiltonian takes the form of (four) Dirac equations, and can be readily diagonalized giving a dispersion relation for the first pair of nodes,

$$E_1(\vec{q}) = \pm\sqrt{(v_F q_x + \lambda)^2 + |\tilde{\Delta}|^2 q_y^2}, \qquad (2.12)$$

and a similar expression with q_x and q_y exchanged for the second pair. Notice that non-zero λ indeed shifts the positions of the nodes.

C. Quantum Fluctuations

Up to this point, we have taken a spatially constant gap function $\tilde{\Delta}(\vec{x})$. To disorder the d-wave superconductor it is necessary to allow for quantum fluctuations of this order parameter. It is tempting to uniformly suppress the complex order parameter, and simply put $\tilde{\Delta} = 0$. But doing so recovers the conventional metallic state with a Fermi surface. Our task is trickier, since we are searching for an *intermediate* phase, which has strong local $d-$wave pairing (which destroys the Fermi surface) but with longer length scale quantum fluctuations destroying the superconducting phase coherence. Our task is similar to the problem of describing the hexatic phase in a classical two-dimensional triangular solid,[17] which is intermediate between the crystalline and liquid phases. Guided by this example and the principle of pairing below T^*, we want to fix the *magnitude* of the complex pair field, and introduce fluctuations of its phase.

Pursuant to this goal, we write

$$\tilde{\Delta} \to v_\Delta e^{i\varphi}, \qquad (2.13)$$

where v_Δ is real and φ can be interpreted as the phase of the complex superconducting order parameter. The BCS gap equation has a degenerate manifold of solutions, for arbitrary phase φ. This degeneracy is responsible for the Goldstone modes, wherein φ varies slowly in both space and time. Our goal is to obtain an effective theory for the space and time dependence of φ, similar in spirit to the non-linear sigma models "derived" for localization.[18] Specifically, we focus on spatial variations of $\varphi(\vec{x})$ on scales longer than Λ^{-1}. Since φ can vary spatially, care is needed in introducing it into the quasiparticle Hamiltonian:

$$\tilde{\Delta}\tau^+ i\partial_y \to v_\Delta\tau^+ e^{i\varphi/2}(i\partial_y)e^{i\varphi/2}. \qquad (2.14)$$

This symmetric form leads to an hermitian Hamiltonian, physical currents, and respects the symmetries of the problem. A careful derivation of Eq. 2.14 is given in

Appendix A. With this prescription, the quasiparticle Hamiltonian becomes

$$\mathcal{H}_{qp} = \sum_{s=\pm} \Psi_1^\dagger [v_F \tau^z i\partial_x + v_\Delta \tau^s e^{is\varphi/2}(i\partial_y)e^{is\varphi/2}]\Psi_1$$
$$+(1 \leftrightarrow 2; x \leftrightarrow y). \tag{2.15}$$

Since φ can also fluctuate with time, it will convenient to consider the time dependence via a Lagrangian formulation. The Lagrangian density is

$$\mathcal{L}_{qp} = \Psi_j^\dagger i\partial_t \Psi_j - H_{qp}. \tag{2.16}$$

The appropriate Lagrangian for the phase of the d-wave order parameter is simply

$$\mathcal{L}_\varphi = \frac{1}{2}\kappa_\mu(\partial_\mu\varphi)^2, \tag{2.17}$$

where the Greek index μ runs over time and two spatial coordinates: $\mu = 0, 1, 2 = t, x, y$. Here κ_0 is equal to the compressibility of the condensate and $\kappa_j = -v_c^2 \kappa_0$ (for $j = 1, 2 = x, y$) with v_c the superfluid sound velocity. We expect that the pair compressibility κ_0 is approximately one half the electron compressibility of the original electron model – in the absence of interactions. If the pairing is electronic in origin, one expects that the scale for the "charge velocity" v_c is the Fermi velocity.

As discussed in the introduction, treatment of quantum phase fluctuations is complicated by the mixing of particle and hole variables via the complex gap function. To isolate the uncertain charge, we therefore perform a change of variables, defining a new set of fermion fields ψ_j via

$$\psi_j = \exp(-i\varphi\tau^z/2)\Psi_j. \tag{2.18}$$

In the superconducting phase, and in the absence of quantum flucutations of the order-paramater phase, one can set $\varphi = 0$, and these new fermions are simply the d-wave quasiparticles. However, when the field φ is dynamical and fluctuates strongly this change of variables is non-trivial. In particular, the new fermion fields ψ are electrically *neutral*, invariant under a global charge $U(1)$ transformation. As we shall see, when the d-wave superconductivity is quantum disordered, these new fields will play a fundamental role, describing low energy gapless excitations, centered at the former nodes. For this reason, we refer to these fermions as *nodons*. For completeness, we quote the symmetry properties of the nodon field under a particle/hole transformation. Since $\varphi \to -\varphi$, one has simply

$$\psi \to \psi^\dagger. \tag{2.19}$$

The full Lagrangian in the d-wave superconductor, $\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_{qp}$, can be conveniently re-expressed in terms of these nodon fields since $\mathcal{L}_{qp} = \mathcal{L}_\psi + \mathcal{L}_{int} + \mathcal{L}_\lambda$ with a free nodon piece,

$$\mathcal{L}_\psi = \psi_1^\dagger [i\partial_t - v_F \tau^z i\partial_x - v_\Delta \tau^x i\partial_y]\psi_1$$
$$+(1 \leftrightarrow 2, x \leftrightarrow y), \tag{2.20}$$

interacting with the phase of the order-parameter:

$$\mathcal{L}_{int} = \partial_\mu\varphi J_\mu. \tag{2.21}$$

Here the electrical 3-current J_μ is given by

$$J_0 = \frac{1}{2}\psi_j^\dagger \tau^z \psi_j, \tag{2.22}$$

$$J_j = \frac{v_F}{2}\psi_j^\dagger \psi_j. \tag{2.23}$$

Because the transformation in Eq. 2.18 is local, identical expressions hold for these currents in terms of the quasiparticle fields, Ψ. The form of the particle/hole asymmetry term remains the same in terms of the nodon fields:

$$\mathcal{L}_\lambda = \lambda\psi_j^\dagger \tau^z \psi_j. \tag{2.24}$$

It is instructive to re-express the components of the currents J_μ back in terms of the original electron operators. One finds

$$J_0 = \frac{1}{2}(c_{K_j}^\dagger c_{K_j} + c_{-K_j}^\dagger c_{-K_j}), \tag{2.25}$$

(with an implicit spin summation) which corresponds physically to the total electron density living at the nodes, in units of the Cooper pair charge. Similarly,

$$J_j = \frac{v_F}{2}(c_{K_j}^\dagger c_{K_j} - c_{-K_j}^\dagger c_{-K_j}) \tag{2.26}$$

corresponds to the *current* carried by the electrons at the nodes. Thus, J_μ can be correctly interpreted as the quasiparticles three-current.

To complete the description of a quantum mechanically fluctuating order parameter phase interacting with the gapless fermionic excitations at the nodes, we minimally couple to an external electromagnetic field, A_μ. Since the nodon fermions are neutral, the only coupling is to the order-parameter phase, via the substitution $\partial_\mu\varphi \to \partial_\mu\varphi - A_\mu$. For simplicity, here and in the rest of the paper, we have set the Cooper pair charge $2e = 1$. The final Lagrangian then takes the form $\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_\psi + \mathcal{L}_{int} + \mathcal{L}_\lambda$, with

$$\mathcal{L}_\varphi = \frac{1}{2}\kappa_\mu(\partial_\mu\varphi - A_\mu)^2, \tag{2.27}$$

$$\mathcal{L}_{int} = (\partial_\mu\varphi - A_\mu)J_\mu, \tag{2.28}$$

and $\mathcal{L}_\psi$ still given by Eq. 2.20.

The time component of the electromagnetic field is proportional to the chemical potential μ, i.e. $A_0 = 2\mu$. For electrons at half-filling one has $\mu = 0$. Doping can be

achieved by changing μ. Long-ranged Coulomb interactions could be readily incorporated at this stage by treating A_0 as a dynamical field and adding a term to the Lagrangian of the form, $\mathcal{L}_{coul} = (1/2)(\partial_j A_0)^2$. The spatial components of the electromagnetic field, A_j, have been included to keep track of the current operator. In particular, the total electrical 3-current is obtained by differentiating the Lagrangian, i.e. $J_\mu^{tot} = \partial\mathcal{L}/\partial A_\mu$, which gives

$$J_\mu^{tot} = \kappa_\mu(\partial_\mu\varphi - A_\mu) + J_\mu. \qquad (2.29)$$

Here the first terms are the Cooper pair 3-current, and the second the quasiparticles current. The equation of motion for the phase of the order-parameter, $\partial\mathcal{L}/\partial\varphi = 0$, implies the continuity equation $\partial_\mu J_\mu^{tot} = 0$.

D. Duality

To quantum-disorder the d-wave superconductor, one must allow for *vortices* in the pair-field phase, φ. We do this using field-theoretic duality, as described, e.g. in Ref. 13. To this end we introduce a vortex 3-current, j_μ^v, which satisfies,

$$j_\mu^v = \epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\varphi. \qquad (2.30)$$

In the presence of vortices, φ is multi-valued, $\partial_\mu\varphi$ is not curl-free, and j_μ^v is non-vanishing. In the desired dual representation, the vortices become the quantized particles, rather than the Cooper pairs. However, even in the dual representation one still needs to conserve the total electrical charge. This can be achieved by expressing the total electrical 3-current as a curl,

$$J_\mu^{tot} = \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda, \qquad (2.31)$$

where we have introduced a "fictitious" dynamical gauge field, a_μ. Upon combining Eqs. 2.29-2.31, one can eliminate the pair-field phase, φ, and relate a_μ to the vortices:

$$j_\mu^v = \epsilon_{\mu\nu\lambda}\partial_\nu[\kappa_\lambda^{-1}\epsilon_{\lambda\alpha\beta}\partial_\alpha a_\beta + A_\lambda - \kappa_\lambda^{-1}J_\lambda], \qquad (2.32)$$

where J_μ is the quasiparticle 3-current defined earlier in Eqs. 2.22-2.23.

A dual description is obtained by constructing a Lagrangian, $\mathcal{L}_D$, depending on a_μ, J_μ and j_μ^v, whose equation of motion, obtained by differentiating with respect to a_μ, leads to the above equation. To assure that the vortex 3-current is conserved, it is useful to introduce a complex field, Φ, which can be viewed as a vortex destruction operator. Since a vortex acquires a 2π phase upon encircling a Cooper pair, the vortex field should be minimally coupled to a_μ. The appropriate dual Lagrangian can be conveniently decomposed as $\mathcal{L}_D = \mathcal{L}_\psi + \mathcal{L}_v + \mathcal{L}_a$, where $\mathcal{L}_\psi$ is given in Eq. 2.20. The vortex piece has the Ginzburg-Landau form,[3]

$$\mathcal{L}_v = \frac{\kappa_\mu}{2}|(\partial_\mu - ia_\mu)\Phi|^2 - V_\Phi(|\Phi|), \qquad (2.33)$$

where Φ is a (dimensionless) complex field for the vortices. The vortex 3-current, following from $j_\mu^v = \partial\mathcal{L}_v/\partial a_\mu$, is

$$j_\mu^v = \kappa_\mu\mathrm{Im}[\Phi^*(\partial_\mu - ia_\mu)\Phi]. \qquad (2.34)$$

For small $|\Phi|$ (appropriate close to a second order transition) one can expand the potential as, $V_\Phi(X) = r_\Phi X^2 + u_\Phi X^4$. The remaining piece of the dual Lagrangian is

$$\mathcal{L}_a = \frac{1}{2\kappa_0}(e_j^2 - b^2) + a_\mu\epsilon_{\mu\nu\lambda}\partial_\nu(A_\lambda - \kappa_\lambda^{-1}J_\lambda), \qquad (2.35)$$

with dual "magnetic" and "electric" fields: $b = \epsilon_{ij}\partial_i a_j$ and $e_j = v_c^{-1}(\partial_j a_0 - \partial_0 a_j)$. It is straightforward to verify that the dual Lagrangian has the desired property that Eq. 2.32 follows from the equation of motion $\partial\mathcal{L}_D/\partial a_\mu = 0$.

III. NODAL LIQUID PHASE

In this section we employ the dual representation of the d-wave superconductor to analyze the quantum disordered phase - a new phase of matter which we refer to as a *nodal liquid*. The dual representation comprises a complex vortex field, which is minimally coupled to a gauge field, as well as a set of neutral nodon fermions. Without the nodons and in imaginary time, the dual Lagrangian is formally equivalent to a classical three-dimensional superconductor at finite temperature, coupled to a fluctuating electromagnetic field. To disorder the d-wave superconductor, we must order the dual "superconductor" – that is, condense the vortices. The nature of the resulting phase will depend sensitively on doping, since upon doping, the dual "superconductor" starts seeing an applied "magnetic field". Below, we first consider the simpler case of half-filling. We then turn to the doped case, where two scenarios are possible depending on whether the dual "superconductor" is Type I or Type II.[3]

A. Half-filling

Specialize first to the case of electrons at half-filling, with particle-hole symmetry. In the dual representation, the "magnetic field", b, is equal to the deviation of the total electron density from half-filling. Thus at half-filling $\langle b \rangle = 0$ and the dual Ginzburg-Landau theory is in zero applied field. The quantum disordered phase corresponds to condensing the vortices, setting $\langle\Phi\rangle = \Phi_0 \neq 0$. In this dual Meissner phase, the vortex Lagrangian becomes

$$\mathcal{L}_v = \frac{1}{2}\kappa_\mu\Phi_o^2 a_\mu^2. \qquad (3.1)$$

It is then possible to integrate out the field a_μ. The full Lagrangian in the nodal liquid phase is then

$$\mathcal{L}_{nl} = \mathcal{L}_\psi + A_\mu I_\mu + \frac{\epsilon_0}{2} E_j^2 - \frac{B^2}{2\mu_0} + O\left[(\partial J)^2\right], \quad (3.2)$$

where we have introduced the physical magnetic and electric fields: $B = \epsilon_{ij}\partial_i A_j$ and $E_j = \partial_j A_0 - \partial_t A_j$. The last two terms describe a dielectric, with magnetic permeability $\mu_0 = \kappa_0\Phi_0^2$ and dielectric constant $\epsilon_0 = (\mu_0 v_c^2)^{-1}$, with the sound velocity entering, rather than the speed of light. The external electromagnetic field is coupled to the 3-current I_μ, which can be expressed as a bi-linear of the nodon fermions as,

$$I_\mu = \frac{\epsilon_0}{\kappa_0^2 v_c^2}[\kappa_\nu \partial_\nu^2 J_\mu - \kappa_\mu \partial_\mu(\partial_\nu J_\nu)]. \quad (3.3)$$

Notice that this 3-current is automatically conserved: $\partial_\mu I_\mu = 0$.

The order $(\partial J)^2$ terms which we have not written out explicitly are quartic in the fermion fields, and also involve two derivatives. Since $\mathcal{L}_\psi$ describes Dirac fermions in $2+1$ space-time dimensions, these quartic fermion terms are highly irrelevant, and rapidly vanish under a rescaling transformation. Thus, in the absence of external electromagnetic fields, the description of the nodal liquid phase is exceedingly simple. It consists of four neutral Dirac fermion fields – two spin polarizations ($\alpha = 1, 2$) for each of the two pairs of nodes.

Despite the free fermion description, the nodal liquid phase is highly *non-trivial* when re-expressed in terms of the underlying electron operators. Indeed, the ψ fermion operators are built from the quasiparticle operators Ψ in the d-wave superconductor, but are electrically neutral, due to the "gauge transformation" in Eq. 2.18. Moreover, in the nodal liquid phase, the Cooper pairs are *not* superconducting, but rather in a dielectric Mott-insulating phase,[19] immobilized by their commensurability with the underlying crystal lattice. Although the ψ fermions are electrically neutral, they do carry a new conserved "charge". In fact, there are *four* new conserved charges, since the Lagrangian is invariant under the global transformations $\psi_{ja\alpha} \rightarrow e^{i\theta_{j\alpha}}\psi_{ja\alpha}$ for arbitrary constant phases, $\theta_{j\alpha}$, with $j = 1, 2$ and $\alpha = 1, 2$. We refer to the ψ fields as *nodon* operators, their quanta as *nodons*, and the associated conserved quantities as "nodon charges". As seen from Eq. 2.20, these conserved charges are related to the quasiparticle current, since $J_j = (v_F/2)\psi_j^\dagger \psi_j$. However, in the nodal liquid phase the electrical current operator is I_μ, not J_μ, since $I_\mu = \partial\mathcal{L}_{nl}/\partial A_\mu$.

1. Spin response

The spin response functions in the nodal liquid are rather straightforward, since the electron spin operators have a simple representation in terms of the nodons. In particular, consider the spin operator for small momentum,

$$\mathbf{S}_q = \frac{1}{2}\sum_k c_{k+q}^\dagger \boldsymbol{\sigma} c_k. \quad (3.4)$$

At low energies in the nodal liquid phase one can focus on momenta near the nodes: $\vec{k} = \pm\vec{K}_j$. The electron operators near the nodes can be rewritten in terms of the nodon operators, and one finds that back in real space the long-wavelength piece of the spin operator, $\mathbf{S}(\vec{x}) = \sum_q \exp(i\vec{q}\cdot\vec{x})\mathbf{S}_q$, is simply

$$\mathbf{S}(\vec{x}) = \frac{1}{2}\psi_{ja}^\dagger(\vec{x})\boldsymbol{\sigma}\psi_{ja}(\vec{x}). \quad (3.5)$$

Spin correlation and response functions can then be computed from the free nodon theory. For example, the uniform spin susceptibility is given by

$$\chi = \int_0^\infty dE(-\partial f/\partial E)\rho_n(E), \quad (3.6)$$

where the nodon density of states is $\rho_n(E) = (const)E/v_F v_\Delta$, and $f(E)$ is a Fermi function. One finds $\chi \sim T/v_F v_\Delta$. There are also low energy spin excitations at wavevectors which span between two different nodes. The associated spin operators can readily be be re-expressed in terms of the nodon fields. For example, the staggered magnetization operator, $\mathbf{S}_{\vec{\pi}}$, is

$$\mathbf{S}_{\vec{\pi}} = \frac{1}{2}\left[\psi^\dagger(\tau^y\boldsymbol{\sigma}\sigma^y)\psi^\dagger + \text{h.c.}\right]. \quad (3.7)$$

Notice that this operator is actually "anomalous" in terms of the conserved nodon charge. We will return to the effects of finite wavevector magnetic fluctuations and ordering in Section IV.

In addition to carrying spin, the nodons carry energy, and so will contribute to the thermal transport. At finite temperature, Umklapp scattering processes (or impurities) give a finite thermal conductivity; in their absence the nodon thermal conductivity is infinite.

2. Charge response

The electrical charge properties in the nodal liquid phase are, however, somewhat trickier. Imagine changing the chemical potential away from $\mu = 0$. In terms of the dual vortex "superconductor" this corresponds to applying an external "magnetic" field, due to the coupling $\mathcal{L}_\mu = -2\mu b$. The vortices, however, are in the "Meissner" phase, and for $\mu \leq \mu_c$ the applied field will be screened out, maintaining the internal field at $b = 0$. That is, the electron density will be pinned at half-filling, until the chemical potential passes through the Mott gap for the insulating system of Bosons (Cooper pairs).[19]

Despite the presence of a charge gap, there are low energy current fluctuations in the nodal liquid. Indeed, in this phase the electrical current operator is I_μ, which is bi-linear in terms of the nodon fermions, ψ. One might

imagine employing this current operator to compute the electrical conductivity in the nodal liquid. For this, one requires computing a two-point correlator of the electrical current operator at zero wavevector (say in the x−direction) $I_x(q = 0) = (\epsilon_0/\kappa_o v_s^2)\partial_t^2 J_x(q = 0)$. But notice that $J_x(q = 0)$ is proportional to a globally conserved nodon charge, since $J_x(\vec{x}) = (v_F/2)\psi_1^\dagger\psi_1$. Thus, when the nodon number is conserved one has $I_x(q = 0) = 0$, and the nodons *do not* contribute to the electrical conductivity. (There will of course be a response at finite frequencies in the imaginary part of the conductivity from the Mott-insulating phase of the Cooper pairs.)

When impurities or Umklapp scattering is present, however, the nodon number is no longer conserved, and the nodons presumably will contribute to the real part of the electrical conductivity, at least at finite frequencies. Specifically, the umklapp scattering term with momentum transfer $2\vec{\pi}$ is given by

$$\mathcal{L}_{umklapp} = u\,\epsilon^{ABCD}\psi_{jA}\psi_{jB}\psi_{jC}\psi_{jD} + \text{h.c.}, \qquad (3.8)$$

where the composite index A runs over $1, 2, 3, 4$ corresponding to $a\alpha = 11, 12, 21, 22$. By power counting, this term is irrelevant by one power of frequency. Hence, it enters scaling forms in the combination $u_{eff} = u\omega$.

According to the Kubo formula,

$$\sigma(\omega) \sim \frac{1}{\omega^3}\langle I_x(q = 0, \omega)I_x(q = 0, -\omega)\rangle$$
$$\sim \omega^3\langle J_x(q = 0, \omega)J_x(q = 0, -\omega)\rangle. \qquad (3.9)$$

From scaling, we expect the latter correlation function to vary as ω. However, as we noted above, it actually vanishes in the absence of Umklapp scattering; therefore it is determined by the correction to scaling, which, naively, is of the form:

$$\sigma(\omega) \sim \omega^3 \cdot \omega \cdot u_{eff}^2 \sim u^2\omega^6. \qquad (3.10)$$

Finally, it is instructive to consider the behavior of the electron Green's function, which can be accessed in photo-emission and tunneling experiments. The electron operator $c_\alpha(\vec{x})$ can be conveniently decomposed in terms of the nodon operators by focusing on momenta near the nodes. For example, near the node at $\vec{K}_j$ one can write,

$$c_\alpha(\vec{x}) = e^{i\vec{K}_j\cdot\vec{x}}e^{i\varphi(\vec{x})/2}\psi_{j1\alpha}(\vec{x}) + ... \qquad (3.11)$$

In the nodal liquid phase, the electron Green's function, $G(\vec{x}, t) = \langle c^\dagger(\vec{x}, t)c(\vec{0}, 0)\rangle$ factorizes as,

$$G(\vec{x}, t) = e^{i\vec{K}_j\cdot\vec{x}}\langle e^{-i\varphi(\vec{x},t)/2}e^{i\varphi(\vec{0},0)/2}\rangle\langle\psi_{j1\alpha}^\dagger(\vec{x}, t)\psi_{j1\alpha}(\vec{0}, 0)\rangle.$$
$$(3.12)$$

Although the nodon correlator is a power law, falling off as $|x|^{-2}$ and t^{-2}, one expects the correlator over exponentials of the pair field phase to fall off exponentially in the nodal liquid, since the Cooper pairs (Bosons) are locked

in a Mott insulating phase. This indicates a gap in the electron spectral function at the Fermi energy, of order $\Delta_1 \sim \mu_c$ at the nodes. If μ_c is small relative to T^*, the corresponding gap will show strong four-fold anisotropy in momentum space, varying from of order T^* down to of order μ_c near the nodon wave-vectors. In the discussion section, we comment briefly on how such a d-wave pseudo-gap feature is likely enhanced when the NL is doped in the presence of impurities.

B. Doping the Nodal Liquid

We now consider the effects of doping charge into the nodal liquid phase. In a grand canonical ensemble this is achieved by changing the chemical potential, $\mu = A_0/2$. In the dual Ginzburg-Landau description of the vortices, a chemical potential acts as an applied dual field, as seen from Eq. 2.35, since

$$\mathcal{L}_\mu = -2\mu b. \qquad (3.13)$$

The dual magnetic field, $b = \epsilon_{ij}\partial_i a_j$, is the total electric charge in units of $2e$. For a hole doping with concentration x, one has $\langle b\rangle = x/2a_0^2$, with a_0 the crystal lattice constant. Provided the applied dual field, 2μ, is smaller than the critical field $(2\mu_c)$ of the Ginzburg-Landau theory, the dual superconductor stays in the Meissner phase – which is the nodal liquid phase at half-filling. But for $\mu \geq \mu_c$ dual flux will penetrate the Ginzburg-Landau superconductor, which corresponds to doping the nodal liquid phase. The form of the dual flux penetration will depend critically on whether the dual Ginzburg-Landau theory is Type I or Type II. Within a mean-field treatment this is determined by the ratio of the dual penetration length, λ_v, to the dual coherence length, ξ_v (where the subscript v denotes vortices). In particular, Type II behavior is expected if $\lambda_v/\xi_v \geq 1/\sqrt{2}$, and Type I behavior otherwise. In the Ginzburg-Landau description λ_v determines the size of a dual flux tube, which is simply a Cooper pair. We thus expect that λ_v will be roughly equal to the superconducting coherence length, ξ, which is perhaps $15 - 20\mathring{A}$ in the cuprates. On the other hand, ξ_v is the size of the "vortex-core" in the dual vortex field, and presumably can be no smaller than the microscopic crystal lattice spacing, $\xi_v \geq 3 - 5\mathring{A}$. This reasoning suggest that λ_v/ξ_v is probably close to unity, so that either Type I or Type II behavior might be possible - and could be material dependent. We first consider such Type II doping, returning below to the case of a Type I Ginzburg-Landau theory.

1. Type II Behavior

The phase diagram of a clean three-dimensional type II superconductor is well understood.[3] Above the lower critical field, H_{c1}, flux tubes penetrate, and form an

Abrikosov flux lattice - usually triangular. As the applied field increases the flux tubes start overlapping, when their separation is closer than the penetration length. Upon approaching the upper critical field H_{c2} their cores start overlapping, the Abrikosov flux lattice disappears, and the superconductivity is destroyed. Mean field theory predicts a second order transition at H_{c2}, but with thermal fluctuations one expects this to become weakly first order.[20] This weak first order transition separates a flux-lattice phase from a non-superconducting flux-liquid.

These results hold equally well for our dual Ginzburg-Landau superconductor, except that now the direction parallel to the applied field is actually imaginary time. Moreover, the Ginzburg-Landau order parameter describes quantum vortices, and the penetrating flux tubes are Cooper pairs. Upon doping the nodal liquid with $\mu > \mu_{c1}$, charge is added to the 2d system, which corresponds to the penetration of dual magnetic flux. In this dual transcription, the resulting Abrikosov flux-lattice phase is a Wigner crystal of Cooper pairs, with one Cooper pair per real space unit cell of the lattice. We denote this paired Wigner crystal phase by PWC. Upon further doping, one passes via a weak first order transition (at $\mu = \mu_{c2}$) into the dual flux-liquid phase. In this phase the lattice of Cooper pairs has melted, and they are free to condense - this is the d-wave superconductor. This latter transition should occur when the spacing between dual flux tubes becomes roughly comparable to the coherence length, ξ_v. Experimentally, superconductivity typically sets in for $x = 0.1$, which corresponds to one Cooper pair for every 20 or so Cu atoms, and a mean pair separation of 4-5a_0. This again suggests that λ_v/ξ_v is probably of order one.

In the Cooper pair Wigner crystal phase, translational symmetry is spontaneously broken. However, in a real material the Wigner crystal will have a preferred location, determined by impurities and perhaps crystal fields, which will tend to pin and immobilize the Wigner crystal. The resulting phase should be an electrical insulator. Moreover, in two-dimensions even weak impurities will smooth the weakly first order transition between the Wigner crystal and superconducting phases. (In the absence of impurities, long-ranged Coulomb interactions preclude phase separation, so a mixed phase would result - see Type I behavior below.)

A striking and unusual feature of the PWC phase, is that it *co-exists* with the nodal liquid, as we now argue. With a weak (commensurate) pinning potential present, the Wigner crystal phase is a dielectric. The charge response is thus essentially the same as that of the undoped phase at half-filling, except with a modified dielectric constant and magnetic permeability. Thus even with doping, it is possible to "integrate out" the charge fluctuations described by the fields a_μ, and arrive at the nodal liquid Lagrangian Eq. 3.2, except with different values of ϵ_0 and μ_0. The only complication is that there will be a background frozen in charge density, from the Wigner crystal of Cooper pairs, so that

$\langle b(\vec{x}) \rangle = n(\vec{x}) = n_0 + \delta n(\vec{x})$. Here, the mean pair density is simply $n_0 = x/2a_0^2$ for doping x, and $\delta n(\vec{x})$ has the periodicity of the Wigner crystal. This "background" field couples to the nodons, and from Eq. 2.35 leads to a term of the form,

$$\mathcal{L}_b = \langle b(\vec{x}) \rangle \kappa_0^{-1} J_0 = \frac{1}{2\kappa_0} n(\vec{x}) \psi_j^\dagger \tau^z \psi_j. \qquad (3.14)$$

What is the effect of this term on the nodons? Consider first the spatially constant piece, proportional to n_0. This term can be absorbed into $\mathcal{L}_\psi$, which contains a term of the form, $\psi_j^\dagger v_F q_j \tau^z \psi_j$, and leads to a momentum space shift of the nodes, with $q_j \rightarrow q_j + (n_0/2v_F\kappa_0)$. Since the pair compressibility $\kappa_0 \sim \partial n_0/\partial \mu$, the shift satisfies $v_F \delta q \sim \delta \mu$, as expected from the change of the area enclosed by the Fermi surface upon doping.

The spatially varying background density, $\delta n(\vec{x})$, which has the periodicity of the underlying Wigner crystal, causes a mixing between nodon states at momentum differing by Wigner crystal reciprocal lattice vectors. When the reciprocal lattice vectors are *larger* than the momentum cutoff, Λ, of the nodons, the $\delta n(\vec{x})$ term cannot scatter within the low energy nodon theory, and can be dropped. At lower pair densities, it becomes necessary over a range of length scales to retain the new periodicity by working with nodon Bloch states, rather than plane waves. (Since only linear derivatives enter into the nodon Lagrangian, Bloch wavefunctions in the lowest band can be readily constructed.) In any event, at length scales larger than the Wigner crystal lattice spacing, the form of the effective theory of the nodons is identical to that at half-filling.

We thereby arrive at a description of a rather remarkable new phase of matter. A Paired Wigner Crystal (PWC) of doped Cooper pairs co-exists with neutral gapless fermionic excitations – the nodons. In this co-existing phase, which we denote as PWC/NL, low energy spin and thermal properties will be dominated by the nodons. The behavior will be qualitatively similar to that in the undoped nodal liquid phase. We propose that this PWC/NL phase is present in the pseudo-gap region of the high T_c cuprates.

2. Type I Behavior

In a classical Type I superconductor, the applied field is expelled until the critical H_c is exceeded.[3] At this point there is a *first order* phase transition from the Meissner phase with all the flux expelled, to a normal metal phase in which (essentially) all the field penetrates. If a thin film type I superconductor is placed in a perpendicular field, screening currents are unable to expel all the flux, and a "mixed" or "intermediate" state occurs. In this mixed phase, regions of superconductivity co-exist with normal metallic regions. In some cases the superconducting regions form stripes, but generally the lowest energy

configurations are determined in large part by material imperfections, and tend to be "history" dependent.

If our dual Ginzburg-Landau theory describing quantized vortices is of type I, then similar properties are expected. Specifically, as the chemical potential increases, the dual field – which is the Cooper pair density – remains at zero until a critical chemical potential μ_c is reached. At this point there is a first order phase transition, between the nodal liquid phase at half-filling, and a d-wave superconductor at finite doping, x_c. At fixed doping $x < x_c$, phase separation is impeded by long-ranged Coulomb interactions between the Cooper pairs. The system will break apart into co-existing "micro-phases" of nodal liquid and d-wave superconductivity. The configuration of the co-existing "micro-phases" will be determined by a complicated competition between the Coulomb energy and the (positive) energy of the domain walls. In practice, impurities will also probably play a very important role. This doping scenario is similar to that envisaged by Emery and Kivelson,[15] who have extensively discussed the possibility of phase separation as a mechanism for high T_c superconductivity. Unfortunately, with the transition being strongly first order in this case, the associated physics is rather non-universal.

IV. ANTIFERROMAGNETISM IN THE NODAL LIQUID

A. Effective Action for Antiferromagnetism

We now turn to the low-doping region of the phase diagram of Fig. 1. Retaining the Nodal Liquid as the underlying description of the low-energy fermionic degrees of freedom, we consider antiferromagnetic ordering. In principle, this can arise in two ways. Antiferromagnetism could stem from interactions between nodons, i.e. physics below the scale T^*. This can be modeled in principle by including simple inter-nodon interactions. The experimental coincidence of T^* with the Néel temperature and magnon bandwidth J suggests that such a separation of scales is not valid. Instead, antiferromagnetic correlations may exist already at (high) energies comparable to T^*. This sort of local AF amplitude could be captured by decoupling spin-spin (or, e.g. on-site Hubbard) interactions in a microscopic model with a Hubbard-Stratonovich transformation. Such a decoupling introduces a conjugate field $\mathbf{M}$, which interacts with the electron operators via a term of the form $H_M = \sum_{\vec{x}} \mathbf{M}(\vec{x}) \cdot c_\alpha^\dagger \frac{1}{2} \boldsymbol{\sigma}_{\alpha\beta} c_\beta$. Integrating out high-energy degrees of freedom generates an effective action for $\mathbf{M}$. We expect dominant ordering tendencies at momentum $\vec{\pi} = (\pi, \pi)$, and so decompose

$$\mathbf{M} \sim \mathbf{M}_0 + \exp(i\vec{\pi} \cdot \vec{x})\mathbf{N}, \qquad (4.1)$$

where $\mathbf{N}$ and $\mathbf{M}_0$ are slowly-varying. The fields $\mathbf{M}_0$ and $\mathbf{N}$ have the physical interpretation of the coarse-grained

uniform and staggered magnetization. Focusing on the Néel ordering, we imagine integrating out $\mathbf{M}_0$ to obtain the Lagrangian

$$\mathcal{L} = \frac{1}{2}K_\mu|\partial_\mu \mathbf{N}|^2 - V_N(|N|) + g\mathbf{N} \cdot \mathbf{S}_{\vec{\pi}}, \qquad (4.2)$$

where $K_0 = K$, $K_1 = K_2 = -v_s^2 K$, with v_s the spin-wave velocity in the AF. Here $\mathbf{S}_{\vec{\pi}}$ is the spin operator at momentum $\vec{\pi}$, expressed as a bi-linear in terms of the electron operators as in Section III. The staggered magnetization operator can be readily re-expressed in terms of the nodons as,

$$\mathbf{S}_{\vec{\pi}} = \frac{1}{2}\left[\psi^\dagger \tau^y \boldsymbol{\sigma} \sigma^y \psi^\dagger + \text{h.c.}\right], \qquad (4.3)$$

given earlier in Eq. 3.7. Near any phase transitions, and for most phenomenological purposes, it is sufficient to take a simple form for the potential: $V_N(|N|) = r_N|N|^2 + u_N|N|^4$. The parameter r_N controls the presence or absence of AF order. In mean-field theory, and neglecting for the moment the nodon coupling g, the ground state passes from long-range to short-range AF order as r_N is tuned from negative to positive. A precise determination of r_N, u_N, and g in terms of t, t', U, x, etc. is the province of a more microscopic theory.

We note that, in principle, Eq. 4.2 allows for the possibility of incommensurate spin-density-wave ordering at wavevectors other than (π, π), which would correspond to a state which spontaneously develops a spatially periodic expectation value for $\mathbf{N}$. We find such ordering unlikely within this model, however, and have therefore not included terms responsible for locking in possible higher-order commensurate magnetic wavevectors. Since incommensurate order seems not to be realized experimentally at low doping, we hope this omission is unimportant.

B. Magnetism and Phases at Half-Filling

Once we have coupled in the Néel order parameter field, we can describe magnetic phases, in addition to the nodal liquid and d-wave superconducting phases of earlier sections. Here we first focus on the situation at half-filling, where our effective field theory already describes a number of magnetic and non-magnetic ground states. It will be useful to further specialize initially to models with particle-hole symmetry, returning later to the half-filled but particle-hole asymmetric case below.

1. Particle-hole symmetric case

The full effective Lagrangian has two order parameter fields, the Néel order parameter, $\mathbf{N}$, and the vortex complex field, Φ, which is minimally coupled to a gauge field, a_μ. The Néel order parameter is directly coupled to the

nodons, whereas the vortex field only sees the nodons indirectly via the gauge field. Ordering of the two fields is determined by the coefficients of the quadratic terms in the Lagrangian, namely r_N and r_Φ. It will be convenient to plot the phase diagram at half-filling in the $r_N - r_\Phi$ plane. The phase diagram with particle-hole symmetry ($\lambda = 0$) is shown in Fig. 3a. Here we briefly discuss each of the four phases.
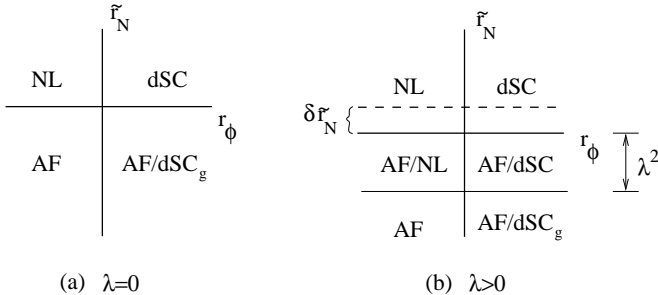

(a) λ=0   (b) λ>0

Fig. 3: Phase diagrams at half-filling for the particle-hole (a) symmetric and (b) asymmetric cases. In (b), both horizontal phase boundaries shift downward with increasing particle/hole asymmetry parameter λ, diminishing the domain of the AF phase.

Consider first r_N large and positive, so that Néel order is not present. For r_Φ negative, the vortices will condense (the Meissner phase in the dual Ginzburg-Landau theory) leaving the nodons as the only low energy gapless excitations. This is the nodal liquid phase. As r_Φ changes sign, the vortices will disorder, entering the "normal" phase of the dual Ginzburg-Landau theory. This phase corresponds to the d-wave superconductor, shown in the upper right quadrant of Fig. 3a. The d-wave superconducting phase can be obtained in a microscopic lattice model even at half-filling, by appropriately choosing the electron interaction terms.[21]

When r_N is large and negative, the model magnetically orders into the antiferromagnetic Néel phase. With particle-hole symmetry the antipodal nodes are separated by the Néel ordering wavevector, (π, π), so the nodons are "nested". This opens a gap in the nodon spectrum, as we now demonstrate. With either sign of r_Φ the gauge field a_μ can be integrated out, generating irrelevant four fermion nodon interaction terms, which can be dropped. Then, upon putting $\langle \mathbf{N} \rangle = N_0 \hat{\mathbf{y}}$ into the effective Lagrangian, we arrive at the following quadratic Lagrangian for the nodon fields,

$$\mathcal{L}_{nodon} = \mathcal{L}_\psi + gN_0(\psi_j^\dagger \tau^y \psi_j^\dagger + \text{h.c.}). \qquad (4.4)$$

This model can be readily diagonalized with an appropriate Bogoliubov transformation, giving energy eigenvalues,

$$E_1(\vec{q}) = \pm\sqrt{(v_F q_x)^2 + (v_\Delta q_y)^2 + (gN_0)^2}, \qquad (4.5)$$

in the $j = 1$ sector, and an identical form with q_x and q_y interchanged for the other pair of nodes ($j = 2$). In all nodon sectors there is a non-zero *gap*, equal to gN_0. In the lower left quadrant of Fig. 3a, with r_Φ negative, this corresponds to the usual Néel antiferromagnet. With the nodons gapped out, the only low energy excitations are the spin waves of the antiferromagnet. For r_Φ positive, in the lower right quadrant, antiferromagnetism co-exists with d-wave superconductivity. In the d-wave superconductor, the nodons become equivalent to the d-wave quasiparticles, so that the d-wave state in this quadrant is rather unusual. In particular, it is a d-wave superconductor with a full single particle gap, and an absence of nodal quasiparticles. We have denoted this with a subscript g – for gap – in the phase of the lower right quadrant.

Before turning to the effects of particle-hole asymmetries, it is interesting to briefly discuss the nature of the phase transitions between the four phases in Fig. 3a. Consider first the vertical phase boundary, separating the superconducting from non-superconducting phases. For r_N negative there are no gapless nodons, and the magnet is ordered in both phases. At the transition, $r_\Phi = 0$, where superconductivity develops, we can employ the dual Ginzburg-Landau theory. Equivalently, we can return to the original representation (before duality) in terms of the pair field phase, φ, which (in the absence of long-ranged Coulomb interactions[22]) is simply the classical three-dimensional XY model. The resulting transition is in the classical $3d$-XY universality class. For r_N positive, we have to worry about the presence of gapless nodons, which might effect the nature of the superconducting transition at $r_\Phi = 0$. However, since the dual Ginzburg-Landau field Φ is only indirectly coupled to the nodons, via the gauge field a_μ, we expect that the transition will still be in the $3d$-XY universality class. Indeed, power counting in d spatial dimensions about the Gaussian fixed point, reveals that the coupling term (of the form $\partial a \psi^\dagger \psi$) has scaling dimension $(3d+1)/2$. Being greater than $d + 1$ for all $d \geq 1$, and much greater near the upper-critical dimension ($d_{uc} = 3$), one expects this coupling to be strongly irrelevant in the two-dimensional case of interest.

Consider next the phase transition into the antiferromagnetic state, upon crossing the horizontal axis. For either sign of r_Φ, since the vortices and gauge field a_μ can be integrated out generating irrelevant four-nodon interactions, the transition is described by the Néel ordering field coupled to the nodon bi-linear, with Lagrangian $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_N$. This is an interesting field theory describing Dirac fermions anomalously coupled to a fluctuating $O(3)$ field. Power counting about the Gaussian fixed point (free nodons and $r_N = u_N = 0$) reveals that the coupling term is relevant in two dimensions, but if the model is suitably generalized to higher dimensions becomes marginal in $d = 3$. This suggests an attack near four space-time dimensions, working perturbatively for small $\epsilon = 4 - (d + 1)$. A complication is that there are

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in fact three independent velocities, two for the nodons (v_F, v_Δ) and one for the Néel spin field (v_s). In Appendix B we first consider the special case in which all three velocities are set equal. This model is then a bona fide relativistic field theory, with Lorentz invariance. A leading order perturbative renormalization group in ϵ reveals the existence of a new non-trivial fixed point. The associated critical properties and exponents are briefly discussed in Appendix B. Remarkably, as we also show in Appendix B, this relativistic fixed point is in fact linearly *stable* to small deviations in the three velocities. A microscopic model with different velocities scales into a relativistic form near criticality. As further discussed in Appendix B, this model is closely related to a remarkable model[23] which has has a non-trivial fixed point in $D = 2+1$ with an exact $(N = 1)$ space-time supersymmetry.[24] Supersymmetry is powerful enough to determine several *exact* critical exponents in $D = 3$,[23] which serve as a useful check on our ϵ expansion results.

It is finally worth mentioning the effects of the coupling to the nodons, on the *location* of the antiferromagnetic ordering transition. With r_N positive one can safely integrate out the nodons. This leads to a supression of r_N, with a "renormalized value", $\tilde{r}_N$, given to leading order by,

$$\tilde{r}_N = r_N - (c\Lambda/v_\Delta)g^2 + O(g^4). \qquad (4.6)$$

Here Λ is the high momentum cutoff on the nodons, and c a cut-off dependent constant of order one. The coupling to the nodons thus tends to *enhance* the magnetic ordering. Within mean-field theory, the antiferromagnetic ordering transition will take place when the "renormalized" coefficient $\tilde{r}_N = 0$, as indicated in Fig. 3a.

2. Effects of Particle-hole asymmetry

Staying at half-filling, we next consider the effects of particle-hole asymmetries, generated for example by a second neighbor electron hopping term in a microscopic square lattice model. As discussed in Section II, a particle-hole asymmetry generates an additional term in the nodon Lagrangian of the form,

$$\mathcal{L}_\lambda = \lambda \psi_j^\dagger \tau^z \psi_j. \qquad (4.7)$$

In the absence of coupling to the Néel field, this simply causes a momentum space shift in the *positions* of the nodes, by an amount $\delta q = \lambda/v_F$. As we shall see, this has a profound effect when the nodons are coupled to the Néel order parameter, since such a shift destroys the "nesting" of the nodons. Indeed, this leads to two *additional* phases at half-filling, as shown in the phase diagram for $\lambda \neq 0$ in Fig. 3b.

Since both additional phases are antiferromagnetically ordered, we once again integrate out Φ and a_μ (valid for non-zero r_Φ), and put $\langle \mathbf{N} \rangle = N_0 \hat{\mathbf{y}}$ into the effective

Lagrangian to arrive at a quadratic nodon Lagrangian of the form $\mathcal{L}_{nodon} + \mathcal{L}_\lambda$, with $\mathcal{L}_{nodon}$ given in Eq. 4.4. Once again a Bogoliubov transformation diagonalizes the quadratic form, and with non-zero λ the energy eigenvalues (in the $j = 1$ nodon sector) satisfy, $E^4 - 2AE^2 + B = 0$ with,

$$A = (gN_0)^2 + (v_F q_x)^2 + (v_\Delta q_y)^2 + \lambda^2, \qquad (4.8)$$

and

$$B = [A - 2\lambda^2] + (2\lambda v_\Delta q_y)^2. \qquad (4.9)$$

If $gN_0 > \lambda$, there is no solution at $E = 0$, so that there is a gap in the nodon spectrum. The resulting phases - antiferromagnet for $r_\Phi < 0$ and co-existing gapped d-wave superconductivity with antiferromagnetism for $r_\Phi > 0$ - occur for large negative r_N, where N_0 is large (given by $N_0 = \sqrt{(-r_N)/2u_N}$ within mean-field theory). But for smaller $|r_N|$, when $gN_0 < \lambda$, zero energy solutions *do* exist, and there are *gapless* nodon states present! This leads to the two new phases present in Fig. 3b.

Specifically, for r_Φ negative, the new phase exhibits gapless nodon excitations co-existing with long-range antiferromagnetic order. The nodons are now incommensurate with the magnetic order (at $\vec{\pi}$), since the zero energy nodon state for $\lambda > gN_0$ occurs at a shifted wavevector:

$$q_x = v_F^{-1} \sqrt{\lambda^2 - (gN_0)^2} \neq 0, \qquad (4.10)$$

and $q_y = 0$. This interesting new phase, which we denote as AF/NL, exhibits gapless incommensurate magnetic fluctuations co-existing with the $\vec{\pi}$ magnons of the Néel state.

For positive r_Φ, where d-wave superconductivity is present, the gapless nodon excitations are simply d-wave quasiparticles, incommensurate with the Néel order. In the new phase, the incommensurate gapless d-wave superconductor co-exists with antiferromagnetic order, as depicted in Fig. 3b

It is worth emphasizing that with particle-hole asymmetry present (ie. non-zero λ) the region of antiferromagnetic order in the phase diagram is *diminished*, relative to the case with $\lambda = 0$, as depicted in Fig. 3b. This occurs because incommensurate nodons are less effective at renormalizing r_N. Specifically, upon integrating out the nodons with non-zero λ one finds the same form as in Eq. 4.6, but with $c \to cF(\lambda/v_F\Lambda)$, where $F(X)$ is a monotonically decreasing function of X with $F(0) = 1$. This leads to a downward shift in the horizontal phase boundary in Fig. 3b by an amount,

$$\delta\tilde{r}_N = [1 - F(\lambda/v_F\Lambda)](c\Lambda/v_\Delta)g^2. \qquad (4.11)$$

Physically, particle-hole asymmetries, such as a second neighbor hopping term, tend to frustrate and weaken the antiferromagnetism at $\vec{\pi}$. With non-zero λ the nodes are shifted off commensurablilty, and the magnetism is indeed weakened. This effect leads to a natural mechanism

for the destruction of antiferromagnetism upon doping, as we describe in the next section.

Finally, we briefly discuss the nature of the phase transitions between the six phases present at half-filling with particle-hole asymmetry. Arguments as in the previous subsection, suggest that the vertical phase boundary separating the three superconducting from the three non-superconducting phases (at $r_\Phi = 0$) should, as before, be in the universality class of the classical $3d$-XY model. Since the nodons are incommensurate at the upper horizontal phase boundary where antiferromagnetism first appears, they will decouple from the critical magnetic fluctutations. Both of these two transitions (for positive and negative r_Φ) should thus be in the universality class of the classical $3d$ Heisenburg model. At the lower horizontal phase boundaries, the gapless nodons disappear. The critical properties are correctly described by the quadratic nodon Lagrangian, considered above. In particular, for $\lambda > gN_0$, one can linearize for small momentum around the shifted zero energy nodes by putting $\delta q_x = q_x - v_F^{-1}\sqrt{\lambda^2 - (gN_0)^2}$, which gives,

$$E = \pm\left[\left(1 - \left(\frac{gN_0}{\lambda}\right)^2\right)(v_F\delta q_x)^2 + v_\Delta q_y^2\right]^{\frac{1}{2}}. \quad (4.12)$$

As the transition is approached, the velocity along the x-direction – i.e. perpendicular to the Fermi surface – vanishes and the nodons become quasi-1D.

C. Doping the Antiferromagnet

We are now in a position to extend our understanding of doping to include AF order at half-filling, in accord with experimental observations. In all high-T_c materials, doping is actually achieved by chemical substitution/depletion of atoms between the CuO_2 layers. While it is generally believed that this process transfers charge to the CuO_2 planes, this charge transfer is not necessarily proportional to the *chemical* doping, defined as the fraction of atoms substituted or removed. To simplify this discussion, however, we shall assume that chemical doping indeed corresponds to hole doping, and consider the phase diagram as a function of hole concentration x. A further assumption concerns the degree of particle/hole asymmetry. Since the composition is changing with doping, the parameter λ should in general be a function of x (in fact, we expect the asymmetry to increase with x). To simplify the discussion, we shall further assume that any explicit dependence of λ on x is weak, and therefore treat the effects of doping solely through the chemical potential μ.

At half-filling, the system sustains long-range AF order. In general, we expect a non-zero particle/hole asymmetry, so that this could correspond to either the AF or AF/NL state, the latter occuring if λ is sufficiently large. We do not believe current experiments distinguish the

two alternative phases in undoped cuprates. Since by assumption we are varying only μ, and μ couples *indirectly* to magnetism via the corresponding dual "flux" in Eq. 2.35, the AF order and nodons are effected only once charge is added to the system. What is the effect of such charge doping on the AF? From Eqs. 2.24,2.35, we see that the added charges act simply to increase the *effective* particle/hole asymmetry of the nodons. In particular,

$$\lambda_{eff} = \lambda + \frac{x}{4\kappa_0 a_0^2}. \quad (4.13)$$

Eq. 4.13 is an extremely useful result. Using it, we can simply trade the doping x for an effective particle/hole asymmetry to determine the fate of the system from the phase diagrams at half-filling, Fig. 3.
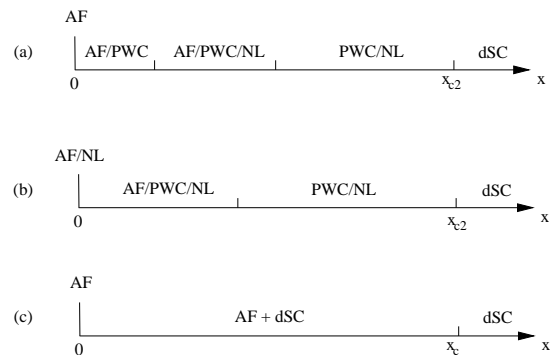

Fig. 4: Possible phase diagrams as a function of doping for the type II (**a** and **b**) and type I (**c**) scenarios. We tentatively identify the PWC/NL phase, which exhibits neither magnetism or superconductivity, with the pseudogap state of the underdoped high-T_c materials.

Consider then first the type II doping scenario. The charge behavior is similar to that obtained when doping the NL, Sec. III. Upon increasing μ from zero, the dual "flux" is first expelled from the sample, and the system remains undoped. Charge first enters above the dual "lower critical field", $\mu > \mu_{c1}$, forming a Paired Wigner Crystal (PWC) with density $x(\mu)$ due to long-range Coulomb interactions. Since x is small at this point, the crystal coexists with the AF, so the actual phase for small x is an AF/PWC. As x increases, so does λ_{eff}, unbinding the nodons into the AF/PWC/NL. This can be understood from the evolution of the phase diagrams at half-filling as a function of λ, as shown in Fig. 3. As x increases further, the NL and AF become increasingly incommensurate, and the energy gain from their coupling is eventually reduced sufficiently to destroy the AF order in a transition to a PWC/NL phase - again a feature of the phase diagrams at half-filling. Finally, when $x \geq x_{c2}$, the upper critical field is reached and the crystal melts into the dSC phase. This progression is shown schematically in Fig. 4a. An alternate type II doping scenario, shown in Fig. 4b, is that the system is an

AF/NL at half-filling, in which case the phase diagram is unchanged except for the absence of the AF/PWC phase.

Another possibility is type I doping. Because this involves a strong first-order transition in the absence of Coulomb interactions, the mixed (micro-phase separated) state could occur as a coexistence between a number of different phases. The simplest phase diagram includes only coexistence between the AF and pure dSC as in Fig. 4c. Because the physics of the mixed state is highly non-universal, we do not discuss it further here.

V. DISCUSSION

The main result of this paper is the Lagrangian, Eqs. 1.1-1.2, which describes the Nodal Liquid phase, its interaction with external electromagnetic fields, and transitions between it and the antiferromagnetic (AF) and superconducting (dSC) phases. This Lagrangian follows directly from disordering the d-wave superconductor. It implies that, in the underdoped region, low-energy fermionic degrees of freedom are located solely at four isolated (Dirac) points in the Brillouin zone – a hypothesis which is strongly supported by ARPES[6,7], NMR[9,10], optical conductivity[11,12], and other experiments[2]. The main consequences of Eqs. 1.1-1.2 are (1) the prediction of a new zero-temperature phase, the Nodal Liquid, which interpolates between the AF and dSC phases; (2) a quantitative description of charge and spin dynamics in this phase; (3) specific predictions for the critical behavior at the AF and dSC ordering transitions; and, above all, (4) a coherent *weak-coupling* framework – with the Nodal Liquid as its foundation – for understanding the underdoped side of the high-T_c phase diagram. We tentatively identify the coexisting Paired Wigner Crystal/Nodal Liquid (PWC/NL) phase as the pseudo-gap state in a hypothetical disorder-free underdoped cuprate. In the real materials, however, disorder will play a role, as we briefly discuss below.

Our description of this part of the phase diagram enjoys kinship with several other approaches. The Nodal Liquid phase is reminiscent of the π-flux state[25] and the $SU(2)$ MFT staggered-flux state,[26] and is the d-wave analog of the short-range[27] resonating valence bond spin-liquid state.[28] These states also involve neutral Dirac fermions interacting with a gauge field, but the coupling to electromagnetic fields, the coupling between the fermions and the gauge fields, the bosonic charged degrees of freedom, and the evolution with doping are all rather different in the Nodal Liquid. Several authors[29,30] have conjectured that the lightly-doped 3-leg Hubbard ladder might serve as a paradigm for the underdoped cuprates and Furukawa and Rice[31] have tried to substantiate these claims with weak-coupling RG calculations on partially nested Fermi liquids. The Nodal Liquid concretely realizes the attractive features of this proposal. The basic idea of bringing AF and dSC under the same rubric, which is expressed in (1.1), is the central theme of Zhang's $SO(5)$ theory.[32] However, there is a direct transition from AF to dSC in the $SO(5)$ theory, whereas the Nodal Liquid intervenes in our theory. There is a further important distinction, namely, that our theory focuses on the zero-temperature quantum phase transitions of the high-T_c materials. This is one reason why our theory accords primary importance to the low-energy *fermionic* degrees of freedom. Finally, our prediction of phase separation at the dSC transition in the type I scenario as well as our interpretation of T^* echoes the ideas of Emery and Kivelson.[15]

There are a number of important issues which we have not addressed in this paper. By restricting our attention to the region underneath the dashed line in Fig. 1, we have skirted one of the most controversial questions in this field: what mechanism drives pair formation at this scale? Presumably this physics must be understood for progress to be made on the part of the phase diagram above and to the right of the dashed line. This would require an investigation complementary (but perhaps orthogonal in spirit) to ours. Also, our discussion of transport was necessarily incomplete because finite-temperature transport can be particularly subtle (for recent examples of this, see Ref. 33) and the transport properties of the Nodal Liquid deserve a thorough exposition of their own, which we defer. Moreover, a pure sample would *melt* at a finite temperature phase transition, although this transition would be rounded by impurities. The melting temperature is expected to vanish upon approaching either zero doping or the PWC/NL to dSC quantum phase transition, and therefore has maximum at some intermediate x in the underdoped regime.

The effects of disorder are also quite subtle, and warrant a full and separate treatment. Nevertheless, a few comments are germane to this discussion. The first, and most basic, is that disorder plays a significant role in the physics of the cuprates. Even in $YBa_2Cu_3O_{7-\delta}$, which is believed to be cleaner than, say, $La_{2-x}Sr_xCuO_4$, doping cannot help but introduce disorder. According to standard arguments,[34] first-order phase transitions will be driven second-order by arbitrarily weak disorder in two dimensions. In particular, we expect the PWC/NL$\rightarrow$dSC transition to be second order with impurities present. Moreover, based on the irrelevance (in the technical sense) of the coupling between the nodons and the superconducting phase in the clean case, we suspect this transition may be in the same universality class as the superconductor–insulator transition. This could explain the experiments of Fukuzumi, *et al.* on $YBa_2Cu_{3-y}Zn_yO_{7-\delta}$.[35] The disorder will also have an effect on the phases themselves. For instance, transport and spin dynamics in the Nodal Liquid will be influenced by disorder. Finally, we note that disorder will transform the PWC into a Bose Glass (BG). One consequence would be power-law suppression (rather than a hard gap) of the low frequency *electron* spectral function at the nodes in the BG/NL phase because the BG is

compressible.[19] Having adopted a panoramic view in the preceeding section, we can afford, in closing, to narrow our focus to the BG/NL (and the PWC/NL from which it descends) because it is our candidate for the $T = 0$ pseudo-gap phase: a phase without a Fermi surface or long-range order but possessing low-energy fermionic excitations centered about four points in the Brillouin zone.

APPENDIX A: MICROSCOPIC APPROACH

In this appendix, we describe techniques to derive the effective field theory from some specific microscopic models. As our purpose is primarily phenomenological, we will consider one of the simplest models which develops antiferromagnetism and d-wave superconductivity. This is a square lattice extended Hubbard model with nearest-neighbor hopping t, on-site electron-electron repulsion U and nearest neighbor attraction V (we emphasize that this model is not realistic, but is chosen for illustrative purposes), i.e.

$$H[c^\dagger, c] = -t \sum_{\langle \vec{x}\vec{x}' \rangle} \left[c_\alpha^\dagger(\vec{x})c_\alpha(\vec{x}') + \text{h.c.} \right] + \tilde{\mu} \sum_{\vec{x}} n(\vec{x})$$
$$+ U\left[n(\vec{x}) \right]^2 - V \sum_{\langle \vec{x}\vec{x}' \rangle} n(\vec{x})n(\vec{x}'). \quad (A1)$$

Here $n(\vec{x}) = c_\alpha^\dagger(\vec{x})c_\alpha(\vec{x})$, and $\tilde{\mu}$ is the microscopic chemical potential, and we have neglected to include terms (e.g. a second-neighbor hopping t') which break particle/hole symmetry at half-filling. Also, for simplicity we measure distances in units of the lattice spacing. The tendency toward AF and dSC states can brought out by using the identities $[n(\vec{x})]^2 = -4/3[\mathbf{S}(\vec{x})]^2 + 2n(\vec{x})$ and $n(\vec{x})n(\vec{x}') = c_\alpha^\dagger(\vec{x})c_\beta^\dagger(\vec{x}')c_\beta(\vec{x}')c_\alpha(\vec{x})$, for $\vec{x} \neq \vec{x}'$. The Hamiltonian can be rewritten as

$$H[c^\dagger, c] = -t \sum_{\langle \vec{x}\vec{x}' \rangle} \left[c_\alpha^\dagger(\vec{x})c_\alpha(\vec{x}') + \text{h.c.} \right] + \mu \sum_{\vec{x}} n(\vec{x})$$
$$-\frac{4}{3}U \sum_{\vec{x}} |\mathbf{S}(\vec{x})|^2 - V \sum_{\langle \vec{x}\vec{x}' \rangle} c_\alpha^\dagger(\vec{x})c_\beta^\dagger(\vec{x}')c_\beta(\vec{x}')c_\alpha(\vec{x}), \quad (A2)$$

where we defined a shifted chemical potential μ, and have neglected an unimportant constant. As usual, the lattice spin operator is defined by $\mathbf{S}(\vec{x}) = \frac{1}{2}c_\alpha^\dagger(\vec{x})\boldsymbol{\sigma}_{\alpha\beta}c_\beta(\vec{x})$. The

angular brackets $\langle \vec{x}\vec{x}' \rangle$ under the two sums indicate sums over all nearest-neighbor pairs of sites.

To derive an effective field theory, it is convenient to use an *imaginary time* path integral formulation. In this case one studies the partition function $Z = \text{Tr}e^{-H/T}$, where T is the temperature. It can be represented using Grassman coherent states as

$$Z = \int [d\overline{c}][dc]e^{-S}, \quad (A3)$$

where the Euclidean action is

$$S = \int d\tau \left\{ \sum_{\vec{x}} \overline{c}_\alpha(\vec{x})\partial_\tau c_\alpha(\vec{x}) + H[\overline{c}, c] \right\}. \quad (A4)$$

We consider here only $T = 0$, for which the τ integration domain is infinite. The possibility of dSC and AF phases can be entertained by decoupling the above action using Hubbard-Stratonovich transformations. One finds that

$$Z = \int [d\overline{c}][dc][d\mathbf{M}][d\Delta][d\overline{\Delta}]e^{-S_1}, \quad (A5)$$

with $S_1 = \int d\tau[\sum_{\vec{x}} \overline{c}_\alpha(\vec{x})\partial_\tau c_\alpha(\vec{x}) + H_{eff}]$. The effective Hamiltonian can be decomposed into $H_{eff} = H_{qp} + H_M + H_\Delta$, with $H_{qp} = H_0 + H_{int}$, and

$$H_0 = -t \sum_{\langle \vec{x}\vec{x}' \rangle} \left[c_\alpha^\dagger(\vec{x})c_\alpha(\vec{x}') + \text{h.c.} \right] + \mu \sum_{\vec{x}} n(\vec{x}) \quad (A6)$$

$$H_{int} = -\sum_{\vec{x}} \mathbf{M}(\vec{x}) \cdot \mathbf{S}(\vec{x})$$
$$+ \sum_{\langle \vec{x}\vec{x}' \rangle} \left[\overline{\Delta}_{\vec{x}\vec{x}'}^{\alpha\beta} c_\beta(\vec{x}')c_\alpha(\vec{x}) + \Delta_{\vec{x}\vec{x}'}^{\alpha\beta} \overline{c}_\alpha(\vec{x})\overline{c}_\beta(\vec{x}') \right], \quad (A7)$$

$$H_M = \frac{3}{8U} \sum_{\vec{x}} |M(\vec{x})|^2, \quad (A8)$$

$$H_\Delta = \frac{1}{V} \sum_{\langle \vec{x}\vec{x}' \rangle} \overline{\Delta}_{\vec{x}\vec{x}'}^{\alpha\beta} \Delta_{\vec{x}\vec{x}'}^{\alpha\beta}. \quad (A9)$$

Eqs. A6-A9 form a basis for studying the original extended Hubbard model. Following the philosophy of Sec. II A , we imagine integrating out high-frequency modes in the functional integral to arrive at an effective field theory for the low-lying degrees of freedom. In the process, one will generate dynamics for the order parameter Δ and the magnetization $\mathbf{M}$. For the most part, symmetry considerations require the corresponding Lagrangians to take the forms given in Sec. II A and Sec. IV, so we choose not to complicate the presentation by explicitly performing these integrations (e.g. diagrammatically).

One subtle point in the analysis of Sec. II A, however, does warrant a more careful treatment. This is the coupling of the nodons to the superconducting phase-gradient, from which follows the expressions for the quasiparticle current, Eqs. 2.22-2.23. In Sec. II A, we derive these using the "symmetric" prescription of Eq. 2.14.

We now show that the currents are indeed obtained correctly using this prescription.

We first specialize to the case of singlet pairing, $\Delta_{\vec{x}\vec{x}'}^{\alpha\beta} = \Delta(\vec{x},\vec{x}')(\delta_{\alpha\uparrow}\delta_{\beta\downarrow} - \delta_{\alpha\downarrow}\delta_{\beta\uparrow})$. Since Δ lives on the bonds, it is convenient to associate two such fields with each site in the square lattice, i.e.

$$\Delta_1(\vec{x}) \equiv \Delta(\vec{x},\vec{x}+\hat{e}_1), \qquad (A10)$$
$$\Delta_2(\vec{x}) \equiv \Delta(\vec{x},\vec{x}+\hat{e}_2), \qquad (A11)$$

where $\hat{e}_1, \hat{e}_2$ are unit vectors along the a and b axes of the square lattice, respectively. Note that at this point, we have specified no particular relation between Δ_1 and Δ_2, so that the model has the potential both for d-wave and s-wave pairing. To distinguish them, we must consider the form of the effective action for $\Delta, \overline{\Delta}$ generated upon integrating out the high-energy modes. By symmetry, the simplest local allowed additional term on the lattice is a sum of $U(1)$-invariant two-bond products around each lattice site, which can be written as $S_2 = \int d\tau\, H_2$, with

$$H_2 = \frac{\tilde{\gamma}}{8}\sum_{\vec{x}}\left\{\overline{\Delta}_1(\vec{x})\Delta_2(\vec{x}) + \overline{\Delta}_2(\vec{x})\Delta_1(\vec{x}-\hat{e}_1)\right.$$
$$\left. +\overline{\Delta}_1(\vec{x}-\hat{e}_1)\Delta_2(\vec{x}-\hat{e}_2) + \overline{\Delta}_2(\vec{x}-\hat{e}_2)\Delta_1(\vec{x}) + \text{c.c.}\right\}. \qquad (A12)$$

Of course, the actual quadratic $\overline{\Delta}\Delta$ interaction terms generated from the high-energy degrees of freedom will be much more complex. However, since the general form of the *long-wavelength* effective action is dictated by symmetry, this example suffices for illustrative purposes. Writing $\Delta_j = \Delta_0 e^{i\phi_j}$, Eq. A12 becomes

$$H_2 = \frac{\gamma}{4}\sum_{\vec{x}}\left\{\cos(\phi_1(\vec{x})-\phi_2(\vec{x})) + \cos(\phi_2(\vec{x})-\phi_1(\vec{x}-\hat{e}_1))\right.$$
$$\left. +\cos(\phi_1(\vec{x}-\hat{e}_1)-\phi_2(\vec{x}-\hat{e}_2)) + \cos(\phi_2(\vec{x}-\hat{e}_2)-\phi_1(\vec{x}))\right\}, \qquad (A13)$$

with $\gamma = (\Delta_0)^2\tilde{\gamma}$. We assume $\gamma > 0$, in which case this interaction favors a relative phase difference of $\phi_1 - \phi_2 = \pi$, i.e. $d_{x^2-y^2}$ order.

We now turn to the fluctuations around the uniform dSC state. To do so, we let $\phi_1 = \varphi$, $\phi_2 = \varphi + \theta + \pi$. The phase φ is precisely the order parameter phase introduced in Sec. II A. The other variable θ represents another branch of *massive* fluctuations around the d-wave state. We can thus assume $\theta \ll 1$, and that φ is slowly varying, i.e. $\partial_j\varphi \ll 1$. Under this assumption, we can take the continuum limit and replace the positional sum in Eq. A13 by an integration. This gives $H_2 = \int d^2x\,\mathcal{H}_2$, with

$$\mathcal{H}_2 = \gamma\left\{\frac{\theta^2}{2} + \frac{1}{4}(\partial_j\varphi)^2 - \frac{1}{4}\partial_x\varphi\partial_y\varphi + \frac{\theta}{2}(\partial_x-\partial_y)\varphi\right\}. \qquad (A14)$$

As expected, the θ field is massive, and can be integrated out. Equivalently, one minimizes $\mathcal{H}_2$ with respect to θ. This process restores isotropy and gives

$$\mathcal{H}_2 \rightarrow \frac{\gamma}{8}(\partial_j\varphi)^2. \qquad (A15)$$

This is just the spatial component of the superfluid Lagrangian, Eq. 2.17, with $\gamma = v_c^2\kappa_0$. The corresponding time component can be obtained similarly.

Finally, consider the coupling of of the phase φ to the fermionic quasi-particle operators. To study this, we take for simplicity $\mu = \mathbf{M} = 0$. Using the definitions in Eqs. A10-A11, the coupling term in Eq. A7 can be rewritten

$$H_{int} = \sum_{j,\vec{x}}\left\{\Delta_j(\vec{x})\left[c_\uparrow^\dagger(\vec{x})c_\downarrow^\dagger(\vec{x}+\hat{e}_j) - \uparrow\leftrightarrow\downarrow\right] + \text{h.c.}\right\}, \quad (A16)$$

where the sum includes all lattice sites and $j = 1,2$. To ease comparison with Sec. II A, we have returned now to a Hamiltonian formalism, replacing $\overline{c}_\alpha$ by $c_\alpha^\dagger$. We are now in a position to take the continuum limit. In this case, it suffices to neglect the massive θ mode, and simply take $\Delta_1 = -\Delta_2 = \Delta_0 e^{i\varphi}$. For agreement with Sec. II A, we define $v_\Delta = 2\sqrt{2}\Delta_0$, or $\Delta_1 = -\Delta_2 = \tilde{\Delta}/2\sqrt{2}$. In addition, we take the continuum limit of the electron fields, using the decompositions

$$c_\uparrow^\dagger \sim \Psi_{111}^\dagger i^{x+y} - \Psi_{122}(-i)^{x+y} + \Psi_{211}^\dagger(-i)^{x-y} - \Psi_{222}i^{x-y},$$
$$c_\downarrow^\dagger \sim \Psi_{112}^\dagger i^{x+y} + \Psi_{121}(-i)^{x+y} + \Psi_{212}^\dagger(-i)^{x-y} + \Psi_{221}i^{x-y},$$

and the hermitian conjugates of these equations. Inserting these into Eq. A16, gradient-expanding the Ψ fields, and rotating 45 degrees to $x-y$ coordinates along the (π,π) and $(-\pi,\pi)$ directions, one obtains $H_{int} = \int d^2x\mathcal{H}_{int}$, with

$$\mathcal{H}_{int} = \left[\frac{\tilde{\Delta}}{2}\left(\Psi_1^\dagger\tau^+ i\partial_y\Psi_1 - (i\partial_y\Psi_1^\dagger)\tau^+\Psi_1\right) + \text{h.c.}\right]$$
$$+(1\leftrightarrow 2, x\leftrightarrow y). \qquad (A17)$$

This form is identical to the $\tilde{\Delta}$ term in Eq. 2.10 when the order parameter $\tilde{\Delta}$ is constant, but the symmetric placement of derivatives is important in the presence of phase gradients. In particular, now let $\tilde{\Delta} = v_\Delta e^{i\varphi}$ and integrate by parts to transfer the derivative in the second term from the $\Psi^\dagger$ to the $\tilde{\Delta}\Psi$ combination. Then, using the operator identity

$$\frac{1}{2}\left(e^{i\varphi}i\partial_y + i\partial_y e^{i\varphi}\right) = e^{i\varphi/2}i\partial_y e^{i\varphi/2}, \qquad (A18)$$

one obtains

$$\mathcal{H}_{int} = \sum_{s=\pm}\Psi_1^\dagger[v_\Delta\tau^s e^{is\varphi/2}(i\partial_y)e^{is\varphi/2}]\Psi_1 + (1\leftrightarrow 2; x\leftrightarrow y). \qquad (A19)$$

Eq. A19 is identical to the symmetrized form of the phase-quasiparticle interaction hypothesized in Eq. 2.15.

APPENDIX B: RENORMALIZATION GROUP ANALYSIS

In this appendix, we present some details of the RG calculations for the transition at half-filling with particle/hole symmetry between the nodal liquid phase and the antiferromagnet. As discussed in Section IV, this transition is described by the *same* field theory as the phase transition between the d-wave superconducting phase and the phase with co-existing antiferromagent and gapped d-wave order. A striking feature of this transition, which we access perturbatively in an $\epsilon = 4 - (d+1)$ expansion, is that it is Lorentz-invariant. Although the model possesses *three* independent velocities, the differences between these scale to zero at the critical point.

The full Lagrangian is given by,

$$
\begin{aligned}
\mathcal{L} = \ & \psi_1^\dagger \left(i\partial_t - v_F \tau^z i\partial_x - v_\Delta \tau^x i\partial_y \right) \psi_1 \\
& + \psi_2^\dagger \left(i\partial_t - v_\Delta \tau^z i\partial_x - v_F \tau^x i\partial_y \right) \psi_2 \\
& + \frac{1}{2} K_\mu |\partial_\mu \mathbf{N}|^2 - r_N \mathbf{N}^2 - u\left(\mathbf{N}^2\right)^2 \\
& + g\mathbf{N} \cdot [\psi^\dagger \tau^y \boldsymbol{\sigma}\sigma^y \psi^\dagger + \text{c.c.}],
\end{aligned} \tag{B1}
$$

where we have suppressed the spin subscripts α, β on the $\boldsymbol{\sigma}$ Pauli matrices, and the particle/hole subscripts a, b on the $\boldsymbol{\tau}$ Pauli matrices. Here $K_0 = K$ and $K_j = -v_s^2 K$, for $j = 1, 2$. Notice that this model has three independent velocities, v_F, v_Δ and v_s.

Now, we can rescale the $\mathbf{N}$ field to set $K = 1$, and rescale $\vec{x}$ to set $v_s = 1$. The Lagrangian can then be rewritten as:

$$
\begin{aligned}
\mathcal{L} = \ & \psi_{j\alpha}^\dagger \left(i\partial_t - \tau^z i\partial_x - \tau^x i\partial_y \right) \psi_{j\alpha} \\
& + \frac{1}{2}(\partial_t \mathbf{N})^2 - \frac{1}{2}(\partial_j \mathbf{N})^2 - r_N \mathbf{N}^2 - u\left(\mathbf{N}^2\right)^2 \\
& + g\mathbf{N} \cdot [\psi^\dagger \tau^y \sigma\sigma^y \psi^\dagger + \text{c.c.}] \\
& - \psi_1^\dagger \left(a_1 \tau^z i\partial_x + a_2 \tau^x i\partial_y \right) \psi_1 \\
& - \psi_2^\dagger \left(a_2 \tau^z i\partial_x + a_1 \tau^x i\partial_y \right) \psi_2
\end{aligned} \tag{B2}
$$

where

$$
\begin{aligned}
a_1 &= \frac{v_F}{v_s} - 1, \\
a_2 &= \frac{v_\Delta}{v_s} - 1.
\end{aligned} \tag{B3}
$$

Power counting about the Gaussian theory reveals that both u and g^2 are relevant in $D = 2+1 = 3$ space-time dimensions, but become marginal in $D = 4$. To implement a perturbative RG calculation thus requires continung the model above $D = 3$. This is a little tricky, due to the Pauli-matrix algebra. One approach is to dimensionally continue the loop integrals, but leave the $3D$ Pauli-matrix algebra unchanged. This turns out to be equivalent to introducing into the Dirac equations an extra Pauli matrix, τ^y, multiplied by another (3rd) spatial dimension. Alternatively, one could replace the two-dimensional Pauli matrices by the $4-$dimensional γ-matrices, appropriate for $4D$ spinors. This latter procedure would be more correct

if we were truly interested in the vicinity of four dimensions, but our choice probably makes more sense given that we are eventually concerned with $\epsilon \to 1$. In any case, the difference is fairly trivial: factors of 2 would be replaced by factors of 4 in traces over the $4D$ γ-matrices. With our convention, we obtain the following one-loop flow equations:

$$
\begin{aligned}
\frac{du}{dl} &= \epsilon u - 44u^2 - 32\lambda u + 32\lambda^2 \\
\frac{d\lambda}{dl} &= \epsilon\lambda - 20\lambda^2 \\
\frac{dr_N}{dl} &\equiv \frac{1}{\nu} r_N = (2 - 20u - 16\lambda)r_N \\
\frac{da_j}{dl} &= -12\lambda a_j,
\end{aligned} \tag{B4}
$$

where $\lambda = g^2$. $\mathbf{N}$ and ψ have anomalous dimensions 8λ and 3λ, respectively. These flow equtions have a fixed point at $O(\epsilon)$: $\lambda = \epsilon/20$, $u = \epsilon\sqrt{371}/440$, $r_N = 0$, $a_j = 0$. This fixed point has the following interesting features:

(1) 'Relativistic Invariance.' Since the velocity differences scale to zero according to (B4), all physically measurable quantities are a function of $x^2 + y^2 - t^2$ in the units which we have chosen, or, upon restoring the velocities, $(x^2 + y^2) - v_s^2 t^2$.

(2) An antiferromagnetic correlation length which diverges as

$$
\xi \sim |r_N - r_N^c|^{-1/(2 - 20u^* - 16\lambda^*)} \tag{B5}
$$

as the transition is approached.

(3) Critical correlation functions with the following power-law decays

$$
\begin{aligned}
\langle N_i(x,t)N_j(0,0)\rangle &\sim \frac{\delta_{ij}}{|\vec{x}^2 - t^2|^{\frac{1}{2} - 16\lambda^*}} \\
\langle \psi^\dagger(x,t)\psi(0,0)\rangle &\sim \frac{1}{|\vec{x}^2 - t^2|^{1 - 6\lambda^*}}
\end{aligned} \tag{B6}
$$

As a check on the reliability of the ϵ-expansion, we consider the following related model[23]:

$$
\begin{aligned}
\mathcal{L} = \ & \psi^\dagger \left(i\partial_t - \tau^z i\partial_x - \tau^x i\partial_y \right) \psi \\
& + \frac{1}{2}|\partial_t \Phi|^2 - \frac{1}{2}|\partial_j \Phi|^2 - u|\Phi|^4 + g[\psi^\dagger \tau^y \psi^\dagger + \text{c.c.}],
\end{aligned} \tag{B7}
$$

where ψ is now a single two-component spinor, and Φ is a complex field. Using the same ϵ-expansion procedure, we find a fixed point at $\lambda^* = u^* = \epsilon/6$; at this fixed point, the fields Φ and ψ have anomalous dimension $\epsilon/6$. It is a remarkable and fortunate fact that this model exhibits $N = 1$ supersymmetry. As a result, the existence of a fixed point is guaranteed and the scaling dimensions of ψ and Φ can be determined exactly, in agreement with the ϵ expansion.

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