Quasiparticle density of states in dirty high-T_c superconductors

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(Received 26 October 1998)

We study the density of quasiparticle states of dirty *d*-wave superconductors. We show the existence of singular corrections to the density of states due to quantum interference effects. We then argue that the density of states actually vanishes in the localized phase as $|E|$ or E^2 depending on whether time-reversal is a good symmetry or not. We verify this result for systems without time-reversal symmetry in one dimension using supersymmetry techniques. This simple, instructive calculation also provides the exact universal scaling function for the density of states for the crossover from ballistic to localized behavior in one dimension. Above two dimensions, we argue that in contrast to the conventional Anderson localization transition, the density of states has critical singularities which we calculate in a $2+\epsilon$ expansion. We discuss consequences of our results for various experiments on dirty high-T_c materials. [S0163-1829(99)00833-4]

I. INTRODUCTION

The question of the quasiparticle density of states of a two-dimensional $d_{x^2-y^2}$ superconductor in the presence of disorder has been a matter of some controversy. Early theoretical work based on approximate self-consistent treatments[1–3] of the disorder demonstrate that a finite Fermi-level density of states is generated for arbitrarily weak disorder. In contrast, some exact results[4] for a simplified model of the disorder which ignores the scattering between the two pairs of antipodal nodal points show that the density of states ($\rho(E)$) vanishes on approaching zero energy (measured from the Fermi energy) as $\rho(E)\sim E^{1/7}$. Claims of rigorous proofs[5] of a constant nonzero density of states have also appeared in the literature.

In a recent paper,[6] we discussed the problem of quasiparticle transport and localization in dirty superconductors ignoring the quasiparticle interactions, and treating the disorder with a nonlinear sigma model field theory. The starting point for the sigma model description is the approximate self-consistent treatment of the disorder which, as mentioned above, generates a finite density of states. We argued that inclusion of small harmonic fluctuations about the self-consistent solution leads to diffusion of the spin and energy densities of the quasiparticles (though not of the charge density). Quantum interference effects finally lead to quasiparticle localization at the longest length scales in two dimensions. In this paper, we consider the behavior of the density of states in the sigma model. We show that in the diffusive regime, quantum interference effects lead to a singular logarithmic suppression of the density of states. We then argue that in the localized spin insulator, the density of states actually vanishes as $|E|$ for superconductors with both spin rotation and time-reversal symmetry. A schematic plot of the density of states as a function of energy is shown in Fig. 1. The linear density of states of the pure $d_{x^2-y^2}$ superconductor gets rounded off at an energy scale E_1 of the order of the elastic scattering rate. This marks the crossover from the ballistic to the diffusive regime. At a lower energy scale $E_2\sim D/\xi^2$, the density of states dips linearly to zero. (Here D is the ''bare'' spin diffusion constant, and ξ is the quasiparticle localization length.) This second energy scale marks the crossover from the diffusive to the localized regime. The ratio of the two crossover scales E_1/E_2 is exponentially large in the bare dimensionless spin conductance, and can be quite large.

Note that our conclusion of the vanishing density of states is only superficially similar to the results of Nersesyan, Tsvelik, and Wenger.[4] In particular, we argue that the localization length approaches a finite constant as $E\to 0$. (In contrast, Nersesyan, Tsvelik, and Wenger find a diverging localization length as $E\to 0$). Our results are also in disagreement with the claims of Ziegler, Hettler, and Hirschfeld.[5] It has, however, been pointed out[7] that the techniques of Ref. 5 give incorrect results in other situations—this signals a flaw in the technique which could invalidate their results.

For a superconductor with spin rotation invariance but no time reversal ($\mathcal{T}$), there is again a logarithmic suppression of the density of states in the diffusive regime. This accounts entirely for the leading logarithmic suppression of the spin conductance found earlier, and provides an explanation of it. In this case, we argue that in the localized phase, the density of states vanishes as E^2.

We provide an explicit verification of some of our general results by exact nonperturbative calculations in one dimension using supersymmetry techniques.

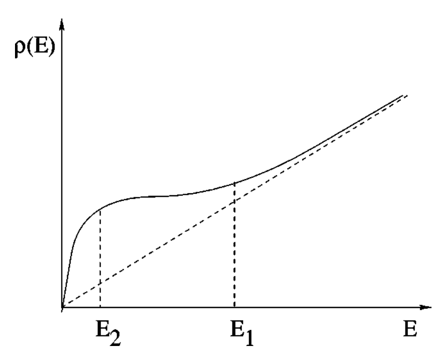

FIG. 1. Density of states of the two-dimensional dirty $d_{x^2-y^2}$ superconductor.

II. DIFFUSIVE REGIME

Consider first a two-dimensional $d_{x^2-y^2}$ superconductor with both spin rotation and $\mathcal{T}$ invariance. In the absence of quasiparticle interactions, and at scales larger than the elastic mean free path, the quasiparticle diffusion[8] is described by the replica field theory[6]

$$S_{\mathrm{NL}\sigma\mathrm{M}}=\int d^2x\frac{1}{2g}\mathrm{Tr}\,(\nabla U\cdot\nabla U^\dagger)-\eta\,\mathrm{Tr}\,(U+U^\dagger). \quad (1)$$

Here $U(x)$ are $2n\times 2n$ unitary matrices $\in \mathrm{Sp}(2n)$, and η is a positive infinitesimal that is introduced to enable calculation of the appropriate Green's functions. The coupling constant g is related to the spin conductance σ_s by $1/g=(\pi/2)\sigma_s$.

The quasiparticle density of states at the Fermi energy is exactly proportional to the uniform spin susceptibility which is the ''order parameter''[6] for this field theory. Thus it is given by

$$\rho=\lim_{n\to 0}\frac{\rho_0}{4n}\langle \mathrm{Tr}(U^\dagger+U)\rangle, \quad (2)$$

where ρ_0 is the bare density of states (DOS) (i.e., its value on the scale of the mean free path). The limit $n\to 0$ is implied. As this is the ''order parameter'' for the field theory, quantum corrections to it can be obtained from the known results[9] for the ''field renormalization'' to one loop order. To perturbatively calculate corrections to ρ, we write $U=1+i\phi-\phi^2/2+\cdots$ with ϕ belonging to the Lie algebra of $\mathrm{Sp}(2n)$, and expand in powers of ϕ. To quadratic order, the action is

$$S_0=\frac{1}{2g}\int d^2x\ \mathrm{Tr}(\nabla\phi)^2. \quad (3)$$

We may choose a basis T^a for the Lie algebra to write $\phi=\phi_a T^a$, and normalize the basis by some choice $\mathrm{Tr}(T^aT^b)=C\delta^{ab}$ with C a positive constant. The action then becomes

$$S_0=\frac{C}{2g}\int d^2x\nabla\phi_a\cdot\nabla\phi_a. \quad (4)$$

(Summation over the index a is implied in the above equations.) The matrices T^a are traceless, and the susceptibility may be expressed to leading order as

$$\frac{\rho}{\rho_0}=\frac{1}{4n}\langle \mathrm{Tr}(2-\phi^2)\rangle=\frac{1}{4n}\langle 4n-\mathrm{Tr}(T^aT^b)\phi_a\phi_b\rangle$$
$$=1-\frac{CN}{4n}G(0).$$

In the last line, we have used the function $G(x)$ defined by $\langle\phi_a(x)\phi_b(0)\rangle=G(x)\delta_{ab}$; N is the number of linearly independent matrices T^a. For $\mathrm{Sp}(2n)$, $N=n(2n+1)$. It is understood that the $n=0$ limit is taken at the end of the calculation. Considering now a finite system size, we get

$$G(0)=\frac{g}{C}\int_{|k|>L^{-1}}\frac{d^2k}{(2\pi)^2}\frac{1}{k^2}.$$

This gives

$$\frac{\rho-\rho_0}{\rho_0}=-\frac{gN}{8\pi n}\ln\left(\frac{L}{l_e}\right). \quad (5)$$

Note that the constant C has dropped out of this result (as it should). The quantity l_e is the elastic mean free path.

The same result is obtained in the $\mathcal{T}$ broken but spin rotation invariant case. This is described by the $\mathrm{Sp}(2n)/U(n)$ field theory[6] with the action

$$S=\int d^2x\frac{1}{2g}\mathrm{Tr}((\nabla Q)^2-\eta Q\sigma_z), \quad (6)$$

where $Q=U^\dagger\sigma_z U$ with $U\in\mathrm{Sp}(2n)$. The density of states is again the order parameter of this field theory, and is given by

$$\rho=\lim_{n\to 0}\frac{\rho_0}{2n}\langle \mathrm{Tr}(Q\sigma_z)\rangle. \quad (7)$$

Calculation similar to the one above gives the result Eq. (5) but with $N=n(2n+1)-n^2$ being the number of independent massless fields. In either case, in the replica limit, we get

$$\frac{\rho-\rho_0}{\rho_0}=-\frac{g}{8\pi}\ln\left(\frac{L}{l_e}\right). \quad (8)$$

Thus to leading order, the suppression of the density of states is independent of whether or not $\mathcal{T}$ is present.

The leading logarithmic correction to the spin conductance in two dimensions was evaluated in Ref. 6:

$$\sigma_s(L)=\sigma_s^0-\frac{1}{2\pi^2}\ln\frac{L}{l_e}, \quad (9)$$

where σ_s^0 is the bare spin conductance, and l_e is the elastic mean free path. If $\mathcal{T}$ is broken, then the correction is reduced by a factor of 2.[6]

The spin conductance satisfies the Einstein relation $\sigma_s=D\rho/4$ with D being the spin diffusion constant. (The factor of 4 arises from the spin of 1/2.) Equations (8) and (9) together with the relation $1/g=(\pi/2)\sigma_s$ imply a logarithmic suppression of the diffusion constant at order $1/g$ when $\mathcal{T}$ is present. Without $\mathcal{T}$ invariance, there is no suppression of the diffusion constant to this order.

It is possible to understand this result in terms of a semiclassical picture[10,11] involving interfering trajectories. To that end, consider, quite generally, a lattice Hamiltonian for the quasiparticles in a singlet superconductor

$$\mathcal{H}=\sum_{i,j}\left[t_{ij}\sum_\sigma c^\dagger_{i\sigma}c_{j\sigma}+\Delta_{ij}c^\dagger_{i\uparrow}c^\dagger_{j\downarrow}+\Delta^*_{ij}c_{j\downarrow}c_{i\uparrow}\right], \quad (10)$$

where i,j refer to the sites of some lattice. Hermiticity implies $t_{ij}=t^*_{ji}$, and spin rotation invariance requires $\Delta_{ij}=\Delta_{ji}$. It is useful conceptually to use the alternate representation in terms of a new set of d operators defined by: $d_{i\uparrow}=c_{i\uparrow}, d_{i\downarrow}=c^\dagger_{i\downarrow}$. The Hamiltonian Eq. (10) then takes the form

$$\mathcal{H}=\sum_{ij}d_i^\dagger\begin{pmatrix} t_{ij} & \Delta_{ij}\\ \Delta^*_{ij} & -t^*_{ij}\end{pmatrix}d_j=\sum_{ij}d_i^\dagger H_{ij}d_j. \quad (11)$$

Writing $t_{ij}=a^z_{ij}+ib_{ij}$, $\Delta_{ij}=a^x_{ij}-ia^y_{ij}$ with $\vec{a}_{ij}=\vec{a}_{ji}$, real symmetric and $b_{ij}=-b_{ji}$, real antisymmetric, we get

$$H_{ij}=ib_{ij}+\overrightarrow{a_{ij}}\cdot\vec{\sigma}. \tag{12}$$

Note that $SU(2)$ invariance requires $\sigma_y H_{ij}\sigma_y=-H^*_{ij}$. This implies that the amplitude $iG_{ij,\alpha\beta}=\langle i\alpha|e^{-iHt}|j\beta\rangle\theta(t)$ for a d particle to go from point j, (pseudo)spin β to point i, spin α satisfies the relations

$$G_{ij,\uparrow\uparrow}(t)=-G^*_{ij,\downarrow\downarrow}(t), \tag{13}$$

$$G_{ij,\uparrow\downarrow}(t)=G^*_{ij,\downarrow\uparrow}(t). \tag{14}$$

The Fourier transform of this amplitude is

$$G_{ij,\alpha\beta}(\omega+i\eta)=\int dt\, e^{i(\omega+i\eta)t}G_{ij,\alpha\beta}(t)$$

$$=\langle i\alpha|\frac{1}{\omega-H+i\eta}|j\beta\rangle.$$

The density of states at the Fermi energy may be obtained from this in the usual manner.

$$\rho=-\frac{1}{\pi}\mathrm{Im}(\overline{G}_{ii,\uparrow\uparrow}(i\eta)+(\uparrow\leftrightarrow\downarrow)) \tag{15}$$

Consider now the return amplitude $G_{ii,\uparrow\uparrow}(t)$. This can be written as a sum over all possible paths for this event. Consider in particular the contribution from the special class of paths where the particle traverses some orbit and returns to the point i in time $t/2$ with spin down, and then traverses the same orbit again in the remaining time and returns with spin up. This contribution to $iG_{ii,\uparrow\uparrow}(t)$ can be written

$$iG_{ii,\uparrow\downarrow}\left(\frac{t}{2}\right)iG_{ii,\downarrow\uparrow}\left(\frac{t}{2}\right)=-\left|G_{ii,\uparrow\downarrow}\left(\frac{t}{2}\right)\right|^2$$

using the symmetry relation Eq. (14). Now $|G_{ii,\uparrow\downarrow}(t/2)|^2$ is just the probability for the event $i\uparrow\rightarrow i\downarrow$ in time $t/2$. For large t, this is half the total return probability which $\sim 1/t$ in two dimensions if the particles are diffusing. This leads to a logarithmic divergence in the density of states which may be cut off by a finite system size. To be precise, this gives

$$\frac{\delta\rho}{\rho_0}=-\frac{1}{\pi^2\rho_0 D}\ln\left(\frac{L}{l_e}\right) \tag{16}$$

in agreement with the field-theoretic result obtained earlier.

In addition, even to leading order the spin conductance for $\mathcal{T}$ invariant systems is suppressed further by the usual constructive interference between paths and their time reverse which explains the larger suppression in that case.

III. LOCALIZED REGIME

Having established the presence of a singular suppression of the density of states in perturbation theory, we now consider the opposite limit of strong disorder when the system is localized. We show that the density of states vanishes at zero energy. To see this heuristically, consider the Hamiltonian (10) in the limit of strong on-site randomness and weak hopping between sites. In the limit of zero hopping, the sites are all decoupled. At each site, the Hamiltonian in terms of the d particles satisfies the $SU(2)$ invariance requirement $\sigma_y H\sigma_y=-H^*$. This takes the form $H=\vec{a}\cdot\vec{\sigma}$ with $\vec{a}$ random. With $\mathcal{T}$ symmetry, we further have $H=H^*$ implying $a_y=0$. Considering now the case where the probability distribution of $\vec{a}$ has finite, nonzero weight at zero, we see immediately that the disorder averaged density of states vanishes as E^2 without $\mathcal{T}$ and as $|E|$ with $\mathcal{T}$. Now consider weak nonzero hopping. In the localized phase, perturbation theory in the hopping strength should converge, and we expect to recover the single site results at asymptotically low energies.

A more formal field-theoretic version of this argument with the same conclusions is as follows. As we are concerned with the properties of the localized phase, we prefer to phrase the argument in terms of a supersymmetric field theory rather than the replica version used before. In the localized phase, we expect that a strong-coupling expansion of this field theory converges. This may be performed, as usual, by regularizing the sigma model on a lattice. The leading term in the strong-coupling expansion is the zero-dimensional limit of the sigma model which is equivalent to the random matrix theory of Hamiltonians with these symmetries. In the random matrix limit, it is known[11] that the density of states vanishes in the manner discussed above.

These results on the localized phase can be verified in great detail in one spatial dimension for systems without $\mathcal{T}$. Consider a lattice Hamiltonian for the d particles in one dimension. In the absence of disorder, we take this to be of the form

$$\mathcal{H}=\sum_i -t(d_i^\dagger\sigma_z d_{i+1}+\mathrm{H.c})-\mu d_i^\dagger\sigma_z d_i. \tag{17}$$

Now consider adding random terms to this Hamiltonian consistent with the required symmetry. For weak disorder, we may just keep the modes near the two Fermi points of the pure system. Linearizing the dispersion near these Fermi points, we arrive, as usual, at a one-dimensional Dirac theory with various sorts of randomness. The resulting Hamiltonian can be written down on symmetry grounds as

$$H=-i\tau_z\partial_x+(\vec{\eta}_1(x)\cdot\vec{\sigma})\left(\frac{1+\tau_z}{2}\right)-(\vec{\eta}_2(x)\cdot\vec{\sigma})\left(\frac{1-\tau_z}{2}\right)$$
$$+t_0(x)\tau_y+(\vec{t}(x)\cdot\vec{\sigma})\tau_x. \tag{18}$$

This is the most general Hamiltonian consistent with the symmetry $\sigma_y H\sigma_y=-H^*$ required by spin rotation invariance. (We have set the Fermi velocity to 1.) The $\vec{\tau}$ are Pauli matrices in the right mover/left mover space and $\eta_1,\eta_2,t_0,\vec{t}$ are random, independently distributed real variables. Green's functions of this Hamiltonian are generated by the action

$$S=\int dx[\overline{\psi}(iH+\omega)\psi+\xi^*(iH+\omega)\xi], \tag{19}$$

where $\psi,\overline{\psi}$ are Grassmann variables, and ξ is a complex scalar field. For a system of finite size L, we impose periodic boundary conditions on all fields. The partition function Z corresponding to this action is exactly equal to one for any L

as the fermionic and bosonic integrals cancel each other. Here ω is chosen to have a positive real part to ensure convergence of the bosonic integral.

The density of states can be obtained from the Green's function through

$$\rho(E)=-\frac{1}{\pi}\mathrm{Im}\ \mathrm{Tr}\overline{G(E+i\eta)}, \tag{20}$$

where the overline indicates disorder averaging and

$$G_{ab}^{\alpha\beta}(x,x';E+i\eta)=\langle a\alpha x|\frac{1}{E-H+i\eta}|b\beta x'\rangle. \tag{21}$$

Its disorder average can be expressed in terms of correlators of the either the Bose or Fermi variables:

$$\overline{iG_{ab}^{\alpha\beta}(x,x';E+i\eta)}=\langle\psi_{a\alpha}(x)\bar{\psi}_{b\beta}(x')\rangle=\langle\xi_{a\alpha}(x)\xi^*_{b\beta}(x')\rangle. \tag{22}$$

(We have set $i\omega=E+i\eta$ in evaluating the correlators.) As we need the Green's function when $x=x'$, there is some subtlety on the relative ordering of x and x'. The correct procedure[12] is to take a symmetrized form:

$$2\pi\rho(E)=\mathrm{Re}[\langle\xi_{a\alpha}(x+\epsilon)\xi^*_{a\alpha}(x)+\xi_{a\alpha}(x-\epsilon)\xi^*_{a\alpha}(x)\rangle], \tag{23}$$

where $\epsilon=0^+$, and summation over a,α is implied. Precisely the same expression with $\xi\to\psi$ holds in terms of the fermionic variables as well. The other physical quantity we will be interested in is the diffusion propagator. This is defined, as usual, in terms of the Green's function by

$$P(x,x')=\sum_{ab,\alpha\beta}\overline{|G_{a\alpha,b\beta}(x,x';i\eta)|^2}. \tag{24}$$

Now the symmetry $\sigma_y H\sigma_y=-H^*$ can be used to show that

$$G_{ab}^{*\alpha\beta}(x,x';i\eta)=-(-1)^{\alpha+\beta}G_{ab}^{\bar{\alpha}\bar{\beta}}(x,x';i\eta), \tag{25}$$

where $\bar{\alpha}=2$ if $\alpha=1$ and vice versa. Thus $P(x,x')$ may be written

$$P(x,x')=-\sum_{ab,\alpha\beta}(-1)^{\alpha+\beta}\overline{G_{ab}^{\alpha\beta}G_{ab}^{\bar{\alpha}\bar{\beta}}} \tag{26}$$

$$=\sum_{ab,\alpha\beta}(-1)^{\alpha+\beta}\langle\psi_{a\alpha}(x)\bar{\psi}_{b\beta}(x')\xi_{a\bar{\alpha}}(x)\bar{\xi}_{b\bar{\beta}}(x')\rangle. \tag{27}$$

We have chosen to write one Green's function in terms of the fermions and one in terms of the bosons. This enables a calculation of the two particle properties using the same formulation needed to calculate the one particle properties.

In the limit where $t^\mu=0$, $\mu=0,1,2,3$, the left/right moving fields decouple for every realization of the disorder. Considering just one of them, say the right movers, we get the action

$$S=\int dx[\bar{\psi}_1(\partial_x+\vec{\eta}_1\cdot\vec{\sigma})\psi_1+\omega(\bar{\psi}_1\psi_1)+(\psi\leftrightarrow\xi)]. \tag{28}$$

We now average over the disorder assuming $\vec{\eta}_1$ to be distributed as $P[\vec{\eta}_1]\propto\exp\{-\int dx[(\vec{\eta}_1)^2/2u]\}$. The resulting translationally invariant action can be interpreted as the coherent state path integral of a zero-dimensional quantum ''Hamiltonian'' in terms of Bose operators $b_{1\alpha}=(b_{1\uparrow},b_{1\downarrow})$ and Fermi operators $f_{1\alpha}=(f_{1\uparrow},f_{1\downarrow})$. Before doing that, we note that the fermionic fields actually satisfy periodic boundary conditions. To get fermion fields that satisfy antiperiodic boundary conditions, we may perform a change of variables $\psi_1\to\psi_1 e^{i\pi x/L},\bar{\psi}_1\to\bar{\psi}_1 e^{-i\pi x/L}$. This adds a term $\int dx(\pi/L)\bar{\psi}_1\psi$ to the action. Thus we get

$$Z=S\,\mathrm{Tr}\,e^{-Lh_R}, \tag{29}$$

$$h_R=u(f_1^\dagger\vec{\sigma}f_1+(f_1\leftrightarrow b_1))^2+\omega(f_1^\dagger f_1+b_1^\dagger b_1). \tag{30}$$

(The subscript R on h is a reminder that this is for the right moving fields alone.) The supertrace operation $S\,\mathrm{Tr}$ is defined through $S\,\mathrm{Tr}\,\mathcal{O}=\mathrm{Tr}((-1)^{f_1^\dagger f_1}\mathcal{O})$. It is necessary to take the supertrace to account for the extra term in the action coming from the change of the fermion boundary conditions.

At zero ω, it is clear that there is a triplet of states with zero energy: the vacuum state with no particles which we denote $|0\rangle$, the state $f_{1\uparrow}^\dagger f_{1\downarrow}^\dagger|0\rangle\equiv|\uparrow\downarrow;0\rangle$, and the state $(1/\sqrt{2})(b_{1\uparrow}^\dagger f_{1\downarrow}^\dagger-b_{1\downarrow}f_{1\uparrow})|0\rangle\equiv|\uparrow;\downarrow\rangle$. All other states have energies at least $\mathcal{O}(u)$. (Nonzero ω, of course, splits the energies of this zero energy triplet.) Similar considerations apply to the left moving sector as well. Thus there are a set of nine states all at zero energy at zero ω in the limit of decoupled right/left sectors.

Now consider coupling the left/right moving sectors. The full action can also be interpreted (after disorder averaging) as the coherent state path integral of a zero-dimensional quantum Hamiltonian. In the limit where the coupling is small, it is sufficient to project the interactions induced between the two sectors to the nine-dimensional space of the ground states of the two decoupled sectors. For simplicity, we assume that the t_μ are Gaussian distributed with

$$\overline{t_\mu(x)t_\nu(x')}=t^2\delta_{\mu\nu}\delta(x-x'). \tag{31}$$

To leading order then, the coupling between the two sites in the nine-dimensional space will be of order t^2/u. To derive the form of this coupling, it is convenient to gauge away $\vec{\eta}_1$ and $\vec{\eta}_2$ by letting $\psi_i\to U_i\psi_i,\bar{\psi}_i\to\bar{\psi}_i U_i^\dagger$, and similarly for ξ with $U_i(x)=T_x[e^{i\int dx\vec{\eta}_i(x)\cdot\vec{\sigma}}]$ for $i=1,2$. (T_x is the x-ordering symbol.) We impose the condition that $U_i(x=L)=1$ to maintain the periodic boundary conditions. Note that the $U_i(x)$ are random $SU(2)$ matrices. The full action can then be written

$$S=\int dx\bar{\psi}(\tau_z\partial_x+\omega)\psi+i(\bar{\psi}B(x)\tau^+\psi+\bar{\psi}B^\dagger(x)\tau^-\psi)$$
$$+(\psi\to\xi). \tag{32}$$

Here $B(x)=U_1(x)(t_0+i\vec{t}\cdot\vec{\sigma})U_2^\dagger(x)$ is a random 2×2 matrix. It is distributed according to

$$\overline{B_{\alpha\beta}^\dagger(x)B_{\gamma\delta}(x')}=t^2e^{-u|x-x'|}\delta_{\alpha\delta}\delta_{\beta\gamma},$$

$$\overline{B_{\alpha\beta}(x)B_{\gamma\delta}(x')} = t^2 e^{-u|x-x'|}(\sigma_y)_{\alpha\delta}(\sigma_y)_{\beta\gamma}.$$

For large u, we may replace $t^2 e^{-u|x-x'|} \to J\delta(x-x')$ with $J = t^2/2u$. It is now convenient to change variables $\psi_{2\alpha} \to -\overline{\psi}_{2\alpha}, \overline{\psi}_{2\alpha} \to \psi_{2\alpha}, \xi_{2\alpha} \leftrightarrow -\xi^*_{2\alpha}$. This changes the action to

$$\int dx[\overline{\psi}\partial_x\psi + \xi^*\partial_x\xi + (-\overline{\psi_1}B(x)\overline{\psi_2} - \psi_2 B^\dagger(x)\psi_1 - \xi_1^* B\xi_2 + \xi_2 B^\dagger \xi_1) + \omega(\overline{\psi}\psi + \xi^*\xi)]. \tag{33}$$

Under this change of variables, the expression Eq. (23) for the density of states remains unchanged (in the limit $\epsilon \to 0^+$).

We may now perform the disorder average to get a translationally invariant action which can be interpreted as the coherent state path-integral of a zero-dimensional quantum problem with Bose operators $b_{a\alpha}$, Fermi operators $f_{a\alpha}(a = 1,2; \alpha = \uparrow,\downarrow)$, and a ''Hamiltonian:''

$$h = h_0 + h_\omega, \tag{34}$$

$$\begin{aligned} h_0 = -J\{&(f_1^\dagger\sigma_y f_1^\dagger)(f_2^\dagger\sigma_y f_2^\dagger) + (f_1\sigma_y f_1)(f_2\sigma_y f_2) \\ &+ 2(f_1^\dagger f_1 - 1)(f_2^\dagger f_2 - 1)] - 2(b_1^\dagger b_1 + 1)(b_2^\dagger b_2 + 1) \\ &+ [2(b_1^\dagger\sigma_y f_1^\dagger)(f_2^\dagger\sigma_y b_2^\dagger) + 2(b_1\sigma_y f_1)(f_2\sigma_y b_2) \\ &- 2(f_1^\dagger b_1)(f_2^\dagger b_2) - 2(b_1^\dagger f_1)(b_2^\dagger f_2)]\}, \end{aligned} \tag{35}$$

$$h_\omega = \omega(f^\dagger f + b^\dagger b). \tag{36}$$

Note that this Hamiltonian is non-Hermitian. Its action on the nine-dimensional subspace is simplified by noting that $f_1^\dagger f_1 - f_2^\dagger f_2$, and $b_1^\dagger b_1 - b_2^\dagger b_2$ commute with h. Thus the six states $|\uparrow;\downarrow\rangle_1 \otimes |0\rangle_2, |0\rangle_1 \otimes |\uparrow;\downarrow\rangle_2, |\uparrow\downarrow;0\rangle_1 \otimes |0\rangle_2, |0\rangle_1 \otimes |\uparrow\downarrow;0\rangle_2, |\uparrow;\downarrow\rangle_1 \otimes |\uparrow\downarrow;0\rangle_2, |\uparrow\downarrow;0\rangle_1 \otimes |\uparrow;\downarrow\rangle_2$ are immediately seen to be eigenstates of the Hamiltonian. The first four have eigenvalues $4J + 2\omega$ and the last two have $4J + 4\omega$. The action of h on the remaining three states $|0\rangle_1 \otimes |0\rangle_2, |\uparrow;\downarrow\rangle_1 \otimes |\uparrow;\downarrow\rangle_2, |\uparrow\downarrow;0\rangle_1 \otimes |\uparrow\downarrow;0\rangle_2$ can be represented in terms of a 3×3 non-Hermitian matrix:

$$4\begin{pmatrix} 0 & -J & -J \\ J & 2J+\omega & J \\ -J & -J & \omega \end{pmatrix}. \tag{37}$$

There is one eigenvalue 0 and two eigenvalues $4J + 4\omega$. With these eigenvalues, it is easy to see that $Z = S\,\mathrm{Tr}\,e^{-Lh} = 1$ for any system size L, as required. We will also need to know the zero energy wave function. However, due to the non-Hermiticity of the Hamiltonian, the left eigenvector $(\vec{V}_L)$ and the right eigenvector $(\vec{V}_R)$ are different. They are easily seen to be

$$\vec{V}_R = \frac{1}{1+\frac{\omega}{J}}\left[1+\frac{\omega}{J}, -1, 1\right],$$

$$\vec{V}_L = \frac{1}{1+\frac{\omega}{J}}\left[1+\frac{\omega}{J},\ \ 1, 1\right].$$

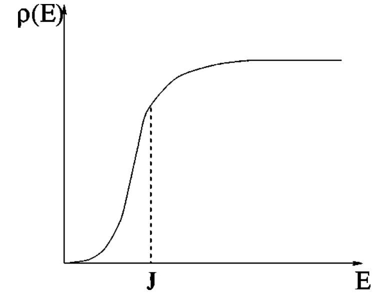

FIG. 2. Density of states in one dimension in the absence of time reversal symmetry; the energy scale $J \sim 1/\xi$.

We have normalized these so that $\vec{V}_L \cdot \vec{V}_R = 1$. There are some subtle questions regarding the resolution of the identity in the basis of right (left) eigenvectors of the Hamiltonian which are addressed at length in the Appendix.

The expression Eq. (23) for the density of states can clearly be interpreted as the following expectation value:

$$2\pi\rho = 2\,\mathrm{Re}\langle 2 + b^\dagger b\rangle, \tag{38}$$

where we calculate expectation values setting $i\omega = E + i\eta$. In the thermodynamic limit only the zero energy state contributes, and the result is

$$\frac{\pi}{2}\rho(E) = 1 - \frac{1 - \frac{E^2}{J^2}}{\left(1 + \frac{E^2}{J^2}\right)^2}. \tag{39}$$

Note that this vanishes as E^2 at small E, entirely consistent with the general arguments presented earlier. For large E, this saturates at $2/\pi$ which is the ballistic result (see Fig. 2).

The diffusion propagator can also be calculated explicitly in this one-dimensional case. The calculation proceeds straightforwardly from Eq. (26). We first perform the change of variables $\psi_{2\alpha} \to -\overline{\psi}_{2\alpha}, \overline{\psi}_{2\alpha} \to \psi_{2\alpha}, \xi_{2\alpha} \leftrightarrow -\xi^*_{2\alpha}$, and then interpret the resulting correlator as an expectation value of an operator in the equivalent quantum problem. We find

$$P(x,x') = \langle T_x(O_1(x)O_2(x'))\rangle,$$

$$O_1 = (f_{1\uparrow}b_{1\downarrow} - f_{1\downarrow}b_{1\uparrow}) + (f_{2\uparrow}^\dagger b_{2\downarrow}^\dagger - f_{2\downarrow}^\dagger b_{2\uparrow}^\dagger),$$

$$O_2 = (f_{1\uparrow}^\dagger b_{1\downarrow}^\dagger - f_{1\downarrow}^\dagger b_{1\uparrow}^\dagger) - (f_{2\uparrow}b_{2\downarrow} - f_{2\downarrow}b_{2\uparrow}).$$

The expectation value is to be taken in the zero energy state. Consider $x > x'$ for definiteness. Thus we write

$$P(x,x') = \langle O_1 e^{-h(x-x')} O_2\rangle. \tag{40}$$

We may evaluate this by inserting a complete set of states. As O_2 acting on the ground state is a state with energy $4J$ (when $\omega \to 0$), $P(x,x')$ decays as $e^{-4J(x-x')}$. The precise result is easily seen to be

$$P(x,x') = 8e^{-4J|x-x'|}. \tag{41}$$

Thus the localization length of the system is $\xi = 1/2J$. In momentum space, this becomes

TABLE I. Properties of the two different symmetry classes of superconductors with spin $SU(2)$ symmetry. WL stands for weak localization. $\rho_{loc}(E)$ is the density of states in the localized phase. The last column gives the critical properties of the density of states above two dimensions as calculated in a $2+\epsilon$ expansion. The distance from the critical point is (δg), and $\rho_{cr}(E)$ is the density of states at the critical point.

Symmetry	WL in $d=2$	$\rho_{loc}(E)$	Critical properties in $d=2+\epsilon$
Spin $SU(2)$ and $\mathcal{T}$	$\frac{\delta\sigma_s}{\sigma_s}=-\frac{1}{2\pi^2\sigma_s}\ln\left(\frac{L}{l_e}\right)$	$\|E\|$	$\rho(E=0)\sim(\delta g)^{1/2}$
	$\frac{\delta\rho}{\rho}=-\frac{1}{4\pi^2\sigma_s}\ln\left(\frac{L}{l_e}\right)$		$\rho_{cr}(E)\sim\|E\|^{\epsilon/4}$
Spin $SU(2)$ and no $\mathcal{T}$	$\frac{\delta\sigma_s}{\sigma_s}=-\frac{1}{4\pi^2\sigma_s}\ln\left(\frac{L}{l_e}\right)$	E^2	$\rho(E=0)\sim(\delta g)$
	$\frac{\delta\rho}{\rho}=-\frac{1}{4\pi^2\sigma_s}\ln\left(\frac{L}{l_e}\right)$		$\rho_{cr}(E)\sim\|E\|^{\epsilon/2}$

$$P(q)=\frac{32J}{(4J)^2+q^2}. \tag{42}$$

Note the difference in structure from conventional localization with a finite density of states where $P(q=0,\omega)$ has a pole at $\omega=0$. In this problem, the density of states vanishes and there is no pole.

The detailed calculation above of the one-dimensional problem is strong evidence in support of our general assertions regarding the vanishing of the density of states in the localized phase. In this case, the crossover from the constant to the vanishing density of states occurs at an energy scale $J\sim 1/\xi$ which is the energy scale for the crossover from the ballistic to the localized regime. We expect that Eq. (39) is a universal scaling function for the density of states associated with this crossover. In two dimensions (or in quasi-one-dimensional situations such as that considered in Ref. 13), the crossover occurs between the diffusive and localized regimes at a scale D/ξ^2 (where D is the diffusion constant). Again, this crossover is expected to be represented by a universal scaling function for the density of states.

IV. ABOVE TWO DIMENSIONS

We now turn to the situation above two dimensions where there is the possibility of a spin metal to spin insulator transition. The density of states is finite in the spin metal phase and vanishes on approaching the transition. Thus, in contrast to usual Anderson localization, the density of states behaves as a conventional order parameter in these universality classes. The order parameter exponent β may be calculated within the $2+\epsilon$ expansion. We find, to leading order in ϵ, $\beta=\frac{1}{2}$ if $\mathcal{T}$ is present, and $\beta=1$ without $\mathcal{T}$. Right at the transition, the density of states vanishes with energy as $\rho(E)\sim E^{1/\delta}$. The exponent $\delta=4/\epsilon, 2/\epsilon$ with and without $\mathcal{T}$, respectively.

V. DISCUSSION

In this paper, we have studied the behavior of the quasiparticle density of states in a dirty $d_{x^2-y^2}$ superconductor ignoring the quasiparticle interactions. We showed the existence of a singular logarithmic suppression of the density of states in the diffusive regime in two dimensions due to quantum interference effects. We then argued that in any dimension in the localized phase the density of states vanishes as $|E|$ if both spin rotation and $\mathcal{T}$ symmetry are present, and as E^2 if spin rotation is the only symmetry. This was verified by a simple explicit calculation in the latter case in one dimension using supersymmetry techniques. Above two dimensions, we showed that the density of states is finite in the spin metal phase, but vanishes on approaching the transition to the insulator. The corresponding critical exponent was calculated in a $2+\epsilon$ expansion. These results are summarized in Table I.

We emphasize that the ultimate vanishing of the density of states at zero energy in two dimensions does not invalidate the use of the nonlinear sigma model field theory.[6] The sigma model description assumes a finite, nonvanishing *bare* density of states. The renormalized value of the density of states is then determined by the properties of the field theory itself. It is useful to make a comparision to a more familiar physical situation—the classical Heisenberg ferromagnet in two dimensions. It is well known that this has no long range order at any finite temperature. Nevertheless, a correct field-theoretic description of this system at low temperature is provided by the $O(3)$ nonlinear sigma model in two dimensions. This field theory assumes the presence of a local order parameter—however, the renormalized value of the order parameter is zero at any finite temperature. The field theoretic description of quasiparticle localization in a superconductor is quite similar with the density of states playing the role of the order parameter. This is, however, quite different from localization in a normal metal where the density of states remains finite in the localized phase.

These results imply that the spin susceptibility, linear temperature coefficient of the specific heat, and the tunneling density of states all have a logarithmic suppression as a function of temperature in the diffusive regime in two dimensions due to quantum interference effects. As pointed out in Ref. 6, inclusion of a Zeeman magnetic coupling drives the system into the usual unitary universality class where there are no singular corrections to the density of states. Thus this logarithmic correction is killed by an external Zeeman field (though not by a purely orbital magnetic field). Experimental verification of this effect may be clouded somewhat due to the presence of quasiparticle interactions. We have shown elsewhere[14] that in the diffusive regime, interaction effects lead to a logarithmic Altshuler-Aronov suppression of the tunneling density of states in the diffusive regime in two dimensions. This therefore adds to the quantum interference correction discussed in this paper. In contrast, the specific heat and spin susceptibility are expected to get logarithmic enhancements due to interactions in two dimensions in the diffusive regime,[15] which too is killed by a Zeeman field. They would thus compete with the quantum interference corrections. Nevertheless, if the interactions are weak, we expect that the quantum interference effects would dominate leading to a logarithmic suppression of the spin susceptibility and specific heat, which can be probed by applying an external Zeeman magnetic field.

The effect of interactions in the localized phase is a more delicate matter. Qualitatively, repulsive interactions tend to favor the formation of local moments leading possibly to a divergent spin susceptibility and linear specific-heat coefficient. This effect will however compete with the vanishing density of states we have discussed above (which tends to produce a vanishing spin susceptibility, etc.). The ultimate fate of the localized phase in the presence of these two competing physical effects is a formidable problem that we will not attempt to answer here.

ACKNOWLEDGMENTS

We are particularly grateful to Martin Zirnbauer for a most useful communication. We also thank Leon Balents, Ilya Gruzberg, and Andreas Ludwig for useful discussions. This research was supported by NSF Grant Nos. DMR-97-04005, DMR95-28578, and PHY94-07194.

APPENDIX: RESOLUTION OF THE IDENTITY

In this appendix, we discuss some subtle questions regarding the resolution of the identity in the eigenbasis of the (super)Hamiltonian Eq. (34). As all the subtleties are associated with the three-dimensional subspace spanned by $|0\rangle_1 \otimes |0\rangle_2, |\uparrow;\downarrow\rangle_1 \otimes |\uparrow;\downarrow\rangle_2, |\uparrow\downarrow;0\rangle_1 \otimes |\uparrow\downarrow;0\rangle_2$, we just focus on these three states. In this subspace, the Hamiltonian h is represented by the 3×3 matrix Eq. (37). The right eigenstates corresponding to the two eigenvalues 0 and $4J+4\omega$ are easily seen to be (in bra/ket notation)

$$|R_1\rangle = a_1 \begin{bmatrix} 1+z \\ -1 \\ 1 \end{bmatrix}; \quad |R_2\rangle = a_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \tag{A1}$$

where a_1, a_2 are normalization constants, and $z=\omega/J$. ($|R_1\rangle$ has eigenvalue 0, and $|R_2\rangle$ has eigenvalue $4J+4\omega$). Note that the right eigenstates do not span the full three-dimensional space. From the structure of the Hamiltonian Eq. (34), it is easy to see that from every right eigenstate $|R\rangle$, a corresponding left eigenstate $\langle L|$ can be obtained by the operation

$$\langle L| = ((-1)^{f_1^\dagger f_1}|R\rangle)^T, \tag{A2}$$

where the symbol T denotes taking the transpose. The left eigenstates corresponding to $|R_1\rangle$ and $|R_2\rangle$ then are

$$\langle L_1| = a_1[1+z, \quad 1, \quad 1]; \quad \langle L_2| = a_2[0, \quad 1, \quad 1] \tag{A3}$$

respectively, as can also be seen by direct calculation. The left eigenstates also do not span the full three-dimensional space. Note that $\langle L_1|R_2\rangle = \langle L_2|R_1\rangle = \langle L_2|R_2\rangle = 0$.

To get a complete set of states, we need to supplement $\langle L_1|$ and $\langle L_2|$ by any other linearly independent bra vector $\langle L_3|$. It is convenient to choose this to be orthogonal to $|R_1\rangle$ and $(\langle L_2|)^T$.

$$\langle L_3| = a_3[-2, \quad -(1+z), \quad (1+z)]. \tag{A4}$$

A corresponding right state $|R_3\rangle$ can be defined using Eq. (A2):

$$|R_3\rangle = a_3 \begin{bmatrix} -2 \\ 1+z \\ 1+z \end{bmatrix}. \tag{A5}$$

Clearly the $|R_i\rangle$ ($i=1,2,3$) form a complete set of states (as do the $\langle L_i|$). By construction, we have the relations $\langle L_3|R_1\rangle = \langle L_1|R_3\rangle = 0$. We now impose the normalization conditions $\langle L_1|R_1\rangle = \langle L_2|R_3\rangle = \langle L_3|R_2\rangle = \langle L_3|R_3\rangle = 1$. This fixes $a_1 = 1/(1+z), a_2 = -1/(1+z), a_3 = 1/2$. It is now possible to construct the resolution of the identity

$$1 = |R_1\rangle\langle L_1| + |R_2\rangle\langle L_3| + |R_3\rangle\langle L_2| - |R_2\rangle\langle L_2|. \tag{A6}$$

This can be checked directly by its action on any vector in the three-dimensional space.

This resolution of the identity can now be used to easily show explicitly that $Z = S\,\mathrm{Tr}\, e^{-hL} = 1$ in the full nine-dimensional space. For the calculation of the density of states or the diffuson, we need to know the action of the Hamiltonian h on $|R_3\rangle$. This is easily seen to be

$$h|R_3\rangle = 4J(1+z)(|R_3\rangle + |R_2\rangle). \tag{A7}$$

Combined with the eigenvalue equation $h|R_2\rangle = 4J(1+z)|R_2\rangle$, this implies that

$$e^{-hL}|R_3\rangle = e^{-4J(1+z)L}(|R_3\rangle - 4JLz(1+z)|R_2\rangle).$$

In the limit $L\rightarrow\infty$, $e^{-hL}|R_3\rangle \rightarrow 0$; similar considerations apply to $\langle L_3|$ as well. Calculation of any correlation function is thus reduced, in the limit of infinite system size to a calculation in the zero energy state with (right) eigenvector $|R_1\rangle$.

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