Dual order parameter for the nodal liquid

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The guiding conception of vortex-condensation-driven Mott insulating behavior is central to the theory of the nodal liquid. We amplify our earlier description of this idea and show how vortex condensation in two-dimensional (2D) electronic systems is a natural extension of 1D Mott insulating and 2D bosonic Mott insulating behavior. For vortices in an underlying superconducting pair field, there is an important distinction between the condensation of flux $hc/2e$ and flux hc/e vortices. The former case leads to spin-charge confinement, exemplified by the band insulator and the charge-density wave. In the latter case, spin and charge are liberated, leading directly to a 2D Mott insulator exhibiting *spin-charge* separation. Possible upshots include not only the nodal liquid, but also an undoped antiferromagnetic insulating phase with gapped excitations exhibiting spin-charge separation. [S0163-1829(99)02924-0]

I. INTRODUCTION

The present paper is rooted in the conviction that the basic property of an insulator is that it insulates. Magnetic order is a secondary effect which, though it may be one of the appurtenances of insulating behavior, is not synonymous with it. This premise underlies our recent discussion of the phase diagram of underdoped cuprate superconductors, where we relied upon the notion of an insulator as a vortex condensate.[1,2] Here, we would like to elucidate and expand upon this paradigm. As a by-product of this approach, we find a precise distinction between spin-charge-separated and confined insulators.

Since correlated insulators often exhibit magnetism, they are typically described by their magnetic order parameters. Furthermore, in commensurate, weak-coupling models, the development of magnetic order can be the mechanism by which a system becomes insulating. This can lead to a conflation of magnetic order with insulating behavior.[3] However, this state of affairs is both unsatisfying and incomplete, since magnetic order can persist even when the system becomes conducting and, conversely, a system can be insulating even in the absence of magnetic order, as suggested by Anderson's original Resonating Valence Bond (RVB) ideas and exemplified by the nodal liquid. Hence it would behoove us to seek an order parameter which is directly related to the electrical properties of an insulator. The difficulty in such a program is that an insulator seems more disordered than a conductor since most correlation functions in the charge sector are short ranged. We interpret this as suggesting a ''dual'' approach based on a ''disorder'' parameter. Distilling the key elements of our nodal liquid construction, we propose that the appearance of a nonzero expectation value for a disorder parameter signals an insulating behavior.

The relation between the conduction properties of a system and spontaneous symmetry breaking in a dual order parameter is perhaps most transparent in the field-theoretical formulation of duality.[5] In this version of the transformation, the dual theory is constructed to implement local charge continuity, which is a *dynamical* consequence of U(1) symmetry in the original Hamiltonian, as a rigid *constraint* (dynamics in the dual theory implies conservation of *vorticity*). This construction is quite familiar from the one-dimensional theory of Luttinger liquids.[6] There the two-current can be written in terms of a phase field $\phi(x)$ as

$$j_i = \partial_i \phi. \tag{1.1}$$

In the alternative, but equivalent, dual description, the current can be written in terms of a dual field, θ:

$$j_i = \epsilon_{ij} \partial_j \theta. \tag{1.2}$$

In a one-dimensional Mott insulator, the θ field acquires a mass, implying a gap in the spectrum of charge excitations, and hence, experimentally, in the optical conductivity. Note that massive dynamics for the *phase* field ϕ is inconsistent, since it would violate charge conservation $\partial_i j_i = 0$. The one-dimensional model has thus communicated an important lesson: insulating behavior occurs when a gap is acquired by the dual field representing the current operator.

This description of Mott insulators generalizes readily to two-dimensional *bosons*. Let us write the boson annihilation operator ψ in terms of its amplitude and phase: $\psi = \sqrt{\rho} e^{i\varphi}$. In the superfluid state, we fix ρ and describe the system by its phase degree of freedom, $\varphi(x)$:

$$\mathcal{L} = \tfrac{1}{2} (\partial_\mu \varphi)^2. \tag{1.3}$$

We can model the destruction of superfluidity by introducing a vortex field Φ; effectively, we are reducing the amplitude degree of freedom to a vortex field which keeps track of the points at which it vanishes. We can represent the current $j_\mu = \partial_\mu \varphi$ as[5,7]

$$j_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda , \tag{1.4}$$

which is the natural extension of Eq. (1.2) to two dimensions. This parametrization is highly redundant as a result of its invariance under the gauge symmetry:

$$a_\mu \rightarrow a_\mu + \partial_\mu \chi \tag{1.5}$$

for an arbitrary function, χ. This gauge symmetry is enormously larger than the analogous global invariance in the one-dimensional Luttinger liquid, $\theta \rightarrow \theta + \text{const}$. The vortices ''see'' the gauge field a_μ according to the Magnus force law, so the dual Lagrangian takes the form

$$\mathcal{L}_D = \tfrac{1}{2} (\partial_\mu a_\nu)^2 + |(\partial_\mu - i a_\mu)\Phi|^2 + V(\Phi). \tag{1.6}$$

In the superfluid state, vortices are gapped and the bosons condense, while, in the Mott insulating state, the vortices condense and the bosons become gapped (at the chemical potential). When the vortices condense, a_μ becomes massive by the Anderson-Higgs mechanism. As a consequence, the system is insulating.

Let us abstract away the archetypal features of this system. We can introduce representation (1.4) for any conserved current in two dimensions, so we can certainly use it for the electrical current in the fermionic system of our choice. In order to carry the rest of this scheme over to electronic systems, we must find a way for the system to conspire to make a_μ massive. The only way that this can happen which is consistent with the gauge symmetry (1.5) is via the Anderson-Higgs mechanism.[8] Since the Anderson-Higgs phenomenon can only take place if there is a condensate which is coupled to a_μ according to the Magnus force law [Eq. (1.6)], we are led to the following question: how do you define a vortex field in a fermionic system? One possibility is to implement statistical transmutation to represent the fermions as bosons coupled to an auxiliary Chern-Simons gauge field.[6] Then we can define vortices in the bosonic field. This approach is probably suitable for describing a conventional antiferromagnet, as discussed very briefly in Sec. IV. But in this paper we pursue a different tack—using Cooper pairs as the bosons. This is quite promising for the cuprates because it is tailor made for insulators which contain the germ of superconductivity.

A question rears its head when we consider an insulator which descends in this way from a superconductor: do the finite-energy excitations inherit their quantum numbers from the superconducting state, or do they simply have the electron quantum numbers? In particular, one can ask what is the energy of an isolated neutral spin-$\frac{1}{2}$ excitation. If this diverges with system size, then spin and charge are confined. If, on the other hand, it is finite, as it is in a superconductor, then the insulator exhibits spin-charge separation. A two-dimensional (2D) band insulator is, of course, a state of the former variety. As we show in Sec. II, this can be understood (rather differently than in elementary textbooks) as resulting from the condensation of flux $hc/2e$ vortices in a state with s-wave pairing. Spin and charge are confined as a result of the Aharonov-Bohm phase, which a spinful excitation acquires as it orbits an $hc/2e$ vortex; an isolated spin $\frac{1}{2}$ has a logarithmically divergent energy in two-dimensions. This spin-charge confinement physics is also present in a charge-density-wave (CDW) insulator, which occurs for example in the negative-u (extended) Hubbard model at half-filling. There, however, the $hc/2e$ vortex condensation leads to translational symmetry breaking (see also the second reference in Ref. 2). In both instances the resulting insulating phase can be viewed as a ''crystal'' of charge $2e$ ''Cooper'' pairs. For 2D electronic systems at or near half-filling with strong on-site repulsion, however, such CDW order is physically unreasonable, and for the cuprate materials can be discarded on phenomenological grounds. For this reason we are led to consider the possibility of the condensation of double strength hc/e vortices in a superconducting pair field.[9]

The Mott insulator which arises at half-filling upon condensation of hc/e vortices has a number of appealing and remarkable properties. The insulating phase is translationally invariant, in contrast to the CDW. Moreover, since there are no Aharonov-Bohm phase factors when a spin $\frac{1}{2}$ is transported around an hc/e vortex, the resulting Mott insulator exhibits *spin-charge* separation.[4,10,11] For a d-wave superconductor appropriate to the cuprates, condensation of hc/e vortices leads directly to the *nodal liquid*.[12] The nodal liquid indeed posseses *gapless* spin $\frac{1}{2}$ but charge-neutral fermions—the nodons—which descend directly from the d-wave quasiparticles. As we shall see, there are also massive charge e spinless boson excitations in the nodal liquid phase. These ''holons'' are solitonic *topological* excitations in the underlying hc/e condensate, a dual analog of Abrikosov flux tubes. The excitations in the nodal liquid have the same quantum numbers as in the spin-charge-separated gauge theories,[13] but are weakly interacting rather than strongly coupled by a gauge field. We suspect that a nodal liquid ground-state requires the retention of the charged degrees of freedom, and cannot occur in a spin-only model.

A peculiar feature of the nodal liquid is that spin-charge separation survives the ordering of the nodal spins into a phase with long-ranged antiferromagnetic order. This phase—denoted AF*—which has gapped nodons is distinct from the conventional Néel antiferromagnet AF which does not have neutral, spin-$\frac{1}{2}$ excitations even at high energies.[14] These two phases are physically very different, as may be seen from simple *gedanken* experiments which make the point that charge e can be physically separated from spin $\frac{1}{2}$ with finite energy cost in the AF* phase but not the AF phase. However, in two dimensions nodon-holon bound states form in the AF* phase, so the spin-charge separation is not so easily found in the electron spectral function.

Our ultimate goal is to describe a spin-charge-separated state, the nodal liquid, and an ordered state which can result from it, AF*. Along the way, however, we will re-examine a number of seemingly quotidian states such as the band insulator, the charge-density wave, and the antiferromagnet. Our broader framework will enable us to understand the physics of doping these insulators from the point of view of creating topological excitations in the disorder parameter.[15] Such a point of view naturally leads to a discussion of the possibility of spin-charge separation in these states. In Secs. II and III A, we will illustrate the physics of flux $hc/2e$ condensation in systems with attractive electron-electron interactions, where we expect the insulating states to be related to *s*-wave superconductivity. In the resulting states, the band insulator and the CDW, spin and charge are confined, as we discuss at length in Sec. II. In Sec. III B, we then consider flux hc/e vortex condensates in a *d*-wave superconductor, filling in a gap in our earlier paper. We introduce an effective lattice model in Sec. III C which incorporates this physics. In Sec. IV, we discuss the spin-charge-separated antiferromagnet AF*, and compare it to the conventional antiferromagnet AF. Possible experimental signatures are analyzed. We conclude with some summary remarks in Sec. V, relegating some supporting technical details to the Appendix.

II. BAND INSULATOR AND SPIN-CHARGE CONFINEMENT

In the absence of electron interactions, a band insulator with two electrons per unit cell corresponds to a filled valence band of noninteracting levels. Provided the interactions are small compared to the energy gap of the conduction band, this should provide a good description of the phase. But even with stronger interactions a band insulator can be adiabatically deformed (without gap closure) back to the noninteracting state. To obtain an order parameter for the band insulator, we will attempt to describe this phase as a ''quantum-disordered'' *s*-wave superconductor.

To this end, consider spinful electrons moving in the two-dimensional continuum. In the presence of a local attractive interaction the Fermi surface is unstable, and presumed to form a spin-singlet *s*-wave superconducting phase, with energy gap Δ. Now imagine introducing a periodic potential with magnitude V and a period corresponding to two electrons per unit cell. For V much smaller than Δ the superconducting phase should be stable, but with increasing V one expects an eventual quantum transition into a band insulator. This transition can be described as a quantum vortex unbinding transition, analogous to the thermally driven Kosterlitz-Thouless transition. At zero temperature the vortices can condense, giving one an order parameter for the band insulator.

Of particular interest is the nature of the gapped excitations in the two phases. As we shall discuss, in a superconductor it is possible to define charge-neutral quasiparticles[16] which carry spin $\frac{1}{2}$. In an *s*-wave superconductor these ''spinon'' excitations are fully gapped,[10] but can be gapless (called nodons) in a *d*-wave superconductor, which we return to in Sec. IV below. Since the Cooper pairs carry no spin in a singlet superconductor, the Cooper pairs and spinons provide a natural spin-charge-separated description of the superconducting phase. On the other hand, the excitations in a band insulator are electrons which of course carry both spin and charge—the band insulator does *not* exhibit spin-charge separation. We would like to try and understand the mechanism whereby the separated spin and charge excitations in the superconductor become ''confined'' upon entering the band insulator.

To address these issues, it is convenient to consider a low-energy effective theory for an s-wave superconductor in which both Cooper pairs and the gapped quasiparticle states near the Fermi surface are retained. The appropriate Lagrangian takes the form $\mathcal{L}=\mathcal{L}_c+\mathcal{L}_\varphi+\mathcal{L}_{\rm int}$, with

$$\mathcal{L}_c=c_\alpha^\dagger(x,\tau)[i\partial_t-\nabla^2-\mu]c(x,\tau), \tag{2.1}$$

$$\mathcal{L}_\varphi=\frac{\kappa_\mu}{2}(\partial_\mu\varphi)^2, \tag{2.2}$$

$$\mathcal{L}_{\rm int}=|\Delta|e^{i\varphi}c_\uparrow(x)c_\downarrow(x)+\text{H.c.} \tag{2.3}$$

Here c_α denotes an electron with spin α, and φ is the phase of the pair field, with magnitude $|\Delta|$. Integrating over high-energy electron states, well away from the Fermi surface, will generate dynamics for the phase field. The appropriate form of $\mathcal{L}_\varphi$ at low energy is essentially determined by symmetry. Here we have retained the leading-order terms in a gradient expansion, with $\kappa_0=\kappa$ the compressibility and $-\kappa_j=v_c^2\kappa_0$ a superfluid stiffness. Henceforth we will set the velocity $v_c=1$. In general a Berry's phase term[6] of the form $\mathcal{L}_{\rm Berry}=n_0\partial_t\varphi$ is allowed (see Sec. III below) but with one Cooper pair (two electrons) per unit cell of the periodic potential $n_0=1$ and the Berry's phase term can be dropped since $\exp(i\int dt\,\mathcal{L}_{\rm Berry})=1$.

Notice that the phase fluctuations are strongly coupled to the electron operators through $\mathcal{L}_{\rm int}$. To decouple this interaction and to exhibit the spin-charge separation, it is convenient to consider the change of variables

$$f_\alpha(x)=e^{i\varphi/2}c_\alpha(x). \tag{2.4}$$

The Lagrangian becomes

$$\mathcal{L}=\mathcal{L}_f+\mathcal{L}_\varphi+\tfrac{1}{2}J_\mu\partial_\mu\varphi, \tag{2.5}$$

with

$$\mathcal{L}_f=f_\alpha^\dagger[i\partial_t-\nabla^2-\mu]f_\alpha+|\Delta|f_\uparrow(x)f_\downarrow(x)+\text{H.c.}, \tag{2.6}$$

and J_μ is a quasiparticle three-current operator:

$$J_0=f_\alpha^\dagger f_\alpha, \quad \vec{J}=if_\alpha^\dagger\vec{\nabla}f_\alpha+\text{H.c.} \tag{2.7}$$

This current is not conserved as a result of the anomalous term (2.3), but the spin currents

$$J_0^i=f_\alpha^\dagger\sigma_{\alpha\beta}^i f_\beta, \quad \vec{J}^i=if_\alpha^\dagger\sigma_{\alpha\beta}^i\vec{\nabla}f_\beta+\text{H.c.} \tag{2.8}$$

are conserved. Here we have assumed that the phase field is slowly varying, and have dropped terms involving two spatial gradients of φ.

The Lagrangian $\mathcal{L}_f$ can be diagonalized as usual by a Bogoliubov transformation, and describes gapped quasiparticles. Since the f operators are electrically neutral but carry spin $\frac{1}{2}$, these excitations are ''spinons.'' The spinons are coupled to the phase fluctuations via a Doppler-shift-type term. In the superconducting phase these phase fluctuations are small, and will generate a weak interaction between the gapped spinon states.

To quantum disorder the superconducting phase and arrive at a description of the band insulator, we will need to allow for vortices in the phase of the pair field. In two dimensions vortices are simply whorls of current swirling around a core region. The circulation of such vortices is *quantized*, since upon encircling the core the phase φ can only change by integer multiples of 2π. The ''elementary'' vortices have a phase winding of $\pm 2\pi$. Since the Cooper pairs have charge $2e$, in the presence of an applied magnetic field these vortices quantize flux in units of $hc/2e$. Inside the core of a vortex the *magnitude* of the complex order parameter $|\Delta|$ vanishes, but is essentially constant outside. Since the position of these ''pointlike'' vortices can change with time, their dynamics requires a quantum-mechanical description. Thus a collection of many vortices can be viewed as a many-body system of ''pointlike'' particles. Since positive $(+1)$ and negative (-1) circulation vortices can annihilate and disappear, they behave as ''relativistic'' particles. There is a conserved vortex ''charge'' in this process, namely, the total circulation and an associated current. A duality transformation can be implemented[5,7] in which the phase φ is replaced by a dual field θ, which is the phase of a vortex complex field $\Phi \sim e^{i\theta}$. In a Hamiltonian description, Φ and $\Phi^\dagger$ can be viewed as vortex quantum field operators—which destroy and create vortices.

A crucial element in the duality transformation is the total electrical three-current, $J_\mu^{\rm tot} = \kappa_\mu \partial_\mu \varphi + J_\mu/2$, which must be conserved even in the dual representation. This is achieved by expressing the current as a curl of a gauge field, a_μ (Ref. 5):

$$J_\mu^{\rm tot} = \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda , \tag{2.9}$$

which automatically implies the continuity equation $\partial_\mu J_\mu^{\rm tot} = 0$. This representation also introduces a gauge symmetry into the problem, $a_\mu \to a_\mu + \partial_\mu \Lambda$. It is this gauge symmetry which is spontaneously broken in the band insulating state.

In Ref. 1 the duality transformation was implemented in the presence of the Doppler-shift interaction between the Cooper pairs and the spinons, giving a dual Lagrangian of the form: $\mathcal{L}_D = \mathcal{L}_f + \mathcal{L}_v + \mathcal{L}_a$ with a vortex piece of the Ginzburg-Landau form,

$$\mathcal{L}_v(a_\mu) = \frac{\kappa_\mu}{2} |(\partial_\mu - i 2\pi a_\mu)\Phi|^2 - r|\Phi|^2 - u|\Phi|^4, \tag{2.10}$$

and

$$\mathcal{L}_a = \frac{1}{2\kappa_0}(e_j^2 - b^2) + \frac{1}{2\kappa_0} J_\mu \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda . \tag{2.11}$$

Here $e_j = (\partial_j a_0 - \partial_0 a_j)$ and $b = \epsilon_{ij}\partial_i a_j$ are dual ''electric'' and ''magnetic'' fields. The dual magnetic field b is simply the total charge density (in units of the Cooper pair charge). The last term in $\mathcal{L}_a$ is the only one coupling the spinons to the vortices. However, this dual Lagrangian is not valid since it ignores a strong statistical gauge interaction between spinons and $hc/2$e vortices. To see this consider taking a spinon (f) around a closed loop which encircles an $hc/2e$ vortex. Along this circuit the phase φ winds by 2π. Due to the $\frac{1}{2}$ in Eq. (2.4), this implies that the spinon f must change sign upon completing this circuit. This can be formally implemented by introducing branch cuts emanating from each and every vortex across which the fermion wave function must change sign. This represents a strong and long-ranged ''statistical'' interaction between the spinons and $hc/2$e vortices.

The presence of this long-ranged interaction clearly invalidates the form of the dual Lagrangian. One is tempted to try and incorporate the branch cuts by introducing two new gauge fields coupled together by a Chern-Simons interaction.[6] The most natural way of doing this is to introduce a coupling $J_\mu \alpha_\mu$ of the spinons to a gauge field α_μ which attaches half of a fictitious flux quantum to each vortex. However, such a coupling is not gauge invariant since the spinon current J_μ is not conserved. To avoid this problem—but at the cost of breaking spin-rotational invariance—we couple α_μ to the z component of the spinon spin current, which is conserved. Specifically, let α_μ and a_μ^s denote statistical gauge fields coupled to the spinons and vortices, respectively, with a modified dual Lagrangian

$$\mathcal{L}_D = \mathcal{L}_f + \mathcal{L}_a + \mathcal{L}_v(a_\mu + a_\mu^s) + \alpha_\mu J_\mu^z + \mathcal{L}_{\rm cs} \tag{2.12}$$

and a Chern-Simons interaction

$$\mathcal{L}_{\rm cs} = 2\alpha_\mu \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda^s . \tag{2.13}$$

The Chern-Simons term effectively attaches flux tubes with strength $\frac{1}{2}$ to each of the spinons and vortices. This follows from the equations of motion obtained from $\partial\mathcal{L}_D/\partial\alpha_\mu = 0$ and $\partial\mathcal{L}_D/\partial a_\mu^s = 0$, which imply, respectively,

$$\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda^s = \tfrac{1}{2} J_\mu^z \tag{2.14}$$

and

$$\epsilon_{\mu\nu\lambda}\partial_\nu \alpha_\lambda = \tfrac{1}{2} j_\mu^v . \tag{2.15}$$

Here, the vortex three-current is given by

$$j_\mu^v = {\rm Im}[\Phi^*(\partial_\mu - i a_\mu)\Phi]. \tag{2.16}$$

Consider now trying to condense the $hc/2e$ vortices. In the ground state the spinon (f fermions) are gapped out with $\langle J_\mu^z \rangle = 0$, so one can presumably set $a_\mu^s = 0$. Setting $\langle \Phi \rangle = \Phi_0$ corresponds to a spontaneous breaking of the gauge symmetry, and leads to an effective Higgs Lagrangian: $\mathcal{L} = \Phi_0^2 \kappa_\mu a_\mu^2/2$. In terms of the dual Ginzburg-Landau theory this describes the ''Meissner state.'' But since the curl of a_μ corresponds to the total electrical current, this phase corresponds to an insulator—the band insulator—with a charge gap.

If the dual Ginzburg-Landau theory is type II, it will exhibit topological excitations corresponding to penetrating quantized ''dual'' flux tubes. In the electronic insulator these correspond to gapped charge $\pm 2e$ spin-zero states, which are two-electron bound states.

But now consider an excited state in the insulator which carries spin $\frac{1}{2}$. This can be created by adding a spinon at the origin by acting with $f_\alpha^\dagger(x=0)$. The presence of a spinon induces a statistical gauge field from the Chern-Simons term:

$$h_s(x) = \epsilon_{ij}\partial_i a_j^s = \tfrac{1}{2} J_0 = \tfrac{1}{2}\delta^2(x). \tag{2.17}$$

Since $h_s(x)$ corresponds to an applied ''magnetic field'' in the dual Ginzburg-Landau theory, adding a spinon is equivalent to the insertion of a solenoid carrying one-half of a (dual) flux quantum. Being in the Meissner state, the dual Ginzburg-Landau theory will tend to screen out this applied magnetic field by generating currents that induce an opposing internal field, $b(x)$. This follows readily from the energy in the Meissner state, which takes the form

$$E=\frac{\Phi_0^2\kappa_0}{2}\int d^2x|\nabla\theta-2\pi\vec{a}-2\pi\vec{a}^s|^2, \tag{2.18}$$

where we have put $\Phi=\Phi_0 e^{i\theta}$. With both $\theta=0$ and $a_\mu=0$ the energy in the presence of the solenoid will diverge logarithmically with system size. Apparently, the energy of an isolated spinon diverges in the thermodynamic limit. But in the presence of an induced internal magnetic field the energy will be finite provided the integrated flux is precisely one-half of the dual flux quantum:

$$\int d^2x\, b(x)=-\tfrac{1}{2}. \tag{2.19}$$

In physical terms this corresponds to an induced cloud of electric charge with magnitude e. Evidently, in the insulating phase an isolated $S_z=\frac{1}{2}$ excitation will bind charge e to form a spin-up electron. The resulting excitation has finite energy. Similarly, an isolated $S_z=-\frac{1}{2}$ will bind charge $-e$ to form a spin-down hole with the same energy as the spin-up electron. This mechanism for confinement of spin and charge is reminiscent of the confinement of charge and flux which occurs in the quantum Hall effect. In the bosonic Chern-Simons formulation of the quantum Hall effect, the confinement of charge and flux also arises via a Higgs mechanism—in this case when the composite Boson condenses.

For the above Ginzburg-Landau–Chern-Simons theory there is another finite-energy configuration in the presence of a single $S_z=\frac{1}{2}$—a 2π winding in the phase θ of the vortex field together with an internal field of one-half quantum which *aligns* with the ''applied flux'' $[\int d^2x\, b(x)=+\frac{1}{2}]$. This creates a finite-energy excitation with spin $\frac{1}{2}$ and charge $-e$, corresponding to a conventional spin-up hole. We can create a spin-down electron of equal energy in a similar manner. On physical grounds the energy of the $S_z=\pm\frac{1}{2}$ states should clearly be degenerate. Unfortunately, for the above Ginzburg-Landau–Chern-Simons theory, while both energies are finite, they will in general be different, due to differences in the core energies (e.g., near the $S_z=-\frac{1}{2}$ state with θ winding, it is necessary to suppress the magnitude of Φ to zero). This signals a clear deficiency in the Chern-Simons formulation of the ''statistical'' interaction between spinons and $hc/2e$ vortices. Since the vortices and spinons can sense the *sign* of the statistical flux (that is, $\pm\frac{1}{2}$ flux quanta are *not* identical) the Chern-Simons fields do not give a faithful representation of the branch cuts. The necessary evil of breaking spin-rotational symmetry is a consequence of this asymmetry in the Chern-Simons formulation.

Currently, we do not have a convenient formulation of interacting $hc/2e$ vortices and spinons which correctly respects this symmetry. Such a formulation would be particularly desirable for the case of a d-wave superconductor where the quasiparticles are gapless. Nevertheless, we believe that the Chern-Simons formulation does help elucidate the correct mechanism behind confinement of spin and charge upon condensation of $hc/2e$ vortices.

The preceding considerations dovetail naturally with the following approach to understanding spin-charge separation. Consider the following *gedanken* experiment, for which we are indebted to Halperin,[14] which probes the existence of spin-charge separation. Consider a totally gapped system which exhibits spin-charge separation, meaning that it has weakly coupled neutral spin-$\frac{1}{2}$ excitations—spinons—and charge e spin less excitations – which we will call holons, following Ref. 10. Imagine imposing two spatially localized perturbations. These take the form of an interaction Hamiltonian

$$H_{\rm int}=\lambda[(Q(\mathbf{x})-e)^2+(S^z(\mathbf{x}))^2] + \lambda[(Q(\mathbf{x}'))^2+(S^z(\mathbf{x}')-\tfrac{1}{2})^2]. \tag{2.20}$$

Here $Q(\mathbf{x})=\Sigma_{\mathbf{y}}\rho(\mathbf{x}+\mathbf{y})f(y,\zeta)$ and $S^z(\mathbf{x})=\Sigma_{\mathbf{y}}s^z(\mathbf{x}+\mathbf{y})\times f(y,\xi)$ are the total charge and z component of the spin within a smoothed region of linear size ξ around the point $\mathbf{x}$. For reasons of mathematical rigor, we choose $f(y,\xi)$ to be a differentiable function of y with $f(0,\xi)=1$ and $f(y,\xi)=\ln\xi/\ln y$ for $y>2\xi$. The perturbation favors localizing a charge e *without spin* near $\mathbf{x}$ and a spin $\frac{1}{2}$ *without charge* near $\mathbf{x}'$. Now imagine taking $|\mathbf{x}-\mathbf{x}'|\gg\xi\gg a$ (the lattice spacing), so that the points are well separated. For small λ, the ground state of the system will be unchanged, since there is a gap to all excitations. Increasing λ will ultimately induce a change in the ground state to take advantage of these perturbations. Provided ξ is taken larger than the size of the spinon and holon, these excitations can come into the system to lower its energy and will be localized in the wells. This change in the ground state will occur at finite λ, since the energy gap to the spinon and holon is finite, and indeed the critical λ will saturate as $|\mathbf{x}-\mathbf{x}'|\to\infty$. One can interpret this critical λ_c as the minimum energy needed to produce an unbound spinon-holon pair.

Now imagine repeating the same experiment on a band insulator or any other state which does not exhibit spin-charge separation. In this case, the spinon and holon are not available to ''fill'' the local perturbations. Instead the system must create a nonlocal superposition of elementary excitations, i.e. develop a polarization, to localize the desired charge e or spin $\frac{1}{2}$. Consider, for example, the region around the point $\mathbf{x}$, in which a charge e should localize. A simple and generic model for a band insulator is a collection of deep potential wells, each containing two electrons. For the low-energy states, the wells may be approximated as quadratic and, in the ground state, all electrons are in the lowest harmonic-oscillator level. The lowest excited states which do not induce local spin textures are constructed simply by moving both electrons in one of the wells from the ground state to one of the first excited states with energy gap ω_0. A convenient basis for these states is the set $|\mu\rangle=a^\dagger_{\uparrow\mu}a^\dagger_{\downarrow\mu}|0\rangle$, where $a^\dagger_{\alpha\mu}=\sqrt{m\omega_0/2}(x_\mu-ip_\mu/m\omega)$ is the raising operator along the i axis in space. Now consider the superposition of the ground and excited states,

$$|\mathbf{u}\rangle=(1+|u|^2/l^2)^{-1/2}\left(|0\rangle+\frac{u^\mu}{l}|\mu\rangle\right), \tag{2.21}$$

where $l=\sqrt{2/m\omega_0}$ is the characteristic spatial width of the harmonic oscillator levels. For small $|u|$, the state $|\mathbf{u}\rangle$ represents a small displacement of the electrons, i.e.,

$$\langle\mathbf{u}|\mathbf{x}|\mathbf{u}\rangle=\mathbf{u}. \tag{2.22}$$

Thus it possesses a dipole moment $\mathbf{d}=-2e\mathbf{u}$, and a local charge may be built up by polarizing the collection of all the electrons near the point $\mathbf{x}$ (i.e. forming a many body state which is a direct product of single-particle states with differing displacements on each site). We consider a slowly varying displacement $\mathbf{u}(\mathbf{x})$ for which a continuum description is adequate (although this is not a necessary restriction). Far away from the point $\mathbf{x}$, the induced charge density $\rho = 2e\nabla\cdot\mathbf{u}$, and hence the charge in a given region R is $\int_R d^d\mathbf{x}\,\rho(\mathbf{x}) = 2e\int_{\partial R}\mathbf{u}\cdot d\hat{\mathbf{n}}$. The radially symmetric configuration

$$\mathbf{u}(\mathbf{x}'') = \frac{1}{2S_d}\frac{\mathbf{x}-\mathbf{x}''}{|\mathbf{x}-\mathbf{x}''|^d} \tag{2.23}$$

thus carries a total charge e. Provided all the electrons are involved in the texture, there is no net spin polarization. This polarized state is not an eigenstate of the unperturbed Hamiltonian, but does couple favorably to the first term in Eq. (2.20). We can, however, determine the expectation value of the (unperturbed) energy in this state. The result is essentially classical: $E(\mathbf{u}) = \int d^d\mathbf{x}\omega_0(u/l)^2$. In two dimensions, this integral is logarithmically divergent, $E(\mathbf{u})\sim \ln L$. The isolated long-range polarization thus cost an *infinite* energy. In the thought experiment, this divergence will be cut off by the finite distance between $\mathbf{x}$ and $\mathbf{x}'$, since we may localize an oppositely charged texture around the point $\mathbf{x}'$ in combination with an added electron with $S^z = \frac{1}{2}$, thus satisfying both perturbations and rendering the energy finite. However, the critical λ_c will grow logarithmically as $|\mathbf{x}-\mathbf{x}'|\to\infty$, and hence it becomes impossible to create the isolated holon and spinon in the thermodynamic limit.

This argument is appealing in that it agrees with earlier Chern-Simons calculations, which suggested logarithmic confinement of holons and spinons. As a means of distinguishing spin-charge-separated from spin-charge-confined phases, however, it is somewhat delicate. In particular, it fails for $d>2$, where the polarization energy to create the charge e texture becomes finite. It is also somewhat unsatisfying because the texture is not an eigenstate of the unperturbed Hamiltonian. Fortunately, the duality formalism allows the two phases to be distinguished instead by the dual order parameter Φ. When Φ condenses, the statistical gauge interactions between spinons and the condensing vortices leads to spin-charge confinement in $d=2$. In this sense Φ is an *order parameter for confinement*, a rather unique feature of the present theory.

III. MOTT INSULATORS, *HC/E* CONDENSATES, AND NODAL LIQUID

We now turn to the more interesting case of insulating, phases with one electron per unit cell. In such Mott insulators, electron interactions are necessary to destroy the metallic state, in contrast to the band insulator which has a smooth noninteracting limit. As in Sec. II, we will obtain a description of the insulating state by quantum disordering an appropriate superconducting phase. A brief discussion of quantum-disordered *s* wave superconductivity at this density illustrates the need to consider ‘‘double-strength’’ vortex condensates in the cuprates. Such hc/e condensates are *nodal liquid* insulators, and the subject of the remainder of this section.

A. Quantum-disordered *s* wave

We begin by considering a system of spinful interacting electrons moving in the 2D continuum, which pair to form a spin singlet *s* wave superconducting phase, and then ‘‘turn on’’ a periodic potential which for simplicity has square symmetry. Here, however, we choose the period to correspond to one electron per unit cell. As in Sec. II, with increasing potential strength the superconducting phase can be destroyed by the unbinding and condensation of $hc/2e$ superconducting vortices. But in this case there is only one-half of a Cooper pair per unit cell, so the resulting insulating phase will be dramatically different from the band insulator. In particular, one expects the formation of a crystalline state of Cooper pairs which exhibits charge-density-wave ordering at wave vector (π,π), and spontaneously breaks (discrete) translational symmetry.

In order to proceed expeditiously to our main interest, the nodal liquid, we only mention a few salient points here. The difference between the CDW and the band insulator in the present approach is that the Berry's phase term[7]

$$\mathcal{L}_{\rm Berry} = n_0\partial_t\varphi \tag{3.1}$$

cannot be dropped. To appreciate the physics of this term, we must return to the lattice, where the lattice Hamiltonian corresponding to Eq. (2.2), together with the Berry's phase term [Eq. (3.1)] takes the form:

$$\mathcal{H}_\varphi = -t_2\sum_{\langle i,j\rangle}\cos(\varphi_i-\varphi_j) + U_2\sum_i (n_i-n_0)^2. \tag{3.2}$$

Here n_i is the Cooper pair number operator which is canonically conjugate to φ. Because, at half-filling, there is, on average, half a Cooper pair per site, one has $n_0 = \frac{1}{2}$.

Implementing duality as before[7] (but now on the lattice) gives a dual Euclidean action which is the lattice analog of Eqs. (2.10) and (2.11), the only difference being that the Maxwell term $(1/2\kappa_0)(e_j^2-b^2)$ term is replaced by

$$\mathcal{S}_a = \frac{u_2}{2}\sum_{j\mu}(\epsilon_{\mu\nu\lambda}\Delta_\nu a_j^\lambda - n_0\delta_{\mu 0})^2. \tag{3.3}$$

The new feature, as compared to the last section, is the ‘‘offset’’ charge n_0 which results from the Berry phase term (3.1). It corresponds to an applied ‘‘magnetic’’ field for the lattice Ginzburg-Landau theory. The insulating CDW phase of the Cooper pairs corresponds to an Abrikosov flux lattice in this dual representation. As in the case of the band insulator, the vortex condensate leads to a charge gap and insulating behavior.

Generally, one expects that a condensation of $hc/2e$ vortices will lead to charge ordering with charge $2e$ per unit cell. This follows from the underlying duality: $hc/2e$ vortices pick up a 2π phase change upon encircling charge $2e$ Cooper pairs—the same phase accumulated when a Cooper pair encircles such a vortex. Thus the liberation and condensation of $hc/2e$ vortices leads to a charge quantization in units of $2e$. For the model with attractive interactions and one electron per unit cell considered above, the resulting state is the CDW at wave vector (π,π), which can be thought of as a (charge $2e$) Cooper pair crystal.

B. Quantum-disordered *d* wave

We now turn to the interesting problem of the quantum-disordered *d-wave* superconductor. *d*-wave superconductivity is thought to arise from the combination of strong on-site Coulomb repulsion and some unspecified (and controversial) longer-range attraction on the scale of a few lattice spacings. Certainly strong local repulsion is a key ingredient of the cuprates. For such systems, the $hc/2e$ vortex condensation described above—which implies considerable double occupancy (at least in a region near half-filling)—is physically unreasonable. It must also be discarded on phenomenological grounds, as the actual undoped antiferromagnet is a Mott insulator without charge ordering.

Although $hc/2e$-flux vortex unbinding is untenable for this case, phase coherence must nevertheless still be destroyed to obtain an insulating state. Charge uniformity and phase disruption can both be achieved together by unbinding bound *pairs* of vortices with flux hc/e instead of isolated ones. We expect such a *double-vortex condensate* to appear in the dual description as a condensate with a doubled dual charge, and hence a halved dual flux quantum.[9] At half-filling, then, the dual lattice Ginzburg-Landau theory has a full 2π flux per plaquette, and thus, as desired, exhibits no translational symmetry breaking.[12]

Having motivated double-vortex condensation in the *d* wave case, we now proceed to discuss its implementation. The calculations are significantly different because of an important additional physical ingredient in the *d*-wave superconductor: gapless fermionic quasiparticles. These excitations arise owing to the vanishing of the amplitude of the pair wave function at its nodes in momentum space. The presence of low-energy fermionic excitations necessitates a careful reinvestigation of the duality transformation and its implications. Much of the necessary calculations and formalism was discussed in detail in Ref. 1, and is briefly recapitulated in the Appendix.

As for the *s* wave case, the analysis of the interactions of vortices with quasiparticles is based on the neutralizing change of variables in Eq. (2.4). The distinctive feature of the *d* wave superconductor is that the neutral spin-$\frac{1}{2}$ particles are *gapless*, and can be described by a Dirac Hamiltonian. Because in this case the spinons near the nodes can contribute to low-energy physics, we attribute to them special significance and the name *nodons*, signifying the low-energy spinons descended from the *d*-wave nodal quasiparticles. Being gapless, they can be described by continuum field theory and a four-component Dirac spinor ψ (the analog of f in Sec. II—see the Appendix for a precise definition).

Having already argued that only hc/e-flux vortices should be considered, we will focus primarily on this simpler case. It is, however, appropriate at this point to reflect briefly on the consequences of these strong gauge interactions should single-strength vortices become important low-energy excitations. As above, $hc/2e$-vortex unbinding gives rise to strong gauge interactions of the spinons, now nodons. The gauge-theoretical arguments given in Sec. II can again be carried through, and we expect confinement of the nodons. Unlike the *s* wave case, however, because the nodons are gapless excitations, their presence or absence has definite consequences on the *ground-state* correlations. For instance, in a pure *d*-wave superconductor, the gapless Dirac excitations lead to static (equal time) power-law spin correlations and a T linear magnetic susceptibility. It is natural to expect that the removal of the nodons from the low-energy spectrum in the insulator will be accompanied by the condensation of some pairing operator (for example, spin-density-wave order is characterized by the order parameter $\langle \psi^\dagger \tau^y \sigma^y \vec{\sigma} \psi^\dagger \rangle \neq 0$; other possibilities are legion), in most cases accompanied by a gap for the reconfined electrons.[17] The formation of such a paired-nodon state is analogous to *chiral symmetry breaking* in QCD, and in that context as well is generally believed to accompany confinement (although the converse need not be true). One can imagine approaching such a state from the superconductor by continuously lowering the vortex ''mass'' to zero, at which point one has a theory in which gapless fermionic nodons interact with gapless bosonic vortices via strong gauge interactions. This putative critical point is a tempting starting point for future systematic studies of such instabilities.

We now return to the problem of hc/e vortex condensation, which, although motivated on energetic grounds, has dramatic consequences for the elementary excitations. Again examining Eq. (2.4) when only double-strength vortices are present, φ is defined modulo 4π, and this transformation defines a single-valued neutral fermion. The nodons thus experience no statistical gauge interactions in this case. The results obtained in Ref. 1, which ignored gauge interactions, hence apply to the hc/e condensate, with the proviso that the fundamental vorticity must be doubled throughout the analysis. The salient result is that when gauge effects are absent, the nodons and vortices interact only via the two-fluid interaction Lagrangian

$$\mathcal{L}_{\rm int} = \partial_\mu \varphi J_\mu, \tag{3.4}$$

where J_μ is the electrical current carried by the quasiparticles, and is bilinear in the ψ fields. Equation (3.4) can be understood as the Doppler shift of the nodon energies in a superflow given by $\partial_\mu \varphi$. This is a much weaker coupling than the statistical gauge interactions in the single-vortex condensate, and controlled analytical calculations are possible. Detailed predictions for this quantum-disordered state, the *nodal liquid* (NL), can be derived by writing a coarse-grained continuum theory for Φ_2 and a_μ, as in Refs. 1 and 18. The key conclusions are: (1) gapless nodons survive into the NL state, carrying spin but neither charge nor current at low frequencies; (2) the NL has gapped charged excitations, the lowest lying of which are expected to be (spinless) charge $\pm e$ holons, which occur as vortices in the dual order parameter Φ_2; (3) the half-filled NL has a uniform charge density, and upon (hole) doping charge is introduced as a spinless Wigner crystal with charge e per unit cell (but see Sec. IV for a discussion of how this may be modified when antiferromagnetism is present).

It is important to emphasize that a connection has been made here between two apparently unrelated phenomena. By *assuming* double-vortex condensation, characterized by the dual order parameters $\Phi_2 = \langle e^{2i\theta_j} \rangle \neq 0$ (double vortices are condensed), $\Phi = \langle e^{i\theta_j} \rangle = 0$ (single vortices are bound), we were led to the persistence of spin-charge deconfinement in the insulator. The single-vortex disorder parameter, Φ

(which also distinguishes translational symmetry breaking at half-filling) can thus be regarded as an *order parameter for confinement*.

C. Lattice model for the nodal liquid

We conclude this section by describing a direct route to the NL at half-filling, by which most of its properties may be derived without the use of the duality mapping. To do so, consider the following lattice regularization, which forbids $hc/2e$-flux vortices from the outset:

$$H_{qp}=\sum_{\langle ij\rangle} -t(c_i^\dagger c_j+c_j^\dagger c_i) + |\Delta|(-1)^{x_i-x_j}e^{-i(\varphi_i+\varphi_j)/2}c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger+\text{H.c.}, \quad (3.5)$$

$$H_\varphi=\sum_{\langle ij\rangle} -J\cos\left(\frac{\varphi_i}{2}-\frac{\varphi_j}{2}\right)+\sum_i \frac{U}{2}(2n_i+c_{i\alpha}^\dagger c_{i\alpha}-1)^2. \quad (3.6)$$

The $\cos(\varphi_i/2-\varphi_j/2)$ term has been chosen to allow $\pm 4\pi$ phase slips but not $\pm 2\pi$ phase slips, hence ''confining'' single-strength vortices. Further, we have made the apparently arbitrary choice of dividing the superconducting pair-field phase among neighboring sites. While this may seem unnatural, provided the continuum d-wave quasiparticle Hamiltonian is an adequate low-energy description, *any* lattice regularization should reproduce identical low-wavelength behavior. Finally, we have included a ''charging energy'' term coupling to the total (Cooper pair plus quasiparticle) charge.

This model has particle-hole symmetry, and at zero chemical potential is thus automatically at half-filling. To determine the properties of the system, we begin by performing the lattice analog of Eq. (2.4): $c_{j\alpha}^\dagger=e^{i\varphi_j/2}f_{i\alpha}^\dagger$. Simultaneously, to avoid nontrivial commutation relations between $f,f^\dagger$, and n, we let $N_i=2n_i+c_i^\dagger c_i$ and $\phi_i=\varphi_i/2$. The f fermion creates neutral, spin-$\frac{1}{2}$ quanta. In these variables, the Hamiltonian becomes $H=H_\varphi+H_f+H_{\text{int}}$, with

$$H_\phi=\sum_{\langle ij\rangle} -J\cos(\phi_i-\phi_j)+\sum_i \frac{U}{2}(N_i-1)^2, \quad (3.7)$$

$$H_f=\sum_{\langle ij\rangle} |\Delta|(-1)^{x_i-x_j}f_{i\uparrow}^\dagger f_{j\downarrow}^\dagger+\text{H.c.}, \quad (3.8)$$

$$H_{int}=\sum_{\langle ij\rangle} -t(e^{i(\phi_i-\phi_j)}f_{i\alpha}^\dagger f_{j\alpha}+\text{H.c.}). \quad (3.9)$$

Note that the nodon-phase coupling has been transferred from the pair-field interaction to the kinetic term by the operator transformation. An insulating state is obtained in the limit $U\gg J,t$, where the charging energy dominates over both pair and single-particle hopping. This state can be studied perturbatively in t and J, expanding around the insulating state with $N_i=1$ *exactly* on each site. At $t=J=0$, however, the f-particle Hamiltonian is still highly degenerate. This degeneracy is broken at second order in t and J, giving the effective Hamiltonian (obtained, e.g., by perturbatively integrating out N_i and ϕ_i in a path-integral formulation) $H_{\text{eff}}=H_{e0}+H_{e1}$, with

$$H_{e0}=\sum_{\langle ij\rangle} -\frac{tJ}{U}(f_{i\alpha}^\dagger f_{j\alpha}+\text{H.c.}) + \sum_{\langle ij\rangle} |\Delta|(-1)^{x_i-x_j}f_{i\uparrow}^\dagger f_{j\downarrow}^\dagger+\text{H.c.}, \quad (3.10)$$

$$H_{e1}=-\frac{2t^2}{U}\sum_{\langle ij\rangle} f_{i\alpha}^\dagger f_{j\alpha}f_{j\beta}^\dagger f_{i\beta}. \quad (3.11)$$

The quadratic Hamiltonian H_{e0} is identical to the mean-field Hamiltonian for d-wave quasiparticles, with a renormalized bandwidth $8tJ/U$. It therefore describes two sets of spin-$\frac{1}{2}$ Dirac fermions at low energies. One thus recovers in this way the NL phase obtained previously via continuum duality. The interaction H_{e1} can be rewritten as a combination of antiferromagnetic exchange and contact repulsion of the f particles. If both are weak (as in the large-U limit), such four-fermion interactions are strongly irrelevant around the noninteracting NL fixed point, due to the linearly vanishing density of states of the Dirac fermions. A slightly refined analysis including a physical external gauge field A_μ allows one to calculate the conductivity explicitly, and show that the f fermions carry no current in the NL state, so it is indeed an insulator.[1]

IV. ANTIFERROMAGNETISM

A. Phenomenology

As discussed in Sec. III, hc/e-flux vortex condensation atop the d-wave superconductor yields the NL, an insulator with charge quantization (charge e per unit cell) appropriate near the half-filled Mott insulator. Unlike the CDW obtained by $hc/2e$-flux vortex condensation or the conventional (fully gapped ''short-range RVB'') spin liquid state,[10] the NL also contains low-lying gapless spin degrees of freedom, the nodons, which contain the germ of true antiferromagnetic order. As described in Ref. 1, the phenomenology described above can be easily extended to include Néel order.

The description to this point has essentially neglected internodon interactions. This approach is justified provided such interactions are weak, as all such terms are perturbatively irrelevant (in the renormalization group sense) in the Dirac theory describing the NL.[1] However, perturbative irrelevance does not imply that *strong* interactions cannot drive quantum phase transitions and hence a qualitative change in behavior. Indeed, interacting Dirac fermion models are known to undergo *chiral symmetry-breaking* transitions as quartic couplings are increased.[19] The nature of the transition incurred depends upon the precise nature of the interactions, and various circumstances can induce antiferromagnetism, spin Peierls, charge density wave, and other types of ordering from the NL Lagrangian. Indeed, in the lattice model above we obtained an interaction [Eq. (3.11)], capable of driving a transition to an antiferromagnetic state if sufficiently large.

Because of the uncertainties and pitfalls of attempting a microscopic justification of such interactions, however, we prefer to follow the strategy of Ref. 1 and take a more phenomenological approach. For simplicity let us focus on the case of half-filling with particle/hole symmetry. Since the cuprate materials are clearly antiferromagnetically ordered, we will *assume* the existence of a triplet collective mode with momentum (π,π). In the NL phase, spin-rotational invariance is not broken, and this magnon mode has a gap. Indeed we also expect a nonzero lifetime since the triplet magnons can decay into pairs of nodons. Thus, strictly speaking, the magnons are not sharply defined elementary excitations in this case. Nevertheless, we may imagine tuning a parameter (e.g., reducing frustrating spin-spin interactions in a lattice model) to reduce the magnon gap. Ultimately when it vanishes the collective mode becomes sharp and condenses to form an antiferromagnetically ordered state. In the antiferromagnetic state, the nonzero Néel vector coherently mixes nodons with opposite quasimomentum and opposite spin, halving the magnetic Brillouin zone as is usual in spin-density-wave systems. As in those more conventional cases, this has the effect of opening up a gap in the nodon spectrum.

$$E_n(\vec{q}) = \pm\sqrt{(v_F q_\perp)^2 + (v_\Delta q_\parallel)^2 + (gN_0)^2}, \tag{4.1}$$

where v_F and v_Δ are the Dirac ''velocities'' perpendicular and parallel to the putative Fermi surface, $\mathbf{q}$ is the momentum measured from a node, N_0 is the mean-field staggered magnetization, and g is a phenomenological coupling constant. Due to this mixing, the only gapless degrees of freedom in the antiferromagnet are the collective spin-wave modes guaranteed by Goldstone's theorem.

In this way we arrive at an effective low-energy field theory for an antiferromagnetic Mott insulator, which we will denote (for reasons which will become apparent) as an AF* phase. The antiferromagnetic spin order, featureless incompressible charge configuration, and gapless spin waves are qualitatively identical to those we would obtain in more conventional antiferromagnetic models, e.g. the nested spin-density wave in the weakly interacting half-filled Hubbard model, or alternatively the t-J-like very large-U limit of the same Hamiltonian. We stress, however, that although the nodons have been lifted away from zero energy, they are not confined by the spontaneous symmetry breaking in the AF* phase, i.e., spin-$\frac{1}{2}$ neutral particles still exist as well-defined elementary excitations. For this reason, we believe that such an interesting antiferromagnetic insulator (AF*) is topologically distinct from (i.e., cannot be adiabatically deformed into) a more conventional antiferromagnetic (AF) state. This conviction is bolstered by the existence of the dual order parameter Φ_2, which we have argued characterizes the nodal liquid and the AF* phase. Since Φ_2 creates hc/e vortices in a *pair* field, the AF* phase contains the germ of superconductivity. In contrast, construction of a dual order parameter for the conventional antiferromagnet probably requires the use of Chern-Simons (charge e) bosons, obtained directly by statistical transmutation from the electrons. For example, condensing elementary hc/e vortices in the spin-up boson to form a charge e crystal which lives on one sublattice, and similarly freezing the spin-down particles onto the other sublattice, should suffice to describe the hidden order of the AF phase. Evidently, the dual order parameters in AF and AF* are very different. Alternatively, one can distinguish AF* and AF by testing for the presence or absence of spin-charge separation, employing the argument in Sec. II. To make the argument precise in this case probably requires adding an easy-axis anisotropy,

$$H_{ea} = \sum_{\mathbf{x}} J_{ea}[(s^x)^2 + (s^y)^2], \tag{4.2}$$

which creates a gap in the magnon spectrum. But it is of more interest to address whether the AF and AF* phases can be distinguished *experimentally*. To address this, we turn to a discussion of the electron spectral function in these two phases.

B. Electron spectral function

In standard many-body systems, the existence of well-defined excitations is ascertained by examination of the relevant spectral function.[20] Unfortunately, a direct probe of spin-charge separation via spectral functions is not possible, since there are no local operators which separately create nodons and holons. On the other hand, the electron spectral function $A(\mathbf{k},\omega)$ is accessible experimentally, and has been intensively studied in the high-temperature superconductors with momentum resolution via angle-resolved photoemission spectroscopy[21–23] and locally (i.e., in momentum-integrated form) via nonlinear tunneling characteristics.[24] It seems natural to suggest that $A(\mathbf{k},\omega)$ might possibly give one a way to distinguish the AF and AF* phases, since in the latter unbound nodon-holon pairs form a two-particle continuum, and in the former the electron is itself the elementary excitation. Unfortunately, the situation is not so simple, as we discuss below.

1. AF

To see how this idea works out in practice, let us first consider in some detail the spectral function in the AF phase. A simple model which captures the qualitative physics of the spectral function is fluctuation-corrected spin-density-wave mean-field theory. The (imaginary time) quasiparticle Lagrangian is

$$L = \int_{\mathbf{k}} c^\dagger_{\mathbf{k}}[\partial_\tau + \epsilon_k - \mu]c_{\mathbf{k}} + \tilde{g}\vec{N}\cdot c^\dagger_{\mathbf{k+Q}}\vec{\sigma}c_{\mathbf{k}}, \tag{4.3}$$

where we have assumed ordering at $\mathbf{Q}=(\pi,\pi)$. For simplicity, let us assume the Fermi surface intersects $(\pi/2,\pi/2)$ with some curvature (it is straightforward to generalize this to other geometries). Choosing new coordinates along the (1,1) and $(1,-1)$ axes, we then write $\mathbf{k}=(\pi/\sqrt{2},0)+\mathbf{q}$ and $\epsilon_{\mathbf{k}} - \mu \approx v_F q_x + q_y^2/2m$ near this point. Performing a similar expansion near the opposite point on the Fermi surface, and defining continuum fields $\tilde{\eta}_\pm(\mathbf{q}) \equiv c_{(\pm\pi/\sqrt{2},0)+\mathbf{q}}$, one finds

$$\mathcal{L}_{Nqp} = \tilde{\eta}^\dagger\left[\partial_\tau + iv_F\tau^z\partial_x - \frac{1}{2m}\partial_y^2\right]\tilde{\eta} + g\vec{N}\cdot\tilde{\eta}^\dagger\vec{\sigma}\tau^x\tilde{\eta}, \tag{4.4}$$

where g is the spin-density-wave coupling constant. We suppress the additional quasiparticles located near $(\pm\pi/2, -\pi/2)$, since these are not coupled to the η fields by the

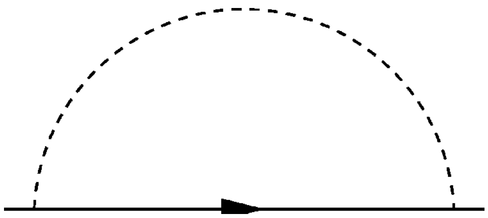

FIG. 1. Self-energy diagram for the charged quasiparticles in the AF phase. The solid line indicates a dressed η-fermion Green's function, while the dashed line is the spin-wave propagator.

ordering wave vector $\mathbf{Q}$. In the antiferromagnetic phase, $\langle \vec{N} \rangle = N_0 \neq 0$, and if fluctuations are ignored the electron states are gapped with energy dispersion,

$$E(\mathbf{q}) = \sqrt{|\Delta|^2 + v_F^2 q_x^2} + q_y^2/2m, \tag{4.5}$$

with $\Delta = gN_0$ the mean-field spin-density wave gap.

Spatial and temporal fluctuations of the Néel field $\vec{N}$ can be described by, e.g., a Landau theory such as Eq. (A10), or by a nonlinear σ model. In the AF phase, we require only the spin-wave expansion for small deviations, $\Pi_i \ll 1$, from perfect alignment, i.e., $\vec{N} = N_0(\Pi_1, \Pi_2, \sqrt{1-\Pi^2})$, for small $\vec{\Pi}$. Since uniform rotations of $\vec{N}$ are equivalent by SU(2) invariance, it is convenient to perform the ''gauge'' rotation

$$\tilde{\eta}(\mathbf{x},\tau) = \exp[i\epsilon_{ij}\sigma_i \Pi_j(\mathbf{x},\tau)]\,\eta(\mathbf{x},\tau). \tag{4.6}$$

In the new variables, the quasiparticle Lagrangian becomes $\mathcal{L}_{Nqp} = \mathcal{L}_\eta + \mathcal{L}_{\Pi-\eta}$, with

$$\mathcal{L}_\eta = \eta^\dagger [\hat{l} + \Delta \sigma^z \tau^x]\,\eta, \tag{4.7}$$

$$\mathcal{L}_{\Pi-\eta} = -i\epsilon_{ij}\eta^\dagger \left[\hat{l}\Pi_i - \frac{1}{m}\partial_y \Pi_i \partial_y \right] \eta. \tag{4.8}$$

Here $\hat{l} \equiv \partial_\tau + i v_F \tau^z \partial_x - (1/2m)\partial_y^2$. Finally, the magnons are governed by the quadratic Lagrangian

$$\mathcal{L}_\Pi = \frac{K}{2}[|\partial_\tau \vec{\Pi}|^2 + v_s^2 |\nabla \vec{\Pi}|^2]. \tag{4.9}$$

The spin waves in Eq. (4.9) are gapless, as required by Goldstone's theorem. Neglecting the coupling to the spin waves (a good approximation if K is very large, so that the $\vec{\Pi}$ fields fluctuate very little), the η particles are noninteracting quasiparticles with a gap Δ, and have a sharp spectral function

$$A_\eta^0(\mathbf{q},\omega) = \pi^{-1}\mathrm{Im}\, G_\eta^0(\mathbf{q}, i\omega_n \to \omega + i\delta) = \delta(\omega - E(\mathbf{q})). \tag{4.10}$$

The spin-wave coupling [Eq. (4.8)], generates a self-energy in the η Green's function. The relevant diagram is indicated in Fig. 1, and a straightforward if tedious evaluation shows that

$$G_\eta(\mathbf{q},\omega) = [i\omega + v_F \tau^z q_x + q_y^2/2m + \Delta \sigma^z \tau^x + \Sigma(\mathbf{q},\omega)]^{-1}, \tag{4.11}$$

$$\mathrm{Im}\,\Sigma(\mathbf{q},\omega) \sim \frac{1}{K v_s^{d+1}} (\delta\omega)^d \Theta(\delta\omega), \tag{4.12}$$

with $\delta\omega = \omega - E(\mathbf{q})$. Equation (4.12) holds provided $|q| < m v_s$. Since $\mathrm{Im}\,\Sigma \ll \delta\omega$ for small $\delta\omega$, the decay rate is negligible at low energies, and we expect a δ-function singularity to survive in the spectral function at $\delta\omega = 0$ (there will also be some small shifts in the energy spectrum itself given by the real part of Σ). When calculating the *electron* spectral function, one needs to include the effects of the SU(2) rotation [Eq. (4.6)]. Since the physical electrons are created by the $\tilde{\eta}$ fields, additional factors of the $\vec{\Pi}$ operators appear in $A(\mathbf{k},\omega)$. The Π operators may be expanded out of the exponential and treated perturbatively. Both this effect and the broadening due to the nonzero self-energy above lead to additional weight for $\delta\omega > 0$, which is often referred to as ''incoherent'' spectral weight. The general expectation for the electron spectral function in the AF phase is illustrated in Fig. 1.

The physical meaning of these results is the following. The minimum energy excitation with charge e and spin $\frac{1}{2}$ is the electron, which in the interacting system is ''dressed'' by magnon excitations that mix with the bare electron. Further, there are higher-energy states involving a dressed electron and unbound excited magnons which are orthogonal to the interacting electron but not the bare one, and thus show up as continua for $\delta\omega > 0$ once interactions are present. The true elementary excitation does not decay however, basically because phase space [which leads to the $\mathrm{Im}\Sigma \sim (\delta\omega)^d$ law above] prevents it. Thus the expected electron spectral function has a resolution-limited dispersing peak at the single-particle gap near its minimum, above which lies continuous spectral weight. Well away from $(\pi/2,\pi/2)$, phase space may (or may not) open up to allow decay even of the single-particle peak, depending upon details of the band structure and interactions.

*2. AF**

In the unconventional antiferromagnet, we expect the presence of unconfined nodons and holons to lead to a two-particle continuum in the electron spectral function. While this is indeed the case, lower-energy features in fact exist due to *nodon-holon bound states*. Such bound states are analogous to excitons in semiconductors, which provide sharp peaks in the optical conductivity despite the existence of an electron-hole continuum at higher energies.

Similar considerations apply here. In particular, if we consider the interaction of a charge e holon with a spin-$\frac{1}{2}$ nodon, it is quite natural to expect that they may experience an attractive interaction leading to a bound state with *both* charge e and spin $\frac{1}{2}$, i.e., an *electron*. To show the existence of such bound states we specialize again to the case of half-filling with particle-hole symmetry. It is convenient to perform a particle-hole (Bogoliubov) transformation on the nodon operators near one pair of nodes,

$$\tilde{\psi}_{a\uparrow} = \psi_{1a\uparrow}, \quad \tilde{\psi}_{a\downarrow} = \psi_{1a\downarrow}^\dagger, \tag{4.13}$$

so that the number operator of the transformed fermions is proportional to the z component of the spin: $S_z = \frac{1}{2}\int d^2x\, \tilde{\psi}^\dagger \tilde{\psi}$. In the presence of Néel order, $\vec{N} = N_0 \hat{z}$, the transformed nodon Hamiltonian density takes the simple form

$$\mathcal{H}_n = \tilde{\psi}^\dagger H_n \tilde{\psi}, \tag{4.14}$$

with a single-particle Hamiltonian

$$H_n(\mathbf{p}_1) = v(\tau_z p_1^x + \tau_x p_1^y) + \Delta_n \tau^y \sigma^y. \tag{4.15}$$

Here the nodon momentum operator $\mathbf{p}_1 = -i\boldsymbol{\nabla}_{r_1}$ is conjugate to the position $\mathbf{r}_1$, and for simplicity we have assumed only a single-nodon velocity. This Hamiltonian describes massive nodon states, with energy gap $\Delta_n = gN_0$. Since the holons are also gapped, the appropriate first quantized Hamiltonian for a single holon (with position $\mathbf{r}_2$ and momentum $\mathbf{p}_2$) is simply

$$H_h(\mathbf{p}_2) = \Delta_h + \frac{\mathbf{p}^2}{2m_h}. \tag{4.16}$$

The form of the interaction between the nodons and holons follows from the dual Lagrangian in Sec. II. For simplicity we only retain the density-density interaction term, proportional to $J_0 \epsilon_{ij}\partial_i a_j$, where $\epsilon_{ij}\partial_i a_j$ is the holon density and the nodon density can be expressed in terms of the transformed fermions as $J_0 = \tilde{\psi}^\dagger \tau^z \sigma^z \tilde{\psi}$. The corresponding first quantized interaction Hamiltonian is then

$$H_{int}(\mathbf{r}_1 - \mathbf{r}_2) = ua^2 \tau^z \sigma^z \delta^{(2)}(\mathbf{r}_1 - \mathbf{r}_2), \tag{4.17}$$

with interaction strength u, and a is a short distance cutoff.

Since the two-body Hamiltonian is independent of the ''center-of-mass'' coordinate $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, the total momentum, $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$, is conserved. For simplicity we consider bound states with $\mathbf{P} = 0$. The Hamiltonian for the *relative* coordinates,

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{p} = (\mathbf{p}_1 + \mathbf{p}_2)/2, \tag{4.18}$$

then takes the simple form

$$H_{rel} = H_n(\mathbf{p}) + H_h(\mathbf{p}) + H_{\text{int}}(\mathbf{r}). \tag{4.19}$$

To solve for bound states with energy E, we recast the Schrödinger equation $H_{\text{rel}}\phi = E\phi$ in the form

$$G^{-1}(\mathbf{q})\phi(\mathbf{q}) = -ua^2 \tau^z \sigma^z \phi(\mathbf{r} = 0), \tag{4.20}$$

with matrix Greens function $G^{-1}(\mathbf{q}) = H_n(\mathbf{q}) + H_h(\mathbf{q}) - E$. Here $\phi(\mathbf{q})$ denotes the Fourier transform of the four-component wave function $\phi(\mathbf{r})$. Upon matrix inversion this can be rewritten as $\mathbf{M}\phi(\mathbf{r}=0) = 0$, with

$$\mathbf{M} = \mathbf{1} + ua^2 \int \frac{d^2q}{(2\pi)^2} G(\mathbf{q}) \tau^z \sigma^z, \tag{4.21}$$

so that the eigenvalue condition is the vanishing of the determinant: $\det(\mathbf{M}) = 0$. Here we are implicitly assuming that the integration is cut off at high momentum by $q_c = 1/a$.

An explicit expression for the bound-state energy E_b can be readily obtained in the $u \to 0$ limit by putting $E_b = \Delta_n + \Delta_h - \epsilon_b$ with *small* binding energy ϵ_b. In this limit one need only retain the contribution to the above integral which is infrared divergent, which gives

$$\mathbf{M} = \mathbf{1} + W(\sigma^z \tau^z - \sigma^x \tau^x), \tag{4.22}$$

with

$$W = -(u/8\pi\epsilon_0)\ln(\epsilon_b/\epsilon_0). \tag{4.23}$$

Here we have defined an energy scale,

$$\epsilon_0 = (m_h v^2 + \Delta_n)/(2m_h \Delta_n a^2). \tag{4.24}$$

Since $\det(\mathbf{M}) = W^2(4 - W^2)$, the eigenvalue condition reduces to $W = 2$, which gives the final result for the bound-state binding energy:

$$\epsilon_b = \epsilon_0 \exp(-16\pi\epsilon_0/u). \tag{4.25}$$

Notice that the binding energy is exponentially small in the interaction strength u, reflecting the two-dimensional constant density of states for free massive holons and nodons. If one were to change the sign of the interaction, there is still a bound state (from $W = -2$) with the *same* energy. In either case, the bound state has the quantum numbers of the electron with $s_z = \frac{1}{2}$ and charge $\pm e$. A spin-down bound state can also be readily found, corresponding to the binding of a holon to a single-nodon ''hole'' in the filled Fermi sea.

Between the threshold energy for generating the electron (E_b) and the energy of the unbound nodon-holon continuum, the electron spectral function should be governed by qualitatively the same physics as in the AF case, except that the total spectral weight of the corresponding feature will be reduced by matrix element factors arising from, e.g., the possibly large spatial extent of the nodon-holon bound-state wave function. Upon reaching the nodon-holon continuum, we expect a much enhanced spectral weight but no sharp feature at the continuum, as it is already lying in the continuum formed by the electron plus spin-wave excitations, and the pair excitation can thereby easily decay.

$A(\mathbf{k},\omega)$ in the AF and AF* phases are thus not qualitatively different, and cannot strictly speaking be used to distinguish the phases. Quantitatively, however, we expect the AF* spectral function to exhibit a very small ''quasiparticle'' peak, with minimal separation from a nodon-holon continuum carrying most of the spectral weight. If we assume that the holon gap greatly exceeds the gap for the nodons, then both features are expected to disperse in approximately d-wave fashion, though the cusp for angles near $\pm 45°$ should be rounded by the nodon gap [Eq. (4.1)]. At nonzero temperatures, thermally excited particles will scatter the injected electron and lead to a broadening of even the threshold peak. In the AF* case, where this feature is expected to lie close to the nodon-holon continuum and have a small weight, such thermal broadening could well remove the quasiparticle peak completely at experimental temperatures.

For a system at half-filling but without particle-hole symmetry, the Néel ordering wave vector is not commensurate with the spacing between antipodal nodes. If this incommensurability is sufficiently large, it is possible for the nodons to remain gapless even in the presence of long-range antiferromagnetic order.[1] In this unusual state, which we denote as AF/NL, gapless spin-$\frac{1}{2}$ nodons coexist with the spin-1 magnons. Since the density of states for the gapless nodons vanishes linearly with energy, a weak interaction with the massive holons is *not* expected to result in a holon-nodon bound state. Angle-resolved photoemission in the AF/NL phase will thus have a number of notable features. Specifically, since the electron will decay into the nodon-holon continuum, one does not expect *any* sharp features in the momentum-resolved spectral function. The lowest energy spectral weight is expected at the nodes, with a threshold energy which disperses linearly away from the nodes as in the d wave superconductor. The spectral weight should rise smoothly above threshold due to the nodon-holon continuum—with no δ-function peaks. This behavior is in fact reminiscent of that observed in the undoped Ca compound by Shen,[23] and is in marked contrast to the δ-function spectral features expected in a conventional antiferromagnetic insulator.

V. DISCUSSION

A small number of examples of condensed matter systems are generally agreed to exhibit exotic quantum numbers, i.e., particles which seem to require ''splitting'' the electron. Both charge fractionalization[25] and spin-charge separation are generic in one dimension.[6] In the two-dimensional quantum Hall effect, fractionally charged particles have been known to exist for some time,[26] and recently have been observed in dramatic shot-noise experiments.[27] In both these examples, fractional charge is connected to topological excitations: solitons or domain walls in one dimension and vortices in two dimensions.

A third example, less widely appreciated, is a superconductor in *any* dimension.[16] For the superconductor the mechanism is different: *Pairing* of electrons into singlets creates a gapless collective (second) sound mode that carries the charge. The sound mode can adjust almost instantaneously to a quasiparticle, effectively neutralizing it, leaving only a bare spin $\frac{1}{2}$. On the face of it this species of spin-charge separation appears considerably different from the other topological varieties.

In this paper we have exploited a dual formulation to show that indeed isolated charges (''holons'') derived from the superconductor can be understood as topological excitations in a vortex condensate.[29] Further, we have described how spin-charge separation can occur in an insulating state which results from the quantum disordering of a superconductor. Of course, propinquity to the superconducting state does not guarantee the inheritance of spin-charge separation: it only occurs when flux hc/e vortices condense. The most interesting example of this phenomenon—from the point of view of high-T_c phenomenology—is the nodal liquid, which we have discussed from this standpoint. The condensation of $hc/2e$ vortices, on the other hand, leads to the confinement of spin and charge. The band insulator and the CDW, for example, can be understood in this way.

One striking consequence of the distinction between spin charge separated, and confined systems is that there are two distinct antiferromagnetic states: one, AF, which is the ordinary antiferromagnet and another, AF*, which is spin charge separated. The latter results from ordering the nodons in a nodal liquid, so it has neutral, spin-$\frac{1}{2}$ excitations. However, the distinction between the AF and AF* phases is experimentally rather elusive. The natural place to look is the electron spectral function, which can be probed through angle-resolved photoemission experiments. Under some conditions (see Sec. IV B), the unconventional antiferromagnet should exhibit *only* a nodon-holon continuum instead of a quasiparticle pole. Unfortunately, the existence of nodon-holon bound states makes the distinction between the AF and AF* phases rather subtle. On the other hand, according to the paradigm presented here, spin-charge separation is a consequence of hc/e vortex condensation. Thus spin-charge separation could be indirectly evidenced by the observation of hc/e vortices near the quantum critical point at which superconductivity is destroyed.

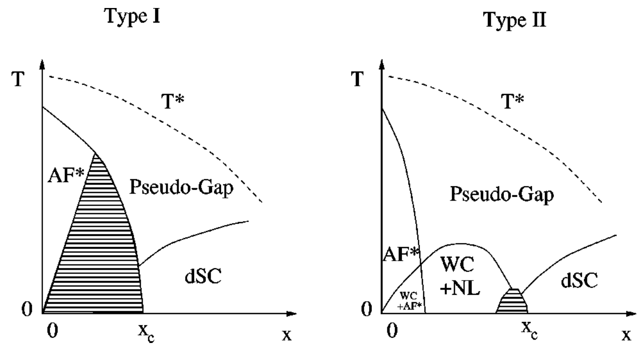

FIG. 2. Phenomenological phase diagrams in the nodal liquid Ginzburg-Landau theory of Ref. 1. Here x is the hole doping, and T is temperature. As discussed in the text, hc/e vortex condensation leads to the unconventional AF* antiferromagnetic Mott insulator. Note that, depending upon the magnitude of particle/hole asymmetry, nodons may remain gapless in an AF*/NL state at half-filling—space constraints prevent us from indicating this on the figure. Depending upon microscopic parameters, two principal phase diagrams occur opon doping. In the type-I scenario, added charge segregates into locally superconducting regions, which coalesce at some critical doping x_c. In the type-II scenario, added charges order into a Wigner crystal (WC) with charge e per period, presumably with some associated spin ordering. After a small amount of doping the antiferromagnetic order is suppressed and the nodons are liberated into a nodal liquid (NL) coexisting with the WC. At x_c this WC melts into the d-wave superconductor (dSC) phase. See Ref. 1 for details. Some modifications are necessary if the effects of impurities are included, some of which are discussed in Ref. 28.

The continuity of spin-charge separation that is embodied in the nodal liquid and its offspring AF* state makes possible a simple phenomenological description of the evolution from the insulator to the superconductor, as espoused in Ref. 1. For example, the simplest Ginzburg-Landau formulation predicts the phase diagrams in Fig. 2. If, on the other hand, spin-charge separation is *absent* in the undoped insulator, there must be a confinement transition between $x=0$ and the superconductor. While we have argued that such a transition is driven by $hc/2e$ vortices, its nature and the phases which it connects are highly nontrivial. The relative simplicity and elegance of the nodal liquid scenario thus argues in favor of its relevance to the cuprates. Despite numerous and interesting differences among different compounds, the phase diagrams of high-temperature superconductors enjoy a remarkable degree of universality. A number of theoretical works have attempted to understand the commonalities and variations among the topology of these phase diagrams phenomenologically.[30] At low temperatures, however, we believe classical phenomenology based only on conventional order parameters misses the important physics of gapless quasiparticles and spin-charge separation that are key in the vicinity of d wave superconductor-insulator transitions. *Any* viable theory of the cuprates must at least address the issue of how spin and charge either remain separated or become confined on approaching the insulator.

In addition to the search for a ''smoking gun'' experiment for spin-charge separation, there are a number of other interesting questions raised by this work. How do we implement the interaction between nodons and $hc/2e$ vortices in an SU(2)-invariant way? How do these formulations of spin-charge separation apply to 3D systems?

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APPENDIX: DIRAC LAGRANGIAN

Here we review the effective Lagrangian for low-energy d-wave quasiparticles, following the notation of Ref. 1. It is most directly written in terms of the appropriate Nambu-Gorkov-like spinor, $\Psi_{ia\alpha}$, with

$$\Psi_{i1\alpha}(\mathbf{k})=c_{\mathbf{K}_i+\mathbf{k},\alpha}, \tag{A1}$$

$$\Psi_{i2\alpha}(\mathbf{k})=i\sigma^{y}_{\alpha\beta}c^{\dagger}_{-(\mathbf{K}_i+\mathbf{k}),\beta}, \tag{A2}$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ are the momenta of the d wave nodes along the Fermi surface. We use index-free notation in which Pauli matrices $\vec{\mu}$, $\vec{\tau}$, and $\vec{\sigma}$ act in the node, particle-hole, and spin $(ia\alpha)$ subspaces, respectively; furthermore, if a single index is given explicitly, it is always the node index. In a particle-hole symmetric model at half-filling, $\mathbf{K}_{1/2}=(\pm\pi/2,\pi/2)$ in the usual (a,b) crystalline coordinate system (i.e., axes along the Cu-O bonds). The separation in Eq. (A2) is well defined provided the momentum is restricted to points near the nodes, i.e., $|\mathbf{k}|<\Lambda$, where Λ is a cutoff.

As in the s-wave case, we must allow for space-time dependence of the superconducting phase φ. For a $d_{x^2-y^2}$ superconductor, one has

$$v\langle c_{i\uparrow}(t)c_{j\downarrow}(t)\rangle'=\Delta_d(\mathbf{x}_i-\mathbf{x}_j)\exp[i\varphi(\overline{\mathbf{x}},t)], \tag{A3}$$

where the prime on the angular brackets indicates an average (path integral) over high-energy electronic states away from the nodes, and $\overline{\mathbf{x}}=(\mathbf{x}_i+\mathbf{x}_j)/2$. The amplitude function $\Delta_d(\mathbf{x})$ is the Fourier transform of the usual momentum-space gap function, $\Delta_{\mathbf{k}}\sim f(|k|)[\cos^2 k_a-\cos^2 k_b]$, and decays on the scale of ξ. It is usually more convenient for us to work in rotated coordinates $x=(x_a+x_b)/\sqrt{2}$, $y=(x_b-x_a)/\sqrt{2}$. The appropriate effective quasiparticle Lagrangian density was derived in Ref. 1:

$$\mathcal{L}_{\Psi}=\sum_{s=\pm}\Psi_1^{\dagger}(i\partial_t+iv_F\tau^z\partial_x + iv_{\Delta}\tau^s e^{is\varphi/2}\partial_y e^{is\varphi/2})\Psi_1+(1\leftrightarrow 2;x\leftrightarrow y). \tag{A4}$$

Equation (A4) is derived on the assumption that the phase φ is slowly varying on the scale of the coherence length, i.e., $|\xi\partial_x\varphi|,|\xi\partial_y\varphi|,|\xi/v\,\partial_\tau\varphi|\ll 2\pi$. However, we expect on grounds of universality that Eq. (A4) and its consequences provide a correct low-energy description of the d-wave superconductor and its quantum-disordered descendents more generally.[31] The analysis of the interactions of vortices with quasiparticles is based on the important change of variables

$$\psi=\exp(-i\varphi\tau^z/2)\Psi. \tag{A5}$$

Inserting Eq. (A5) into Eq. (A4), one finds $\mathcal{L}_\Psi=\mathcal{L}_\psi+\mathcal{L}_{\rm int}$, with

$$\mathcal{L}_{\psi}=\psi_1^{\dagger}[i\partial_t+v_F\tau^z i\partial_x+v_{\Delta}\tau^x i\partial_y]\psi_1+(1\leftrightarrow 2,x\leftrightarrow y). \tag{A6}$$

The nodon field ψ interacts with the phase of the order parameter as in Eq. (3.4). Here the electrical three-current J_μ is given by

$$J_0=\tfrac{1}{2}\psi_j^{\dagger}\tau^z\psi_j, \tag{A7}$$

$$J_j=-i\frac{v_F}{2}\psi_j^{\dagger}\psi_j. \tag{A8}$$

Compared to the statistical gauge interaction with $\pm hc/2e$-flux vortices, Eq. (3.4) represents a much weaker *two-fluid* interaction between the quasiparticle or nodon current J_μ and the superfluid current $\partial_\mu\varphi$. A continuum duality transformation appropriate for such a coupling was described in detail in Ref. 1, and on the lattice in Ref. 18. Noting that the (Euclidian) nodon current J^μ couples to the superfluid current $\partial_\mu\varphi$ in a manner directly generalizing the Berry's phase coupling $in_0\dot{\varphi}$, the dual (Euclidian) lattice action can be determined simply by replacing $in_0\delta_{\mu 0}\to in_0\delta_{\mu 0}+iJ_\mu$, i.e.,

$$\mathcal{S}_a\to\tilde{\mathcal{S}}_a=\frac{u_2}{2}\sum_{j\mu}(\epsilon_{\mu\nu\lambda}\Delta_\nu a_j^\lambda-N_0\delta_{\mu 0}-J_\mu)^2. \tag{A9}$$

The incorporation of antiferromagnetism was also described in Ref. 1. A low-energy effective Lagrangian describing the magnon mode and its coupling to the nodons is

$$\mathcal{L}=\tfrac{1}{2}K_\mu|\partial_\mu\mathbf{N}|^2-V_N(|N|)+g\mathbf{N}\cdot\mathbf{S}_{\vec{\pi}}, \tag{A10}$$

where $K_0=K$, and $K_1=K_2=-v_s^2K$, with v_s the spin-wave velocity in the AF. Here

$$\mathbf{S}_{\vec{\pi}}=\frac{1}{2}[\psi_j^{\dagger}\tau^y\boldsymbol{\sigma}\sigma^y\psi_j^{\dagger}+\text{H.c.}] \tag{A11}$$

is the spin operator at momentum $\vec{\pi}$. Near any phase transitions, and for most phenomenological purposes, it is sufficient to take a simple form for the potential: $V_N(|N|)=r_N|N|^2+u_N|N|^4$. The parameter r_N controls the presence or absence of AF order. In mean-field theory, and neglecting for the moment the nodon coupling g, the ground state passes from long-range to short-range AF order as r_N is tuned from negative to positive. We include only the most relevant coupling of the Néel field to the nodons allowed by symmetry,

$$\mathcal{L}_{\rm nodon}=\mathcal{L}_\psi+g\mathbf{N}_0\cdot\mathbf{S}_{\vec{\pi}}, \tag{A12}$$

with $\mathbf{N}_0 = \langle \mathbf{N} \rangle$.

A compelling feature of the above description is the resulting low-lying spectrum in the antiferromagnet. The model can be readliy diagonalized with an appropriate Bogoliubov transformation, giving the energy eigenvalues, in Eq. (4.1). In all nodon sectors there is a nonzero *gap*, equal to gN_0. The nodons having been lifted to finite energy, the only remaining gapless excitations in the AF* phase are the spin waves (slow rotations of **N**) dictated by Goldstone's theorem.

[1] L. Balents, M. P. A. Fisher, and C. Nayak, Int. J. Mod. Phys. B **12**, 1033 (1998).

[2] M. P. A. Fisher and D. H. Lee, Phys. Rev. B **39**, 2756 (1989); D. H. Lee and R. Shankar, Phys. Rev. Lett. **65**, 1490 (1990).

[3] R. B. Laughlin, preprint, cond-mat/9709195.

[4] P. W. Anderson, Science **235**, 1196 (1987).

[5] M. Peskin, Ann. Phys. (N.Y.) **113**, 122 (1978); P. O. Thomas and M. Stone, Nucl. Phys. B **144**, 513 (1978); X. G. Wen and A. Zee, Int. J. Mod. Phys. B **4**, 437 (1990).

[6] See E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, Reading, MA, 1991), and references therein.

[7] C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. **47**, 1556 (1981); M. P. A. Fisher and D. H. Lee, Phys. Rev. B **39**, 2756 (1989).

[8] There are actually two loopholes. The first is if the effective action contains a Chern-Simons term for a_μ. This can only happen in a system without time-reversal symmetry, and the result is the quantum Hall effect. The other possibility occurs if the effective action for a_μ contains a nonlocal term such as $(\partial_\mu a_\nu)^2/\partial^2$. This can only happen if there are gapless degrees of freedom coupled to j_μ, as in a metal. It is hard to see how such a system can be insulating, but we cannot rule it out.

[9] The stability of hc/e vortices was discussed by S. Schadev, Phys. Rev. B **45**, 389 (1992).

[10] S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 8865 (1987).

[11] I. Affleck and J. B. Marston, Phys. Rev. B **37**, 3774 (1988); **39**, 11 538 (1989).

[12] For an *s*-wave superconductor, we believe that hc/e vortex condensation would lead to a spin-charge-separated Mott insulator with fully gapped, fermionic spin-$\frac{1}{2}$ excitations.

[13] See X. G. Wen and P. A. Lee, Phys. Rev. Lett. **76**, 503 (1996); D. H. Kim and P. A. Lee, preprint, cond-mat/9810130, and references therein.

[14] B. I. Halperin (private communication).

[15] For another perspective on ''topological doping,'' see S. A. Kivelson and V. J. Emery, Synth. Met. **80**, 151 (1996).

[16] S. A. Kivelson and D. S. Rokhsar, Phys. Rev. B **41**, 11 693 (1990).

[17] We cannot rule out the possibility that translational symmetry is broken through the development of ''staggered flux phase'' ordering, $\langle c^\dagger_{\alpha k+Q} c_{\alpha k} \rangle = \cos k_x - \cos k_y$, in which case no gap is opened.

[18] M. P. A. Fisher, preprint, cond-mat/9806164.

[19] For a pedagogical introduction, see, e.g., A. M. Polyakov, *Gauge Fields and Strings* (Harwood, Chur, 1987).

[20] See, e.g., J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, Redwood City, CA, 1988).

[21] See, e.g., A. G. Loeser, Z.-X. Shen, D. S. Dessau, D. S. Marshall, C. H. Park, P. Fournier, and A. Kapitulnik, Science **273**, 325 (1996), and references therein.

[22] M. R. Norman, H. Ding, M. Randeria, J. C. Campuzano, T. Yokoya, T. Takeuchi, T. Takahashi, T. Mochiku, K. Kadowaki, P. Guptasarma, and D. G. Hinks, Nature (London) **392**, 157 (1998), and references therein.

[23] Z.-X. Shen (private communication).

[24] N. Miyakawa, J. F. Zasadzinski, L. Ozyuzer, P. Guptasarma, D. G. Hinks, C. Kendziora, and K. E. Gray, preprint, cond-mat/9809398, and references therein.

[25] See M. P. A. Fisher and L. I. Glazman, in *Mesoscopic Electron Transport*, edited by L. L. Sohn *et al.* (Kluwer, Dordrecht, 1997), and references therein.

[26] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[27] L. Saminadayar, D. C. Glattli, Y. Jin, and B. Etienne, Phys. Rev. Lett. **79**, 2526 (1997); R. de-Picciotto, M. Reznikov, M. Heilblum, V. Umansky, G. Bunin, and D. Mahalu, Nature (London) **389**, 162 (1997).

[28] T. Senthil, M. P. A. Fisher, L. Balents, and C. Nayak, Phys. Rev. Lett. **81**, 4704 (1998).

[29] For alternative constructions, which lead to topological excitation which are semionic holons, see, e.g., D. S. Rokhsar, Phys. Rev. Lett. **65**, 1506 (1990); V. Kalmeyer and R. B. Laughlin, *ibid.* **59**, 2095 (1987).

[30] S. C. Zhang, Science **275**, 1089 (1997); preprint, cond-mat/9808309.

[31] It is of course possible that different physics, neglected here, could be obtained in the vortex cores and lead to distinct quantum-disordered insulating states with additional degrees of freedom. However, there appears to be no compelling argument why this *must* be the case, particularly given the extremely short coherence lengths of the high-T_c materials. We therefore leave the incorporation of ''core states,'' should they exist, as an interesting open problem, focusing here on the more fundamental physics in their absence.