

CHAPTER 9

QUANTUM CONFINEMENT AND CUPRATE CRITICALITY

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ABSTRACT

Theoretical attempts to explain the origin of high-temperature superconductivity are challenged by the complexity of the normal state, which exhibits three regimes with increasing hole doping: a pseudo-gap regime when underdoped, strange power laws near optimal doping and more conventional metallic behavior when heavily overdoped. We suggest that the origin of this behavior is linked to a zero-temperature quantum phase transition separating the overdoped Fermi liquid from a spin-charge separated underdoped phase. Central to our analysis is a new Z_2 gauge theory formulation, which supports topological vortex excitations - dubbed visons. The visons are gapped in the underdoped phase, splitting the electron's charge and Fermi statistics into two separate excitations. Superconductivity occurs when the resulting charge e boson condenses. The visons are condensed in the overdoped phase, thereby confining the charge and statistics of the electron leading to a Fermi liquid phase. Right at the quantum confinement transition the visons are in a critical state, leading to power-law behavior for both charge and spin.

9.1 INTRODUCTION

Despite the remarkable progress [1] in the experimental characterization of the cuprate high- T_c materials, a theoretical consensus on the important underlying physics remains elusive. Experiments have revealed a rich phase diagram as the temperature and chemical doping are varied, with low temperature spin and charge ordering in addition to superconductivity. The normal phase at elevated temperatures is equally varied, exhibiting a pseudo-gap in the underdoped regime

and strange power laws at optimal doping. In this paper, we propose a theoretical picture that provides a description of the basic aspects of all parts of the cuprate phase diagram.

It gives us great pleasure dedicating this paper to Phil Anderson in honor of his innumerable insights into condensed matter physics and - in particular - his profound clairvoyance in discerning the mysteries of high-temperature superconductivity. The approach we expound below shares many qualitative features with Phil's early RVB ideas. Towards the end of the paper we specifically address some of the commonalities and distinctions with RVB theory.

9.2 EXPERIMENTS

We begin with a brief discussion of experiment. In the last few years, angle-resolved photoemission spectroscopy (ARPES) has emerged as an important experimental probe [2] of the cuprates. The ARPES spectra provide a direct experimental measurement of the electron spectral function. In *any* conventional phase (such as a Fermi liquid, band insulator, spin density wave or superconductor), a sharp quasiparticle peak is expected as a function of frequency ω at some momentum \vec{k} in the Brillouin zone. The experimental results in the underdoped [3] (*and* undoped [4]) cuprates in the non-superconducting state are in striking contrast to these expectations: the electron spectral function is highly smeared with no trace of a sharp quasiparticle peak. A sharp peak does appear, however, upon cooling *into* the superconducting state [5, 6]. With increasing doping the normal-state ARPES spectra sharpen somewhat, but even near optimal doping, the observed peak is far too broad to be consistent with a conventional quasiparticle description. Some representative data may be found in Figs. 9.1 and 9.2.

We take the absence of a quasiparticle peak in the ARPES data to be strong evidence that the electron *decays* into other exotic excitations in the underdoped cuprates. Further evidence for this comes from transport measurements. The *c*-axis d.c. resistivity shows "insulating" behavior increasing rapidly upon cooling, whereas the in-plane resistivity is typically "metallic" and much smaller in magnitude. Moreover, in a.c. transport a Drude peak is observed in the *ab* plane, but not along the *c*-axis. This strangely anisotropic behavior, difficult to understand within a conventional framework, follows naturally if the electron decays into exotic excitations which reside primarily in the *ab* plane. Transport along the *c*-axis requires hopping of *electrons* from layer to layer which is strongly suppressed at low energies.

9.3 NOVEL EXCITATIONS

If the electron indeed decays into other excitations, what is their character? There are two distinct possibilities: (a) The electron may decay into two or more other exotic particles, each of which carries some fraction of the quantum numbers

of the electron (for instance, into separate spin and charge carrying excitations), or (b) The exotic excitations may admit no "particle" description at all - this is known to happen generically at quantum critical points. We hypothesize that (a) is realized in the underdoped cuprates. There are two reasons for doing so. First, the experiments strongly suggest that the electron decays throughout the underdoped region - fine-tuning to a critical point as in possibility (b) appears unnecessary. Second, as detailed below, the emergence of a sharp ARPES peak in the superconducting state points to the electron decaying into separate spin- and charge-carrying particle excitations.

9.4 THE Z_2 GAUGE THEORY

In a recent paper [7] we introduced a new theoretical formulation of strongly interacting electrons based on a Z_2 gauge theory, that enabled us to reliably demonstrate the possibility of electron "fractionalization" in two spatial dimensions. The theory is closely linked to an earlier "vortex field theory" approach by Balents et. al. [8], but is formulated in terms of particle excitations - a charge e , spin 0 boson (a chargon) and a charge 0, spin 1/2 fermion (a spinon), which are minimally coupled to a fluctuating Z_2 gauge field. Of particular importance to issues of fractionalization are point-like vortex excitations in this Z_2 gauge field, called "visons". Fractionalization is obtained whenever the visons are gapped. When the visons *condense* the chargons and spinons are *confined*, effectively "glued" together to form an electron. This results in a conventional phase where the excitations are electrons (or electron composites such as a magnon or a Cooper pair).

The apparent decay of the electron in the underdoped cuprates, strongly suggests that the visons are gapped in this part of the phase diagram. On the other hand, in the heavily overdoped region Fermi liquid behavior is expected, implying a condensation of visons. Together, this implies that with increasing doping there must be a zero-temperature phase transition where the visons first condense. The existence of such a novel "quantum confinement transition" is essentially implied by the experimental data - the transition interpolates between the deconfined underdoped region (with no quasiparticle peak) and the heavily overdoped Fermi liquid regime.

9.5 PHASE DIAGRAM

A schematic zero temperature phase diagram paying attention only to the gross feature of whether the visons are gapped or condensed is shown in Fig. 9.3. Of particular interest is the quantum critical point associated with the confinement phase transition. It is clear that, at finite temperature, the crossover from underdoped to overdoped regions will be determined by the properties of the quantum

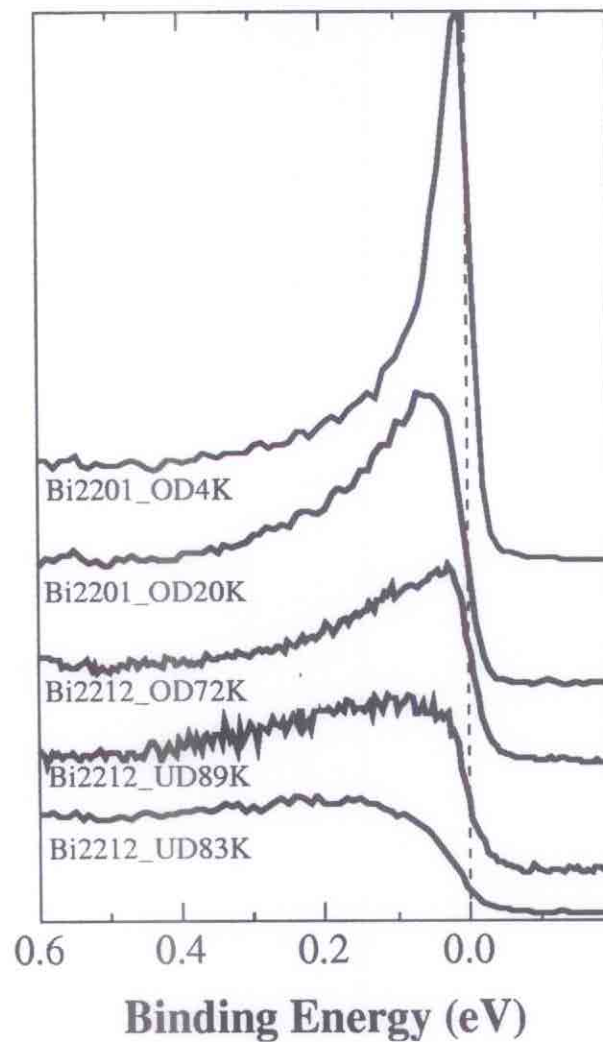


Figure 9.1: Evolution of normal state ARPES lineshape with doping at momentum $(\pi, 0)$. The two lower curves are for underdoped samples while the three upper curves are for overdoped samples. The data are from the group of J.C. Campuzano.

critical region associated with this quantum phase transition. In Fig. 9.4 we sketch the expected finite-temperature crossovers in the vicinity of this phase transition.

The existence of a quantum confinement critical point controlling the crossover from the underdoped to heavily overdoped regimes is in qualitative agreement with a number of experiments. It is well-known that this region is characterized experimentally by power-law temperature or frequency dependences of various physical quantities, as expected at a quantum critical point. But more specifically, the sharpening of the ARPES spectra on moving from the underdoped to the overdoped region strongly suggests that the critical point is associated with a *confinement* transition.

We now discuss the character of the two phases on either side of the confinement transition. For $x > x_c$, and below the finite temperature crossover

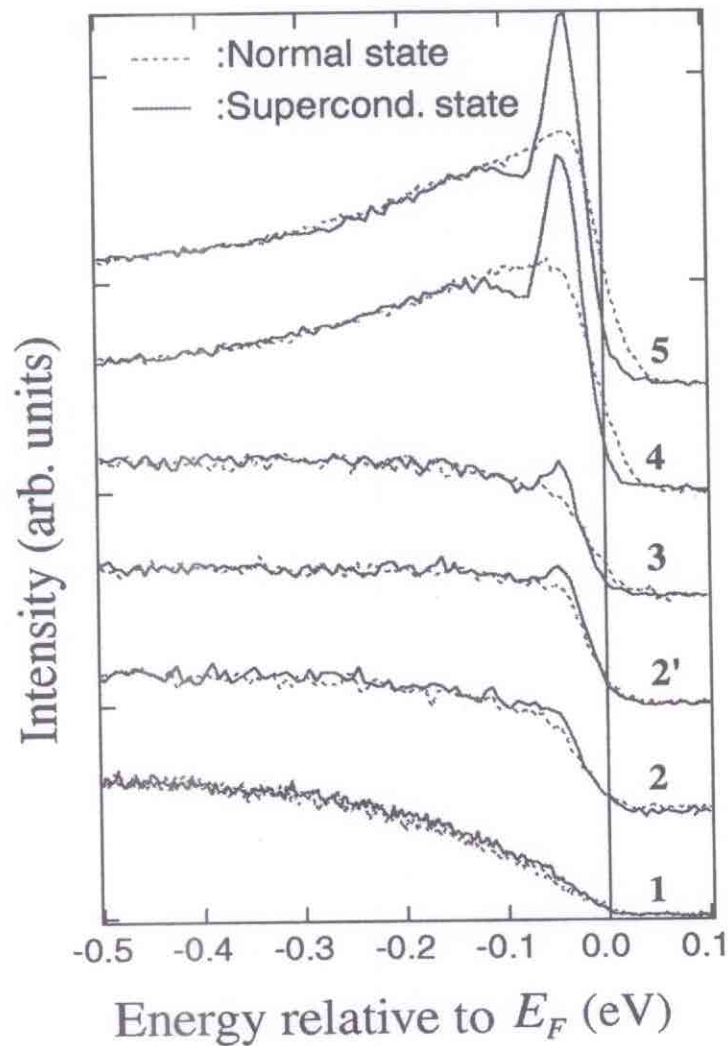


Figure 9.2: ARPES spectra of BSCCO-2212 at momentum $(\pi, 0)$. The data is from the group of Z.X. Shen. The dashed lines are spectra in the normal state, and the solid lines are in the same sample in the superconducting state. The highest curve corresponds to optimal doping with $T_c = 90$ K. The lower curves correspond to underdoped samples with each successive curve corresponding to a lower value of T_c .

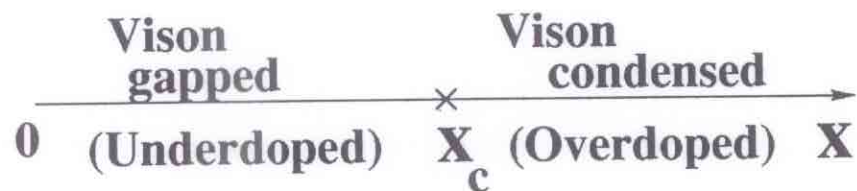


Figure 9.3: Schematic zero temperature phase diagram as a function of doping x

line, the system is presumably well-described by Landau Fermi liquid theory. In this theory the low-energy quasiparticle excitations near the Fermi surface are essentially electrons - they carry the electron quantum numbers, spin 1/2 and charge e - perhaps with a renormalized effective mass. As such, the electron spectral function should exhibit sharp quasiparticle peaks for all momenta lying on the Fermi surface, which sharpen into delta functions at zero temperature. Unfortunately, samples are difficult to grow in this heavily overdoped regime, so that experimental data is rather limited.

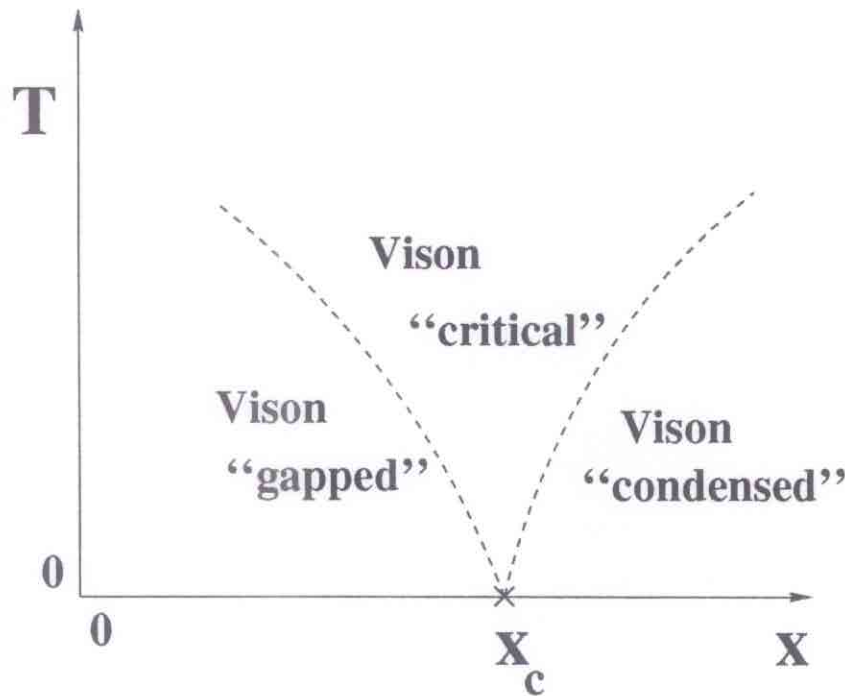


Figure 9.4: Finite temperature crossovers in the vicinity of the quantum confinement transition at $x = x_c$ and $T = 0$. The dashed lines denote crossovers, rather than finite temperature transitions.

But what is the character of the phase for $x < x_c$ where the “vison” excitations are gapped out? This follows readily by inspecting the effective action [7] for the Z_2 gauge theory:

$$S = S_c + S_s + S_K + S_B, \quad (9.1)$$

$$S_c = -t_c \sum_{\langle ij \rangle} \sigma_{ij} (b_i^* b_j + c.c.), \quad (9.2)$$

$$S_s = - \sum_{\langle ij \rangle} \sigma_{ij} (t_{ij}^s \bar{f}_{i\alpha} f_{j\alpha} + t_{ij}^\Delta f_{i\uparrow} f_{j\downarrow} + c.c.) - \sum_i \bar{f}_{i\alpha} f_{i\alpha} \quad (9.3)$$

$$S_K = -K \sum_{\square} \prod_{\square} \sigma_{ij}. \quad (9.4)$$

Here, b_i^\dagger creates a spinless, charge e bosonic excitation - the chargon - and f_i^\dagger creates the spinon, a fermion carrying spin $1/2$ but no charge. When created together, these two excitations comprise the electron. The field σ_{ij} is a gauge field that lives on the links of the space-time lattice, and takes on two possible values: $\sigma_{ij} = \pm 1$. The kinetic term for the gauge field, S_K , is expressed in terms of plaquette products. Here, S_B is a Berry's phase [7] term which depends on the doping x . The vison excitations are vortices in the Z_2 gauge field. Specifically, consider the product of the gauge field σ around an elementary plaquette, which can take on two values, plus or minus one. When this product is negative, a vison excitation is present on that plaquette. Thus, when the visons are gapped and absent in the ground state, all the plaquette products equal plus one, and one can therefore put $\sigma_{ij} = 1$ on every link. In this case the chargon and spinon can propagate *independently*, and the electron is *fractionalized*. Once the electron is thus splintered, the character of the low temperature phase will depend sensitively on the doping. Based on knowledge of bosonic systems, one expects that the chargons will condense into a superconducting phase upon cooling, with $\langle b^\dagger \rangle \neq 0$. But this condensation can be easily impeded by commensurability effects from the underlying Copper-Oxygen lattice acting in concert with the long-ranged Coulomb interaction. Specifically, in the undoped limit with $x = 0$, there is one charge e chargon per unit cell, and the chargons are expected to lock into a Mott insulating phase, rather than condensing. For very small x with the doped holes well separated, the chargon motion will still be greatly impeded by the near commensurability, and the long-ranged Coulomb interactions should drive charge ordering into an insulating state. Thus, one only expects the chargons to condense into a superconducting phase for x just less than x_c , as depicted schematically in Fig. 9.5, and consistent with experiment.

9.6 CHARGON CONDENSATION AND SUPERCONDUCTIVITY

What is the nature of the chargon condensed superconducting phase? In a conventional BCS description of superconductivity, two electrons near the Fermi surface pair, and the resulting charge $2e$ Cooper pairs condense. Within this charge $2e$ boson condensed superconductor, the flux quantum is halved, given by $\phi_0 = (1/2)(hc/e) = hc/2e$. This is the value of the observed flux quantum in the Cuprate superconductors, suggesting that the superconducting phase itself is of the BCS variety. But the chargons carry the electron charge e , so one might have thought that the BCS superconductor would be equivalent to a chargon-pair condensate - with $\langle b^2 \rangle \neq 0$ yet $\langle b \rangle = 0$ - rather than a single chargon condensate, with $\langle b \rangle \neq 0$. But, quite remarkably, this is not the case. As detailed in Ref. [7], it is the condensation of single charge e chargons that corresponds to the

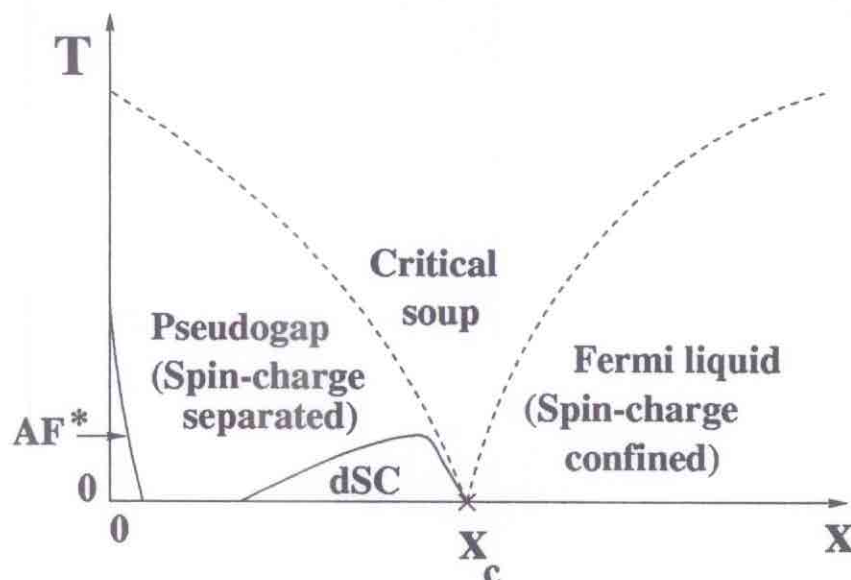


Figure 9.5: Schematic finite temperature phase diagram as a function of doping x , with the quantum confinement transition at $x = x_c$. For x just less than x_c and at low temperature, the superconducting state arises as an *inevitable consequence* of the fractionalization of the electron. At small x and low temperature, the system orders antiferromagnetically. The resulting phase (denoted AF^*) is nevertheless spin-charge separated - see discussion in the text.

conventional BCS superconductor, whereas the chargon pair condensate describes an exotic non-BCS superconducting phase.

This remarkable fact indicates a new route to superconductivity, very different from a Cooper pairing of electrons. Instead, via a fractionalization process the electron charge is liberated from its Fermi statistics - resulting in bosonic charge e particles. A direct condensation of these chargons gives the conventional BCS superconducting phase. Since fractionalization is tantamount to a gapping of the vison excitations, this occurs below the crossover line depicted in Fig. 9.5. Thus, below this crossover line one has “preformed” superconductivity, with liberated chargons poised to condense. The electron spin is carried by fermionic spinons in this regime, which are presumed to be gapped throughout the Brillouin zone, except for four gapless nodal points. This leads naturally to a gapping of spin excitations upon fractionalization. Thus, the non-superconducting vison “gapped” regime can account for the observed “pseudo-gap” phase in the underdoped cuprates.

9.7 QUANTUM CONFINEMENT CRITICAL POINT

Finally, we discuss the regime intervening between the pseudo-gap and Fermi-liquid phases, centered around $x = x_c$. In this regime the visons are neither

gapped nor condensed, but in a critical state. The chargons and spinons which are separated in the vison gapped regime, and confined into the electron when the visons have condensed, are in a state of limbo near $x = x_c$. They cannot move as independent free excitations since they are both strongly coupled to the critical fluctuations of the visons, but they also cannot move together as a confined electron. The precise behavior in this critical regime will be controlled by the nature of the zero temperature quantum phase transition, at $x = x_c$ in the Fig. 9.3.

To our knowledge, the possibility and implications of a direct quantum phase transition between a d-wave superconductor and a Fermi liquid phase has not been discussed previously. Within conventional BCS theory there is *no* quantum phase transition separating the Fermi liquid and superconducting phases. Rather, the Fermi liquid phase in the presence of arbitrarily weak phonon mediated attraction between the electrons is *unstable* to the formation of Cooper pairs which then condense leading to superconductivity. Within a modern renormalization group framework, one would say that the fixed point describing the Fermi liquid phase is unstable and crosses over to the superconducting fixed point, as depicted schematically in Fig. 9.6. We are suggesting an alternate possibility interconnecting these two phases. As depicted in Fig. 9.7, we imagine the existence of an unstable fixed point, denoted QCCP (quantum confinement critical point), which controls the nature of a strong coupling zero temperature phase transition between the Fermi liquid and superconducting phases. The existence of such a fixed point is strongly implied by our Z_2 gauge theory formulation. To see this, imagine initially decoupling the chargons and spinons in Eqn. 9.1, by setting $t_c = t_s = t_\Delta = 0$ and putting $S_B = 0$. The remaining theory describes a pure Z_2 gauge theory, which has two phases [9] - a phase with gapped visons for $K > K_c$, and a vison condensed phase when $K < K_c$. Now recouple the chargon and spinon fields. When the visons are gapped, the chargons and spinons can propagate independently, forming a Bose and Fermi fluid, respectively. Presuming one is not too close to a strongly commensurate filling, the fluid of bosonic chargons should condense at low temperatures giving superconductivity. On the other hand, when the *visons* condense, they *confine* the spinons and chargons, giving fermionic charge e carriers - the electron. Forming a fermionic fluid, these electrons of course cannot condense. Rather, one expects that away from commensurate fillings they will form a conventional metallic Fermi liquid. Finally, right at $K = K_c$ the visons will be in a critical state - described [9] by the classical 3d Ising model when the spinon and chargon coupling is ignored. Here, one expects the spinons and chargons to be strongly scattering off these critical fluctuations, forming a strongly interacting "soup".

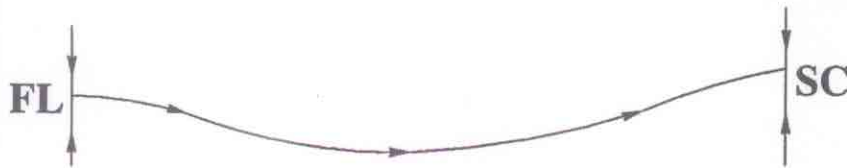


Figure 9.6: A two dimensional section of the renormalization group flow diagram showing the instability of a Fermi liquid in the presence of arbitrary weak attractive interactions. The resulting Cooper pairs condense, leading to superconductivity.

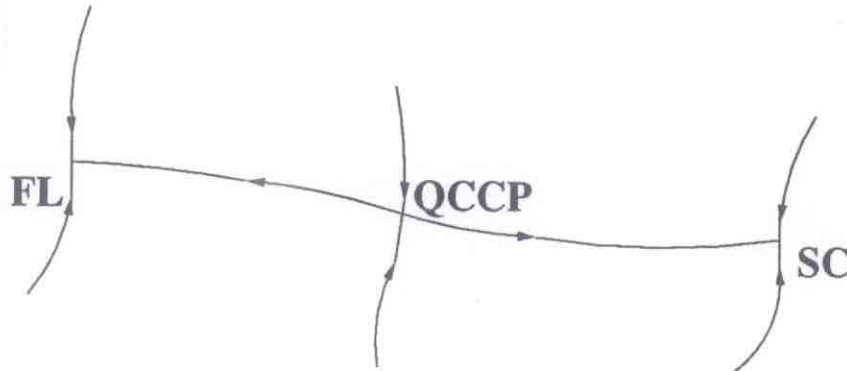


Figure 9.7: A two dimensional section of the renormalization group flow diagram illustrating the different route to superconductivity envisaged in this paper. Central to the proposal is the existence of an unstable fixed point (QCCP) controlling a quantum phase transition at the point of instability of a Fermi liquid toward fractionalization. The resulting spinless charge e bosons condense, leading to superconductivity.

9.8 EXPERIMENTAL IMPLICATIONS

We now turn briefly to a few experimental implications of the above scenario, focussing initially on the vison gapped regime for $x < x_c$. Here, the electron is fractionalized - an electron added to the system will decay into a spinon and chargon. This has direct implications for electron photoemission experiments. Since the electron decays, one does not expect a sharp spectral feature in photoemission. More formally, in this regime the electron propagator, $G(r, \tau)$, can be roughly expressed as a product of the chargon and spinon propagators, G_c and G_s :

$$G(r, \tau) \approx G_c(r, \tau)G_s(r, \tau). \quad (9.5)$$

The spectral functions for the spinons and chargons ($A(k, \omega) = -\frac{1}{\pi} \text{Im}G(k, \omega)$) will have sharp spectral features since these particles can propagate coherently when the visons are gapped, but the *electron* spectral function is a convolution of these two and will hence not exhibit any sharp spectral features. This is exactly as seen in the normal state ARPES spectra in the underdoped samples. Now

consider cooling the system into the superconducting state. As explained above, this requires condensation of the chargons so that

$$G_c(r, \tau) \approx |\langle b \rangle|^2. \quad (9.6)$$

Then, from Eqn. 9.5, the electron Green's function just reduces to

$$G(r, \tau) \approx |\langle b \rangle|^2 G_f(r, \tau), \quad (9.7)$$

and is simply proportional to the spinon Green's function inside the superconductor. Since the spinons propagate coherently, a sharp quasiparticle peak is expected - exactly as seen in the experiments [10]. Moreover, since the *amplitude* of the peak is proportional to $|\langle b \rangle|^2$, it should become smaller as the superconductivity weakens, for instance, by reducing the doping. This is also borne out by the photoemission data [6] - see Fig. 9.2. Thus, the qualitative trends in the underdoped photoemission experiments can be well-explained by assuming the electron decays into a chargon and a spinon.

For $x > x_c$, the low temperature properties of the system should be those of a Fermi liquid. This is commonly believed to be true. It would, however, be useful to have more detailed experimental support.

Now consider the "quantum critical" regime with $x \approx x_c$. As is usual near critical points, power-law temperature dependences are expected for various physical quantities. It is well-known that this is seen in a variety of experiments near optimal doping. In particular, the resistivity in the *ab* plane exhibits a striking linear temperature dependence. In our scenario, the scattering of the chargons off the critical visons is expected to give a power law resistivity $\rho(T) \sim T^\alpha$ with an exponent α that is at present unknown. Calculation of this and other universal properties of this quantum confinement transition is an important challenge that we leave for future work.

Thus far we have primarily focussed on the doping regimes near the superconducting phase. We now turn to the *highly* underdoped and undoped materials. As discussed previously, upon approaching half-filling the condensation of the chargons is expected to be inhibited by commensurability effects together with the long-range Coulomb interactions. Instead, the chargons will localize. Away from half-filling, the charge localization will break the translational symmetry of the lattice. This is qualitatively consistent with the several experiments that observe stripe formation in this region at low temperatures. It is important to stress, however, that in our scenario charge localization and translational symmetry breaking *coexist* with electron fractionalization.

What is the fate of the gapped visons in the undoped material? It is very well established that the undoped insulator has antiferromagnetic long-ranged order. But magnetic order, just like charge order, is conceptually independent of whether or not the electron is fractionalized, in other words, whether the vison is gapped or not. One can therefore contemplate two possibilities - (a) the visons are gapped in

the undoped antiferromagnetic insulator, denoted AF^* , or (b) the visons are condensed leading to a conventional antiferromagnetic insulator, denoted AF , with the electron in the spectrum. Note that the excellent description of the *low energy* spin physics by the quantum Heisenberg spin model is not sufficient to dispose of this question. In fact, the two alternatives are distinguished by the nature of the *gapped* excitations. Experimental evidence for possibility (a) follows from recent photoemission experiments [4] on undoped cuprates, which do not exhibit a sharp quasiparticle peak at *any* momentum in the Brillouin zone. Following Balents et. al. [8], we thus suggest that the electron decays even in the undoped material, and that the visons are gapped. Further qualitative support is provided by mid-infrared optical absorption [11] and Raman [12] measurements in the undoped material which exhibit broad spectral features out to rather high energies, not expected for the simple Heisenberg model.

9.9 COMPARISON WITH ANDERSON'S RVB STATE

Since the original discovery of high temperature superconductivity, literally thousands of theories have been put forward to explain the phenomena. The scenario we describe above has some overlap with many earlier approaches, but is perhaps closest in spirit to the original Resonating Valence Bond (RVB) theory of Anderson [1]. Here, we briefly mention the key similarities and differences with the RVB theory. In the original RVB theory, spin-charge separation is intimately connected with the presence of a "spin-liquid" Mott insulating state, which was argued to support neutral spin one-half spinon excitations. It was soon established, however, that the undoped parent compounds are not spin-liquids but rather antiferromagnetically ordered. It then appeared that the RVB state, if present at all, required the presence of doped holes. In sharp contrast, within our Z_2 gauge theory approach spin-charge separation - or more generally electron fractionalization - is *not* directly linked to magnetic ordering. Rather, electron fractionalization occurs whenever the visons are gapped. This is possible even in the presence of long-range magnetic order, in which case the gapless magnons co-exist with gapped spinon excitations. We believe that this is a likely situation in the undoped cuprates.

The original motivation for the RVB approach was based on an analogy with the physics of spinons in one-dimension. But our approach demonstrates that spin-charge separation in two-dimensions requires the existence of a deconfined phase of the underlying Z_2 gauge theory. In one-dimension this gauge theory always confines, and spin-charge separation occurs via a different solitonic mechanism. Apparently, this solitonic mechanism of spin-charge separation encapsulated within the RVB approach [14], is not generally operative in higher dimensions. In more formal terms, one can attempt [15] to implement RVB theory directly in two-dimensions with a U(1) or SU(2) gauge theory. But despite the apparent similarity with our Z_2 gauge theory, these continuous gauge theories

do not have a deconfined phase, and are thus apparently incapable of describing spin-charge separation.

Within the RVB framework, it is common to describe the valence bonds as being "Cooper pairs" pre-formed in the insulator [1], which become mobile upon doping and condense into the superconducting phase. In contrast, in our Z_2 gauge theory the picture underlying the superconductivity is the liberation of the electron's charge from its Fermi statistics, to form bosonic charge e particles - the chargons. Upon doping the chargons become mobile and can condense giving rise to superconductivity.

An entirely new aspect of our approach is the suggestion of a quantum confinement transition separating the spin-charge separated pseudo-gap regime from the heavily overdoped Fermi liquid phase. At this transition the visons are neither gapped nor condensed, but rather in a gapless critical state. Similarly, the spinons and chargons can neither propagate coherently as independent excitations nor as a confined electron. We believe that this quantum confinement transition might well account for much of the novel behavior observed near optimal doping in the cuprates. Developing a theoretical approach to access the properties of such confinement transitions remains as an important yet challenging task.

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