Problem 1. (10) Prove that if a differentiable relationship \( F(x, y, z) = 0 \) holds between three variables, the following identities are valid:

\[
\left( \frac{dx}{dy} \right)_z \left( \frac{dy}{dx} \right)_z = 1 \tag{1}
\]

\[
\left( \frac{dx}{dy} \right)_z \left( \frac{dy}{dz} \right)_x \left( \frac{dz}{dx} \right)_y = -1 \tag{2}
\]

Problem 2. (20) A heat pump is essentially a Carnot refrigerator, for which the heating of the high temperature reservoir is of the primary interest. Suppose a building is heated to 25°C by a heat pump operating between the building and the underground spring at 5°C. By how much is the energy input multiplied in this device, as compared to a simple electric heating system which converts the energy input directly into heat.

Problem 3. (30) Consider a situation where entropy \( S(U, V) \) of a system with fixed number of particles is given by

\[
S(U, V) = aU^2 + bUV + cV^2 \tag{3}
\]

where \( a, b, c \) are some constants.

Find the pressure and temperature by taking appropriate partial derivatives. Determine the Helmholtz and Gibbs free energies \( F(T, V) \) and \( G(T, p) \) as explicit functions of their arguments. Verify that

\[
\left( \frac{\partial F}{\partial T} \right)_V = -S \quad \text{and} \quad \left( \frac{\partial G}{\partial p} \right)_T = V \tag{4}
\]

Problem 4 (40) In solids and liquids one often measures the expansion coefficient

\[
\kappa_E = V^{-1} \left( \frac{\partial V}{\partial T} \right)_p \tag{5}
\]

and isothermal compressibility

\[
\kappa_T = -V^{-1} \left( \frac{\partial V}{\partial p} \right)_T \tag{6}
\]
Show that specific heats \( c_p = T \left( \frac{\partial S}{\partial T} \right)_p \) and \( c_V = T \left( \frac{\partial S}{\partial T} \right)_V \) are related via

\[
c_p - c_V = V T \kappa_E^2 \kappa_T^{-1}
\]  

(7)

Start by regarding the entropy as a function of \( T \) and \( V \) even though these are not the "natural" variables for entropy.

Some "hints":

1) you may want to remember the existence of an equation of state \( (p = p(T, V)) \) even though you don’t need to know what it is.

2) you may want to use a formula for differentiation under constraint. For an arbitrary \( \Phi(x, y) \) and some constraint \( f(x, y) = q \):

\[
\left( \frac{\partial \Phi}{\partial x} \right)_q = \left( \frac{\partial \Phi}{\partial x} \right)_y + \left( \frac{\partial \Phi}{\partial y} \right)_x \left( \frac{\partial y}{\partial x} \right)_q
\]

(8)

3) you may want to use "Maxwell relation"

\[
\left( \frac{\partial V}{\partial T} \right)_p = - \left( \frac{\partial S}{\partial p} \right)_T
\]

(9)

which follows from

\[
\frac{\partial}{\partial T} \left( \frac{\partial G(T, P)}{\partial p} \right) = \frac{\partial}{\partial p} \left( \frac{\partial G(T, P)}{\partial T} \right)
\]

(10)