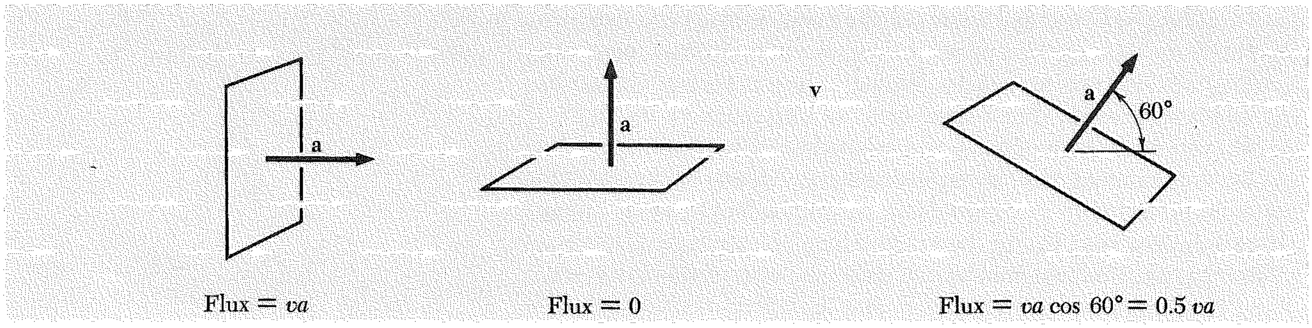


FIGURE 1.14
The flux through the frame of area **a** is **v** · **a**, where **y** is the velocity of the fluid. The flux is the volume of fluid passing through the frame, per unit time.

imagine a vector function which represents the velocity of motion in a fluid—say in a river, where the velocity varies from one place to another but is constant in time at any one position. Denote this vector field by **v**, measured, say, in meters/sec. Then, if **a** is the oriented area in square meters of a frame lowered into the water, **v** · **a** is the *rate of flow* of water through the frame in cubic meters per second (Fig. 1.14). We must emphasize that our definition of flux is applicable to any vector function, whatever physical variable it may represent.

Now let us add up the flux through all the patches to get the flux through the entire surface, a scalar quantity which we shall denote by Φ:

$$\Phi = \sum_{\text{All } j} \mathbf{E}_j \cdot \mathbf{a}_j \tag{16}$$

Letting the patches become smaller and more numerous without limit, we pass from the sum in Eq. 16 to a surface integral:

$$\Phi = \int_{\substack{\text{Entire}\\ \text{surface}}} \mathbf{E} \cdot d\mathbf{a} \tag{17}$$

A surface integral of any vector function **F**, over a surface S, means just this: Divide S into small patches, each represented by a vector outward, of magnitude equal to the patch area; at every patch, take the scalar product of the patch area vector and the local **F**; sum all these products, and the limit of this sum, as the patches shrink, is the surface integral. Do not be alarmed by the prospect of having to perform such a calculation for an awkwardly shaped surface like the one in Fig. 1.13. The surprising property we are about to demonstrate makes that unnecessary!

GAUSS'S LAW

1.10 Take the simplest case imaginable; suppose the field is that of a single isolated positive point charge q and the surface is a sphere of radius r centered on the point charge (Fig. 1.15). What is the flux Φ through this surface? The answer is easy because the magnitude of $\mathbf{E}$ at every point on the surface is q/r^2 and its direction is the same as that of the outward normal at that point. So we have

$$\Phi = E \times \text{total area} = \frac{q}{r^2} \times 4\pi r^2 = 4\pi q \tag{18}$$

The flux is independent of the size of the sphere.

Now imagine a second surface, or balloon, enclosing the first, but *not* spherical, as in Fig. 1.16. We claim that the total flux through this surface is the same as that through the sphere. To see this, look at a cone, radiating from q, which cuts a small patch $\mathbf{a}$ out of the sphere and continues on to the outer surface where it cuts out a patch $\mathbf{A}$ at a distance R from the point charge. The area of the patch $\mathbf{A}$ is larger than that of the patch $\mathbf{a}$ by two factors: first, by the ratio of the distance squared $(R/r)^2$; and second, owing to its inclination, by the factor $1/\cos\theta$. The angle θ is the angle between the outward normal and the radial direction (see Fig. 1.16). The electric field in that neighborhood is reduced from its magnitude on the sphere by the factor $(r/R)^2$ and is still radially directed. Letting $\mathbf{E}_{(R)}$ be the field at the outer patch and $\mathbf{E}_{(r)}$ be the field at the sphere, we have

$$\begin{aligned} \text{Flux through outer patch} &= \mathbf{E}_{(R)} \cdot \mathbf{A} = E_{(R)} A \cos\theta \\ \text{Flux through inner patch} &= \mathbf{E}_{(r)} \cdot \mathbf{a} = E_{(r)} a \end{aligned} \tag{19}$$

$$E_{(R)} A \cos\theta = \left[E_{(r)} \left(\frac{r}{R}\right)^2 \right] \left[a \left(\frac{R}{r}\right)^2 \frac{1}{\cos\theta} \right] \cos\theta = E_{(r)} a$$

This proves that the flux through the two patches is the same.

Now every patch on the outer surface can in this way be put into correspondence with part of the spherical surface, so the total flux must be the same through the two surfaces. That is, the flux through the new surface must be just $4\pi q$. But this was a surface of *arbitrary* shape and size.† We conclude: The flux of the electric field through *any* surface enclosing a point charge q is $4\pi q$. As a corollary we can say that the total flux through a closed surface is *zero* if the charge lies *outside* the surface. We leave the proof of this to the reader, along with Fig. 1.17 as a hint of one possible line of argument.

There is a way of looking at all this which makes the result seem obvious. Imagine at q a source which emits particles—such as bullets or photons—in all directions at a steady rate. Clearly the flux of particles through a window of unit area will fall off with the inverse square of the window's distance from q. Hence we can draw an analogy between the electric field strength E and the intensity of particle

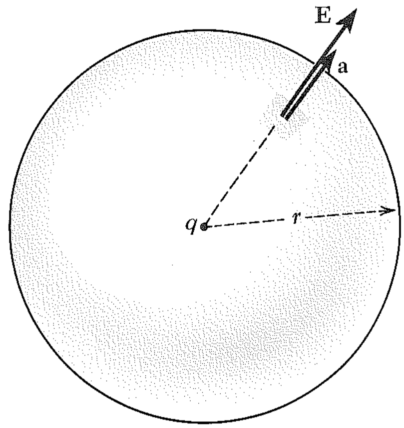


FIGURE 1.15
In the field $\mathbf{E}$ of a point charge q, what is the outward flux over a sphere surrounding q?

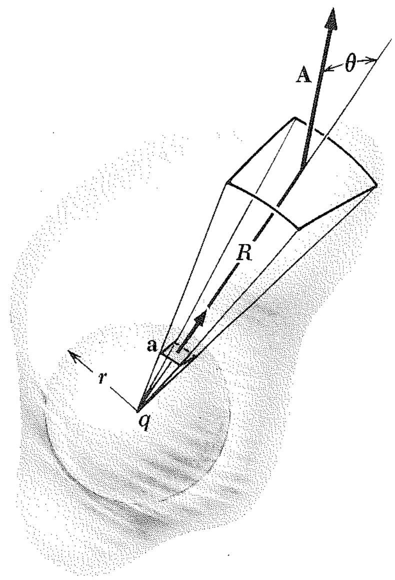


FIGURE 1.16
Showing that the flux through any closed surface around q is the same as the flux through the sphere.

†To be sure, we had the second surface enclosing the sphere, but it didn't have to, really. Besides, the sphere can be taken as small as we please.

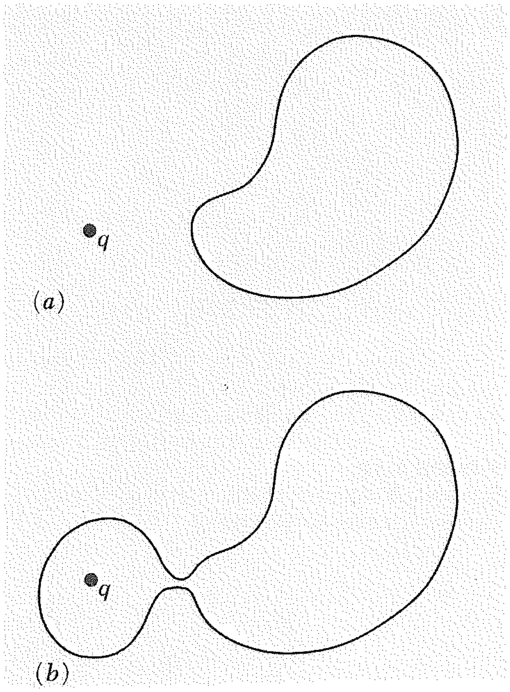


FIGURE 1.17
To show that the flux through the closed surface in (a) is zero, you can make use of (b).

flow in bullets per unit area per unit time. It is pretty obvious that the flux of bullets through any surface completely surrouding q is independent of the size and shape of that surface, for it is just the total number emitted per unit time. Correspondingly, the flux of E through the closed surface must be independent of size and shape. The common feature responsible for this is the inverse-square behavior of the intensity.

The situation is now ripe for superposition! Any electric field is the sum of the fields of its individual sources. This property was expressed in our statement, Eq. 13, of Coulomb's law. Clearly flux is an additive quantity in the same sense, for if we have a number of sources, $q_1, q_2, \ldots, q_N$, the fields of which, if each were present alone, would be $\mathbf{E}_1, \mathbf{E}_2 \ldots, \mathbf{E}_N$, the flux Φ through some surface S in the actual field can be written

$$\Phi = \int_S \mathbf{E} \cdot d\mathbf{a} = \int_S (\mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_N) \cdot d\mathbf{a} \qquad (20)$$

We have just learned that $\int_S \mathbf{E}_n \cdot d\mathbf{a}$ equals $4\pi q_n$ if the charge q_n is inside S and equals zero otherwise. So every charge q inside the surface contributes exactly $4\pi q$ to the surface integral of Eq. 20 and all charges outside contribute nothing. We have arrived at Gauss's law:

> The flux of the electric field $\mathbf{E}$ through any closed surface, that is, the integral $\int \mathbf{E} \cdot d\mathbf{a}$ over the surface, equals 4π times the total charge enclosed by the surface:
>
> $$\int \mathbf{E} \cdot d\mathbf{a} = 4\pi \sum_i q_i = 4\pi \int \rho \, dv \qquad (21)$$

We call the statement in the box a *law* because it is equivalent to Coulomb's law and it could serve equally well as the basic law of electrostatic interactions, after charge and field have been defined. Gauss's law and Coulomb's law are not two independent physical laws, but the same law expressed in different ways.†

†There is one difference, inconsequential here, but relevant to our later study of the fields of moving charges. Gauss' law is obeyed by a wider class of fields than those represented by the electrostatic field. In particular, a field that is inverse-square in r but not spherically symmetrical can satisfy Gauss' law. In other words, Gauss' law alone does not imply the symmetry of the field of a point source which is implicit in Coulomb's law.

Looking back over our proof, we see that it hinged on the inverse-square nature of the interaction and of course on the additivity of interactions, or superposition. Thus the theorem is applicable to any inverse-square field in physics, for instance, to the gravitational field.

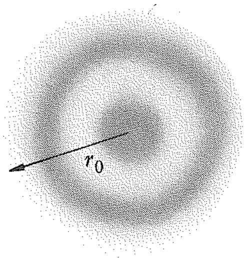
It is easy to see that Gauss's law would *not* hold if the law of force were, say, inverse-cube. For in that case the flux of electric field from a point charge q through a sphere of radius R centered on the charge would be

$$\Phi = \int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{R^3} \cdot 4\pi R^2 = \frac{4\pi q}{R} \tag{22}$$

By making the sphere large enough we could make the flux through it as small as we pleased, while the total charge inside remained constant.

This remarkable theorem enlarges our grasp in two ways. First, it reveals a connection between the field and its sources that is the converse of Coulomb's law. Coulomb's law tells us how to derive the electric field if the charges are given; with Gauss's law we can determine how much charge is in any region if the field is known. Second, the mathematical relation here demonstrated is a powerful analytic tool; it can make complicated problems easy, as we shall see.

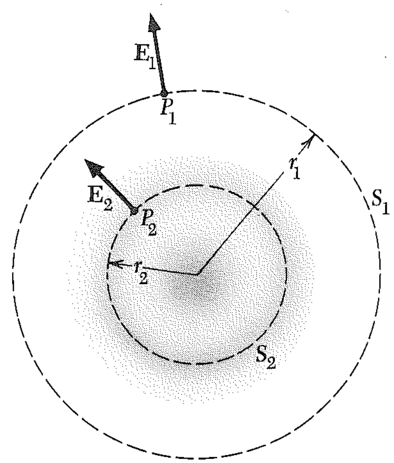
FIGURE 1.18
A charge distribution with spherical symmetry.



FIELD OF A SPHERICAL CHARGE DISTRIBUTION

1.11 We can use Gauss's law to find the electric field of a spherically symmetrical distribution of charge, that is, a distribution in which the charge density ρ depends only on the radius from a central point. Figure 1.18 depicts a cross section through some such distribution. Here the charge density is high at the center, and is zero beyond r_0. What is the electric field at some point such as P_1 outside the distribution, or P_2 inside it (Fig. 1.19)? If we could proceed only from Coulomb's law, we should have to carry out an integration which would sum the electric field vectors at P_1 arising from each elementary volume in the charge distribution. Let's try a different approach which exploits both the symmetry of the system and Gauss's law.

FIGURE 1.19
The electric field of a spherical charge distribution.



Because of the spherical symmetry, the electric field at any point must be radially directed—no other direction is unique. Likewise, the field magnitude E must be the same at all points on a spherical surface S_1 of radius r_1, for all such points are equivalent. Call this field magnitude E_1. The flux through this surface S_1 is therefore simply $4\pi r_1^2 E_1$, and by Gauss's law this must be equal to 4π times the charge enclosed by the surface. That is, $4\pi r_1^2 E_1 = 4\pi$ (charge inside S_1) or

$$E_1 = \frac{\text{charge inside } S_1}{r_1^2} \tag{23}$$

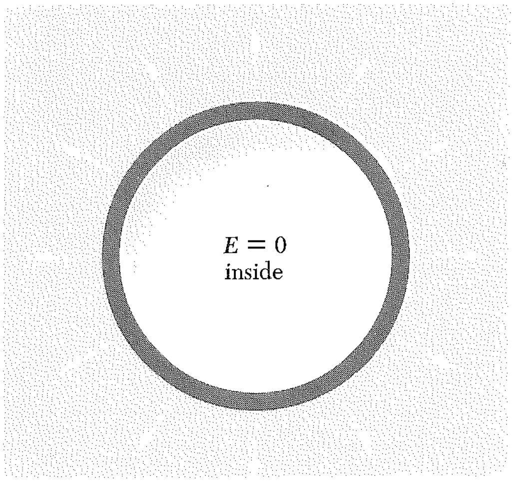


FIGURE 1.20
The field is zero inside a spherical shell of charge.

Comparing this with the field of a point charge, we see that *the field at all points on S_1 is the same as if all the charge within S_1 were concentrated at the center.* The same statement applies to a sphere drawn *inside* the charge distribution. The field at any point on S_2 is the same as if all charge within S_2 were at the center, and all charge *outside* S_2 absent. Evidently the field inside a "hollow" spherical charge distribution is zero (Fig. 1.20).

The same argument applied to the gravitational field would tell us that the earth, assuming it is spherically symmetrical in its mass distribution, attracts outside bodies as if its mass were concentrated at the center. That is a rather familiar statement. Anyone who is inclined to think the principle expresses an obvious property of the center of mass must be reminded that the theorem is not even true, in general, for other shapes. A perfect cube of uniform density does *not* attract external bodies as if its mass were concentrated at its geometrical center.

Newton didn't consider the theorem obvious. He needed it as the keystone of his demonstration that the moon in its orbit around the earth and a falling body on the earth are responding to similar forces. The delay of nearly 20 years in the publication of Newton's theory of gravitation was apparently due, in part at least, to the trouble he had in proving this theorem to his satisfaction. The proof he eventually devised and published in the *Principia* in 1686 (Book I, Section XII, Theorem XXXI) is a marvel of ingenuity in which, roughly speaking, a tricky volume integration is effected without the aid of the integral calculus as we know it. The proof is a good bit longer than our whole preceding discussion of Gauss's law, and more intricately reasoned. You see, with all his mathematical resourcefulness and originality, Newton lacked Gauss's theorem—a relation which, once it has been shown to us, seems so obvious as to be almost trivial.

FIELD OF A LINE CHARGE

1.12 A long, straight, charged wire, if we neglect its thickness, can be characterized by the amount of charge it carries per unit length. Let λ, measured in esu/cm, denote this *linear charge density.* What is the electric field of such a line charge, assumed infinitely long and with constant linear charge density λ? We'll do the problem in two ways, first by an integration starting from Coulomb's law.

To evaluate the field at the point P, shown in Fig. 1.21, we must add up the contributions from all segments of the line charge, one of which is indicated as a segment of length dx. The charge dq on this element is given by $dq = \lambda\, dx$. Having oriented our x axis along the line charge, we may as well let the y axis pass through P, which is r cm from the nearest point on the line. It is a good idea to take advantage of symmetry at the outset. Obviously the electric field at P must

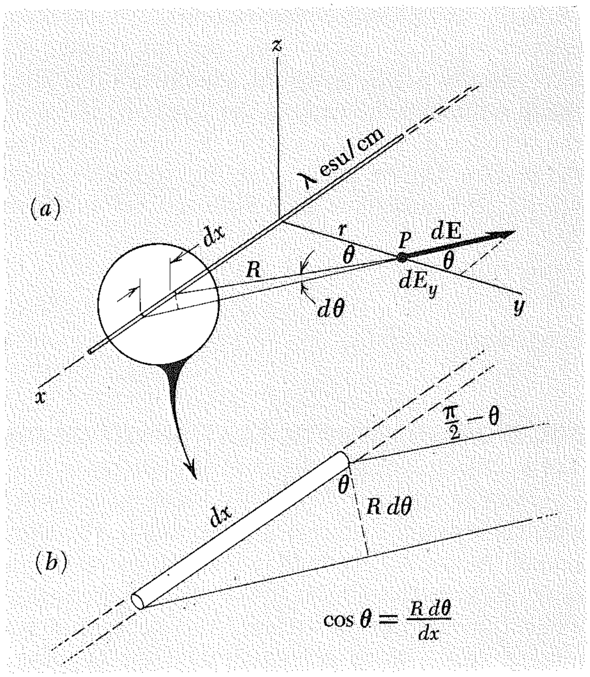


FIGURE 1.21
(*a*) The field at *P* is the vector sum of contributions from each element of the line charge. (*b*) Detail of (*a*).

point in the y direction, so that E_x and E_z are both zero. The contribution of the charge dq to the y component of the electric field at P is

$$dE_y = \frac{dq}{R^2}\cos\theta = \frac{\lambda\,dx}{R^2}\cos\theta \tag{24}$$

where θ is the angle the vector field of dq makes with the y direction. The total y component is then

$$E_y = \int dE_y = \int_{-\infty}^{\infty} \frac{\lambda\cos\theta}{R^2}\,dx \tag{25}$$

It is convenient to use θ as the variable of integration. Since $R = r/\cos\theta$ and $dx = R\,d\theta/\cos\theta$, the integral becomes

$$E_y = \int_{-\pi/2}^{\pi/2} \frac{\lambda\cos\theta\,d\theta}{r} = \frac{\lambda}{r}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta = \frac{2\lambda}{r} \tag{26}$$

We see that the field of an infinitely long, uniformly dense line charge is proportional to the reciprocal of the distance from the line. Its direction is of course radially outward if the line carries a positive charge, inward if negative.

Gauss' law leads directly to the same result. Surround a segment of the line charge with a closed circular cylinder of length L and radius r, as in Fig. 1.22, and consider the flux through this surface. As we have already noted, symmetry guarantees that the field is radial, so the flux through the ends of the "tin can" is zero. The flux through the cylindrical surface is simply the area, $2\pi rL$, times E_r, the field at the surface. On the other hand, the charge enclosed by the surface is just λL, so Gauss's law gives us $2\pi rLE_r = 4\pi\lambda L$ or

$$E_r = \frac{2\lambda}{r} \tag{27}$$

in agreement with Eq. 26.

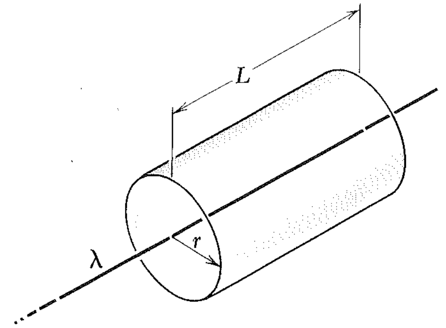


FIGURE 1.22
Using Gauss' law to find the field of a line charge.

FIELD OF AN INFINITE FLAT SHEET OF CHARGE

1.13 Electric charge distributed smoothly in a thin sheet is called a *surface charge distribution.* Consider a flat sheet infinite in extent, with the constant surface charge density σ. The electric field on either side of the sheet, whatever its magnitude may turn out to be, must surely point perpendicular to the plane of the sheet; there is no other unique direction in the system. Also because of symmetry, the field must have the same magnitude and the opposite direction at two points P and P' equidistant from the sheet on opposite sides. With these facts established, Gauss's law gives us at once the field intensity, as follows: Draw a cylinder, as in Fig. 1.23, with P on one side and P' on the other, of cross-section area A. The outward flux is found only at the ends, so that if E_P denotes the magnitude of the field at P, and $E_{P'}$ the magnitude of P', the outward flux is $AE_P + AE_{P'} = 2AE_P$. The charge enclosed is σA. Hence $2AE_P = 4\pi\sigma A$, or

$$E_P = 2\pi\sigma \tag{28}$$

We see that the field strength is independent of r, the distance from the sheet. Equation 28 could have been derived more laboriously by calculating the vector sum of the contributions to the field at P from all the little elements of charge in the sheet.

The field of an infinitely long line charge, we found, varies inversely as the distance from the line, while the field of an infinite sheet has the same strength at all distances. These are simple consequences of the fact that the field of a point charge varies as the inverse square of the distance. If that doesn't yet seem compellingly obvious, look at it this way: Roughly speaking, the part of the line charge that is mainly responsible for the field at P, in Fig. 1.21, is the near part—the charge within a distance of order of magnitude r. If we lump all this together and forget the rest, we have a concentrated charge of magnitude $q \approx \lambda r$, which ought to produce a field proportional to q/r^2, or λ/r. In the case of the sheet, the amount of charge that is "effective," in this sense, increases proportionally to r^2 as we go out

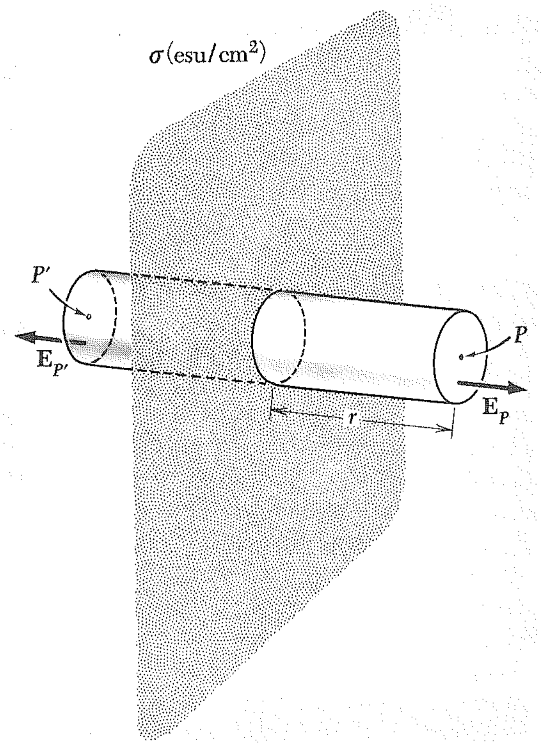


FIGURE 1.23
Using Gauss' law to find the field of an infinite flat sheet of charge.

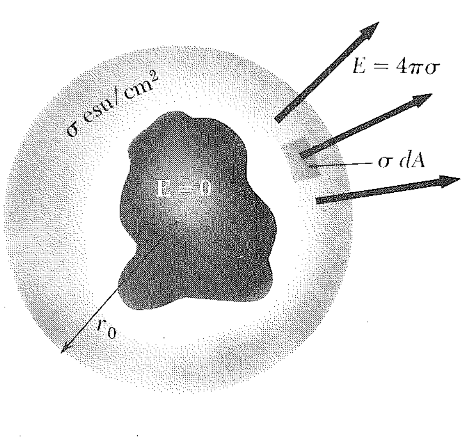
from the sheet, which just offsets the $1/r^2$ decrease in the field from any given element of charge.

THE FORCE ON A LAYER OF CHARGE

1.14 The sphere in Fig. 1.24 has a charge distributed over its surface with the uniform density σ, in esu/cm^2. Inside the sphere, as we have already learned, the electric field of such a charge distribution is zero. Outside the sphere the field is Q/r^2, where Q is the total charge on the sphere, equal to $4\pi r_0^2 \sigma$. Just outside the surface of the sphere the field strength is $4\pi\sigma$. Compare this with Eq. 28 and Fig. 1.23. In both cases Gauss' law is obeyed: The *change* in E, from one side of the layer to the other, is equal to $4\pi\sigma$.

What is the electrical force experienced by the charges that make up this distribution? The question may seem puzzling at first because the field **E** arises from these very charges. What we must think about is the force on some small element of charge dq, such as a small patch of area dA with charge $dq = \sigma\, dA$. Consider, separately, the force on dq due to all the other charges in the distribution,

FIGURE 1.24
A spherical surface with uniform charge density σ.



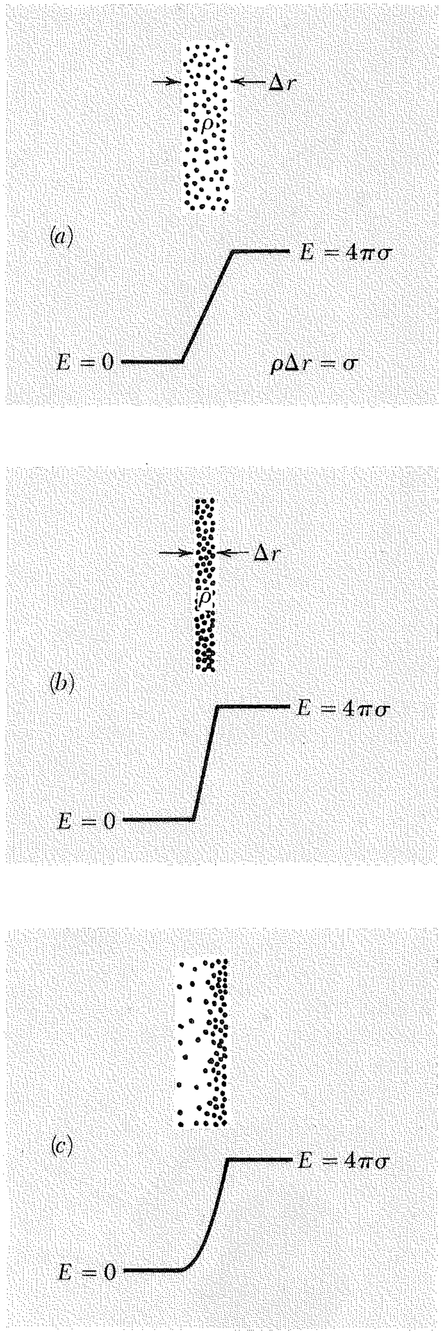


FIGURE 1.25
The net change in field at a charge layer depends only on the total charge per unit area.

and the force on the patch due to the charges within the patch itself. This latter force is surely zero. Coulomb repulsion between charges within the patch is just another example of Newton's third law; the patch as a whole cannot push on itself. That simplifies our problem, for it allows us to use the entire electric field $\mathbf{E}$, *including* the field due to all charges in the patch, in calculating the force $d\mathbf{F}$ on the patch of charge dq:

$$d\mathbf{F} = \mathbf{E}\, dq = \mathbf{E}\, \sigma\, dA \tag{29}$$

But what E shall we use, the field $E = 4\pi\sigma$ outside the sphere or the field $E = 0$ inside? The correct answer, as we shall prove in a moment, is the *average* of the two fields.

$$dF = \tfrac{1}{2}(4\pi\sigma + 0)\, \sigma\, dA = 2\pi\sigma^2\, dA \tag{30}$$

To justify this we shall consider a more general case, and one that will introduce a more realistic picture of a layer of surface charge. Real charge layers do not have zero thickness. Figure 1.25 shows some ways in which charge might be distributed through the thickness of a layer. In each example the value of σ, the total charge per unit area of layer, is the same. These might be cross sections through a small portion of the spherical surface in Fig. 1.24 on a scale such that the curvature is not noticeable. To make it more general, however, we have let the field on the left be E_1 (rather than 0, as it was inside the sphere), with E_2 the field strength on the right. The condition imposed by Gauss's law, for given σ, is in each case

$$E_2 - E_1 = 4\pi\sigma \tag{31}$$

Now let us look carefully within the layer where the field is changing continuously from E_1 to E_2 and there is a volume charge density $\rho(x)$ extending from $x = 0$ to $x = x_0$, the thickness of the layer (Fig. 1.26). Consider a much thinner slab, of thickness $dx \ll x_0$, which contains per unit area an amount of charge $\rho\, dx$. The force on it is

$$dF = E\rho\, dx \tag{32}$$

Thus the total force per unit area of our charge layer is

$$F = \int_0^{x_0} E\rho\, dx \tag{33}$$

But Gauss's law tells us that dE, the change in E through the thin slab, is just $4\pi\rho\, dx$. Hence $\rho\, dx$ in Eq. 33 can be replaced by $dE/4\pi$, and the integral becomes

$$F = \frac{1}{4\pi}\int_{E_1}^{E_2} E\, dE = \frac{1}{8\pi}(E_2^2 - E_1^2) \tag{34}$$

Since $E_2 - E_1 = 4\pi\sigma$, the result in Eq. 34, after being factored, can be expressed as

$$F = \tfrac{1}{2}(E_1 + E_2)\sigma \tag{35}$$

We have shown, as promised, that for given σ the force per unit area on a charge layer is determined by the mean of the external field on one side and that on the other.† This is independent of the thickness of the layer, as long as it is small compared to the total area, and of the variation $\rho(x)$ in charge density within the layer.

The direction of the electrical force on an element of the charge on the sphere is, of course, outward whether the surface charge is positive or negative. If the charges do not fly off the sphere, that outward force must be balanced by some inward force not included in our equations, which can hold the charge carriers in place. To call such a force "nonelectrical" would be misleading, for electrical attractions and repulsions are the dominant forces in the structure of atoms and in the cohesion of matter generally. The difference is that these forces are effective only at short distances, from atom to atom, or from electron to electron. Physics on that scale is a story of individual particles. Think of a charged rubber balloon, say, 10 cm in radius, with 20 esu of negative charge spread as uniformly as possible on its outer surface. It forms a surface charge of density $\sigma = 20/400\pi = 0.016$ esu/cm^2. The resulting outward force, per cm^2 of surface charge, is $2\pi\sigma^2$, or 0.0016 dynes/cm^2. In fact our charge consists of about 4×10^{10} electrons attached to the rubber film. As there are about 30 million extra electrons per cm^2, "graininess" in the charge distribution is hardly apparent. However, if we could look at one of these extra electrons, we would find it roughly 10^{-4} cm—an enormous distance on an atomic scale—from its nearest neighbor. This electron would be stuck, electrically stuck, to a local molecule of rubber. The rubber molecule would be attached to adjacent rubber molecules, and so on. If you pull on the electron, the force is transmitted in this way to the whole piece of rubber. Unless, of course, you pull hard enough to tear the electron loose from the molecule to which it is attached. That would take an electric field many thousands of times stronger than the field in our example.

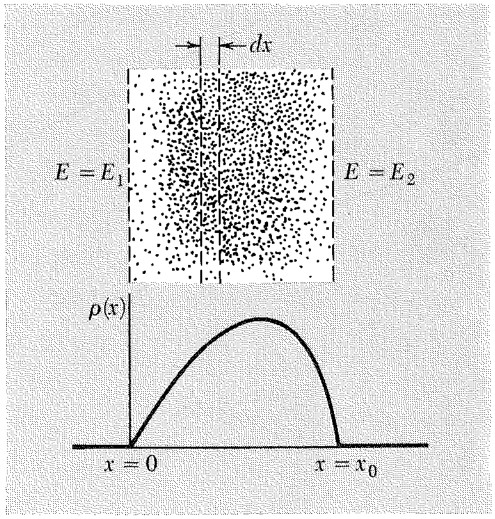
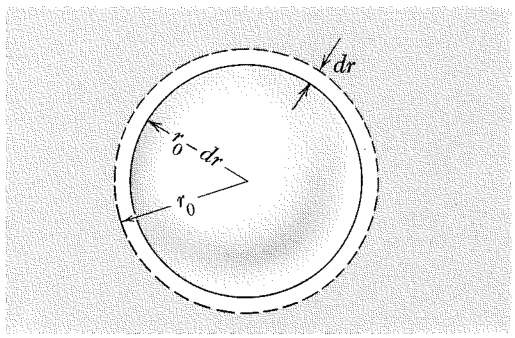


FIGURE 1.26
Within the charge layer of density $\rho(x)$, $E(x + dx) - E(x) = 4\pi\rho\, dx$.

ENERGY ASSOCIATED WITH THE ELECTRIC FIELD

1.15 Suppose our spherical shell of charge is compressed slightly, from an initial radius of r_0 to a smaller radius, as in Fig. 1.27. This requires that work be done against the repulsive force, $2\pi\sigma^2$ dynes for

FIGURE 1.27
Shrinking a spherical shell or charged balloon.



†Note that this is *not* necessarily the same as the average field within the layer, a quantity of no special interest or significance.

each square centimeter of surface. The displacement being dr, the total work done is $(4\pi r_0^2)(2\pi\sigma^2)\ dr$, or $8\pi^2 r_0^2 \sigma^2\ dr$. This represents an *increase* in the energy required to assemble the system of charges, the energy U we talked about in Section 1.5:

$$dU = 8\pi^2 r_0^2 \sigma^2\ dr \tag{36}$$

Notice how the electric field E has been changed. Within the shell of thickness dr the field was zero and is now $4\pi\sigma$. Beyond r_0 the field is unchanged. In effect we have created a field of strength $E = 4\pi\sigma$ filling a region of volume $4\pi r_0^2\ dr$. We have done so by investing an amount of energy given by Eq. 36 which, if we substitute $E/4\pi$ for σ, can be written like this:

$$dU = \frac{E^2}{8\pi}\, 4\pi r_0^2\ dr \tag{37}$$

This is an instance of a general theorem which we shall not prove now: *The potential energy U of a system of charges, which is the total work required to assemble the system, can be calculated from the electric field itself simply by assigning an amount of energy $(E^2/8\pi)\ dv$ to every volume element dv and integrating over all space where there is electric field.*

$$\boxed{U = \frac{1}{8\pi} \int_{\substack{\text{Entire}\\ \text{field}}} E^2\ dv} \tag{38}$$

E^2 is a scalar quantity, of course: $E^2 \equiv \mathbf{E} \cdot \mathbf{E}$.

One may think of this energy as "stored" in the field. The system being conservative, that amount of energy can of course be recovered by allowing the charges to go apart; so it is nice to think of the energy as "being somewhere" meanwhile. Our accounting comes out right if we think of it as stored in space with a density of $E^2/8\pi$, in ergs/cm^3. There is no harm in this, but in fact we have no way of identifying, quite independently of anything else, the energy stored in a particular cubic centimeter of space. Only the total energy is physically measurable, that is, the work required to bring the charge into some configuration, starting from some other configuration. Just as the concept of electric field serves in place of Coulomb's law to explain the behavior of electric charges, so when we use Eq. 38 rather than Eq. 9 to express the total potential energy of an electrostatic system, we are merely using a different kind of bookkeeping. Sometimes a change in viewpoint, even if it is at first only a change in bookkeeping, can stimulate new ideas and deeper understanding. The notion of the electric field as an independent entity will take form when we study the dynamical behavior of charged matter and electromagnetic radiation.

We run into trouble if we try to apply Eq. 38 to a system that contains a point charge, that is, a finite charge q of zero size. Locate q at the origin of the coordinates. Close to the origin E^2 will approach q^2/r^4. With $dv = 4\pi r^2\, dr$, the integrand $E^2\, dv$ will behave like dr/r^2, and our integral will blow up at the limit $r = 0$. That simply tells us that it would take infinite energy to pack finite charge into zero volume—which is true but not helpful. In the real world we deal with particles like electrons and protons. They are so small that for most purposes we can ignore their dimensions and think of them as point charges when we consider their electrical interaction with one another. How much energy it took to make such a particle is a question that goes beyond the range of classical electromagnetism. We have to regard the particles as supplied to us ready-made. The energy we are concerned with is the work done in moving them around.

The distinction is usually clear. Consider two charged particles, a proton and a negative pion, for instance. Let $\mathbf{E}_p$ be the electric field of the proton, $\mathbf{E}_\pi$ that of the pion. The total field is $\mathbf{E} = \mathbf{E}_p + \mathbf{E}_\pi$, and $\mathbf{E} \cdot \mathbf{E}$ is $E_p^2 + E_\pi^2 + 2\mathbf{E}_p \cdot \mathbf{E}_\pi$. According to Eq. 38 the total energy in the electric field of this two-particle system is
tem of elementary charged particles. Indeed, we want to omit them.

$$
\begin{aligned}
U &= \frac{1}{8\pi}\int E^2\, dv \\
&= \frac{1}{8\pi}\int E_p^2\, dv + \frac{1}{8\pi}\int E_\pi^2\, dv + \frac{1}{4\pi}\int \mathbf{E}_p \cdot \mathbf{E}_\pi\, dv
\end{aligned}
\tag{39}
$$

The value of the first integral is a property of any isolated proton. It is a constant of nature which is not changed by moving the proton around. The same goes for the second integral, involving the pion's electric field alone. It is the third integral that directly concerns us, for it expresses the energy required to assemble the system *given* a proton and a pion as constituents.

The distinction could break down if the two particles interact so strongly that the electrical structure of one is distorted by the presence of the other. Knowing that both particles are in a sense composite (the proton consisting of three quarks, the pion of two), we might expect that to happen during a close approach. In fact, nothing much happens down to a distance of 10^{-13} cm. At shorter distances, for strongly interacting particles like the proton and the pion, nonelectrical forces dominate the scene anyway.

That explains why we do not need to include "self-energy" terms like the first two integrals in Eq. 39 in our energy accounts for a system of elementary charged particles. Indeed, we want to omit them. We are doing just that, in effect, when we replace the actual distribution of discrete elementary charges (the electrons on the rubber balloon) by a perfectly continuous charge distribution.